Quantum Black Holes

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[Disclaimer: QFT à la "theoretical physics" (not "mathematical physics")]

I. Brief Recap of Classical Field Theory

General Relativity

$\hbar = c = 1$

- Line-element & metric of a curved s-t:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad\qquad g \equiv det(g_{\mu\nu})$$

- Einstein-Hilbert action:

$$S_{GR}[g_{\mu\nu}] = \frac{1}{16\pi G} \int_\Omega d^4x \, \sqrt{-g} \, (R + 2\Lambda)$$

Ω: region of s-t

R: Ricci scalar

Λ: Cosmological const.

- Einstein-Hilbert action with matter action:

$$S[\phi, g_{\mu\nu}] = S_{GR}[g_{\mu\nu}] + S_m[\phi, g_{\mu\nu}]$$

Action principle:

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \longrightarrow$$

Einstein field eqs.:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Stress-energy tensor: $T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$

T_{00} : energy density

T_{0j} : momentum density $\qquad (\mu, \nu = 0, 1, 2, 3; \qquad j = 1, 2, 3)$

T_{ij} : (normal & shear) stress

Matter: Scalar Field

For a real (minimally-coupled) scalar field of mass m:

$$S_m[\phi; g_{\mu\nu}] = \int_\Omega d^4x \sqrt{-g} \underbrace{\left[-\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}m^2\phi^2\right]}$$

$\mathcal{L}_m$: Lagrangian density

$$\xrightarrow{\delta S_m/\delta g^{\mu\nu}} \quad T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi^{;\alpha}\phi_{;\alpha} - \frac{1}{2}m^2\phi^2 g_{\mu\nu}$$

It's conserved: $\nabla_\nu T^{\mu\nu} = 0$

$\delta S_m/\delta\phi = 0$

$\longrightarrow$ Klein-Gordon eq.: $\Box\phi - m^2\phi = 0$

$$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$$

Scalar Field in *Flat* Spacetime

- Klein-Gordon eq. for a massive real scalar classical field in flat s-t:

$$\Box\phi - m^2\phi = 0 \qquad\qquad \Box = -\partial_t^2 + \vec{\nabla}^2$$

- Apply spatial Fourier transform in Cartesian coords.

The modes $\displaystyle \phi_{\vec{k}}(t) \equiv \int_{\mathbb{R}^3} \frac{d^3\vec{x}}{(2\pi)^{3/2}}\, e^{-i\vec{k}\vec{x}}\phi(\vec{x},t)$

satisfy the eq. for the simple harmonic oscillator:

$$\frac{d^2\phi_{\vec{k}}}{dt^2} + \omega_k^2\phi_{\vec{k}} = 0$$

oscillator frequency: $\omega_k \equiv \sqrt{k^2 + m^2}$

$$k \equiv ||\vec{k}||$$

The *general* sln. may be expressed via the inverse Fourier transform as

$$\phi(x) = \int_{\mathbb{R}^3} d^3\vec{k} \left(a_{\vec{k}}\, u_{\vec{k}}(x) + \underbrace{a^*_{\vec{k}}\, u^*_{\vec{k}}(x)}_{\text{so that } \phi(x) \in \mathbb{R}} \right)$$

↑ Fourier coefficients

$$u_{\vec{k}}(x) \equiv \frac{e^{-i\omega_k t + i\vec{k}\vec{x}}}{\sqrt{16\pi^3 \omega_k}}$$

are mode slns. which are positive frequency wrt ∂_t :

$$\partial_t u_{\vec{k}} = -i\, \underset{0}{\underset{\vee}{\omega_k}}\, u_{\vec{k}}$$

In particular, the Hamiltonian is that for a set of harm. oscs.:

$$H = \int_{\mathbb{R}^3} d^3\vec{x}\; T_{00} = \frac{1}{2}\int_{\mathbb{R}^3} d^3\vec{x}\;\left(\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$= \int_{\mathbb{R}^3} d^3\vec{k}\;\underbrace{\frac{\omega_k}{2}\left[a^*_{\vec{k}}a_{\vec{k}} + a_{\vec{k}}a^*_{\vec{k}}\right]}_{\text{it's the Hamiltonian of the harm. osc.}}$$

The field may be viewed as an infinite set of decoupled simple harm. oscs., one for each $\vec{k}$

II. QFT in Flat Space-time

(a) Formalism

Canonical Quantization

- We've just seen: a real scalar field may be viewed as an infinite collection of decoupled simple harm. oscs., one for each $\vec{k}$

- In order to quantize the field (in the Heisenberg picture) we quantize the harm. oscs.:

$a_{\vec{k}} \rightarrow \hat{a}_{\vec{k}}$: annihilation ops. $\quad a^*_{\vec{k}} \rightarrow \hat{a}^\dagger_{\vec{k}}$: creation ops.

$$\Longrightarrow \phi(x) \rightarrow \hat{\phi}(x) = \int_{\mathbb{R}^3} d^3\vec{k} \, \left(\hat{a}_{\vec{k}} \, u_{\vec{k}}(x) + \hat{a}^\dagger_{\vec{k}} \, u^*_{\vec{k}}(x) \right)$$

mode slns.

$$\Longrightarrow T_{\mu\nu}(x) \rightarrow \hat{T}_{\mu\nu}(x)$$

(formally)

- The Minkowski vacuum state is defined via $\hat{a}_{\vec{k}} \mid M\rangle = 0, \ \forall \vec{k} \in \mathbb{R}^3$

Commutation Relations

Commutation rlns. for *harm. osc.*: $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{\mathbb{I}}$

which are equivalent to: $[\hat{q}(t), \hat{p}(t)] = \hat{\mathbb{I}}\, i \qquad \forall t$

So, here for the *scalar field*: $\left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] = \hat{\mathbb{I}}\; \delta^{(3)}\left(\vec{k} - \vec{k}'\right)$

which are equivalent to: $\left[\hat{\phi}(\vec{x}, t), \hat{\Pi}(\vec{x}', t)\right] = \hat{\mathbb{I}}\; i\, \delta^{(3)}(\vec{x} - \vec{x}') \qquad \forall t$

↑ equal-time commutation rlns.

where canonical field momentum $\Pi(x) \equiv \dfrac{\partial \mathcal{L}_m}{\partial \dot{\phi}}$

Inner Product

- Define $$(\phi_1, \phi_2) \equiv i \int_{\mathbb{R}^3, t=t_0} d^3\vec{x}\, [\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*]$$

- it is a scalar product (it's conjugate-symmetric & linear in 2nd argument)

- if ϕ_1 & ϕ_2 are slns. of K-G eq, then it is independent of t_0

- The Minkowski modes $u_{\vec{k}}(x)$ are orthonormal:

$$\left(u_{\vec{k}}, u_{\vec{k}'}\right) = \delta^{(3)}\left(\vec{k} - \vec{k}'\right) = -\left(u^*_{\vec{k}}, u^*_{\vec{k}'}\right) \qquad \left(u_{\vec{k}}, u^*_{\vec{k}'}\right) = 0$$

the pos. freq. modes have positive norm

the negat. norm modes have negative norm

So (,) is an inner prod. when restricted to the pos. freq. modes $u_{\vec{k}}$

Quantization Procedure

(1) Choose a set $\{u_i(x), u_i^*(x), \forall i\}$ of slns. of field eq. that is complete and orthonormal:

$$(u_i, u_j) = \delta_{ij} = -\left(u_i^*, u_j^*\right), \quad \left(u_i, u_j^*\right) = 0$$

(2) Then any sln. of the field eq. may be expanded as

$$\phi(x) = \sum_i a_i u_i(x) + a_i^* u_i^*(x), \quad a_i = (u_i, \phi), \quad a_i^* = -\left(u_i^*, \phi\right)$$

(3) Quantize by promoting to ops. $a_i \to \hat{a}_i, \quad a_i^* \to \hat{a}_i^\dagger, \quad \phi \to \hat{\phi}$

and impose comm. rlns. $\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \hat{\mathbb{I}}\,\delta_{ij}$

↑ it may be a Dirac-δ

(4) Define a vacuum by $\hat{a}_i\,|0\rangle = 0, \forall i$ excited states by $\left(\hat{a}_i^\dagger\right)^n |\,0\rangle$ etc

Choice of vacuum depends (via $\hat{a}_i$) on choice of complete set of slns. u_i !

QFT Divergences

$\hat{\phi}$ is an operator-valued distribution - See $\left[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}\right] = \hat{\mathbb{I}}\; \delta^{(3)}\left(\vec{k} - \vec{k}'\right)$

Strictly, one should integrate it against a test-function $f(x)$ in order to get a well-defined operator:

$$\hat{\phi}(f) \equiv \int d^4x\; f(x)\hat{\phi}(x)$$

So products like $\hat{\phi}^2(x)$ in $\hat{T}_{\mu\nu}(x)$ are not well-defined and plague the QFT with divergences

An *example* of these divergences is in the zero-point energy density:

$$\int_{\mathbb{R}^3} d^3\vec{k}\ \frac{\omega_k}{2}\left[\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{k}}\right]$$

$$\|$$

$\langle M \mid \hat{H} \mid M \rangle$ per unit volume ~

$$\frac{1}{2}\int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3}\omega_k = \frac{1}{4\pi^2}\int_0^\infty dk\ k^2\sqrt{k^2+m^2}$$

has ultra-violet divergence since there is an infinite amount of harm. oscs., each with *non*zero zero-point energy $\frac{\omega_k}{2}$

Normal Ordering

- We remove these divergences in a "physically meaningful" way via a 'renormalization' procedure
- In *flat* s-t, we measure energies only as differences wrt a vacuum energy -> renormalize by subtracting a vacuum energy and experiments indicate that such vacuum is the Minkowski vacuum:

$$\langle \psi \,|\!: \hat{\phi}(x)^2 :\!|\, \psi \rangle = \langle \psi \,|\, \hat{\phi}(x)^2 \,|\, \psi \rangle - \langle M \,|\, \hat{\phi}(x)^2 \,|\, M \rangle$$

$$\langle M \,|\!: \hat{H} :\!|\, M \rangle = 0$$

This is equivalent to normal ordering: place all annihilation ops. to the right of the creation ops.

$$\hat{H} = \int_{\mathbb{R}^3} d^3\vec{k}\, \frac{\omega_k}{2} \left[\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \hat{a}_{\vec{k}} \hat{a}^\dagger_{\vec{k}} \right] \longrightarrow :\hat{H}: = \int_{\mathbb{R}^3} d^3\vec{k}\, \omega_k \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}}$$

Bogolyubov Transformations

Consider two complete and orthonormal sets of slns. of the field eqs.: $\{u_i(x), u_i^*(x), \forall i\}$ and $\{v_i(x), v_i^*(x), \forall i\}$

Then

$$\hat{\phi}(x) = \sum_i \hat{a}_i u_i(x) + \hat{a}_i^\dagger u_i^*(x) = \sum_r \hat{b}_r v_r(x) + \hat{b}_r^\dagger v_r^*(x)$$

and, since $\{u_i(x), u_i^*(x), \forall i\}$ form a complete set,

$$v_r(x) = \sum_i \alpha_{ri}\ u_i(x) + \beta_{ri}\ u_i^*(x) \longleftarrow \text{Bogolyubov transf.}$$

Bogolyubov coeffs.: $\alpha_{ri},\ \beta_{ri} \in \mathbb{C}$

Vacuum States

- The relationship between the creation/annihilation ops. is then

$$b_r = (v_r, \phi) \longrightarrow \quad \ldots \quad \longrightarrow \hat{b}_r = \sum_i \alpha^*_{ri} \, \hat{a}_i - \beta^*_{ri} \, a^\dagger_i$$

- Different choices of complete sets yield different quantum 'vacuum' states of the matter field:

$$\hat{a}_i \,|0\rangle_u = 0, \quad \forall i \qquad \text{and} \qquad \hat{b}_r \,|0\rangle_v = 0, \quad \forall r$$

- The number of quantum v-particles of mode 'type r' in the u-vac. is

$${}_u\langle 0 \mid \hat{b}^\dagger_r \hat{b}_r \mid 0\rangle_u = \cdots = \sum_i |\beta_{ri}|^2$$

The two vac. are equivalent iff $\beta_{ri} = 0, \; \forall r, i$

(i.e., no mixing between pos. and negat. freq. modes)

[N.B.: For a rigorous approach: algebraic QFT]

II. QFT in Flat Space-time

(b) Unruh Effect

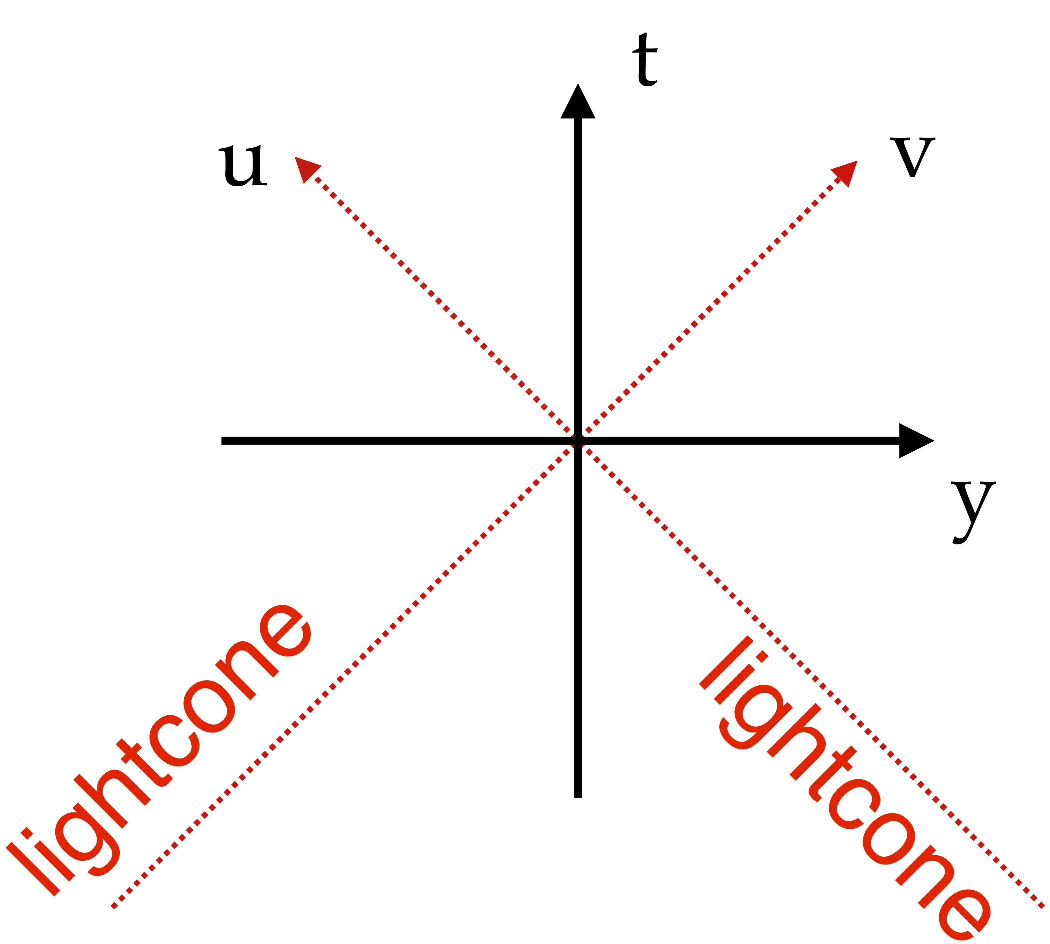
Flat Spacetime

Consider 2-D flat s-t. Line element: $ds^2 = -dt^2 + dy^2$

Inertial coords.: $t, y \in \mathbb{R}$ t is the proper time of inertial observers

Null coords.: $u \equiv t - y, \quad v \equiv t + y \quad \in \mathbb{R}$

$$ds^2 = -dudv$$



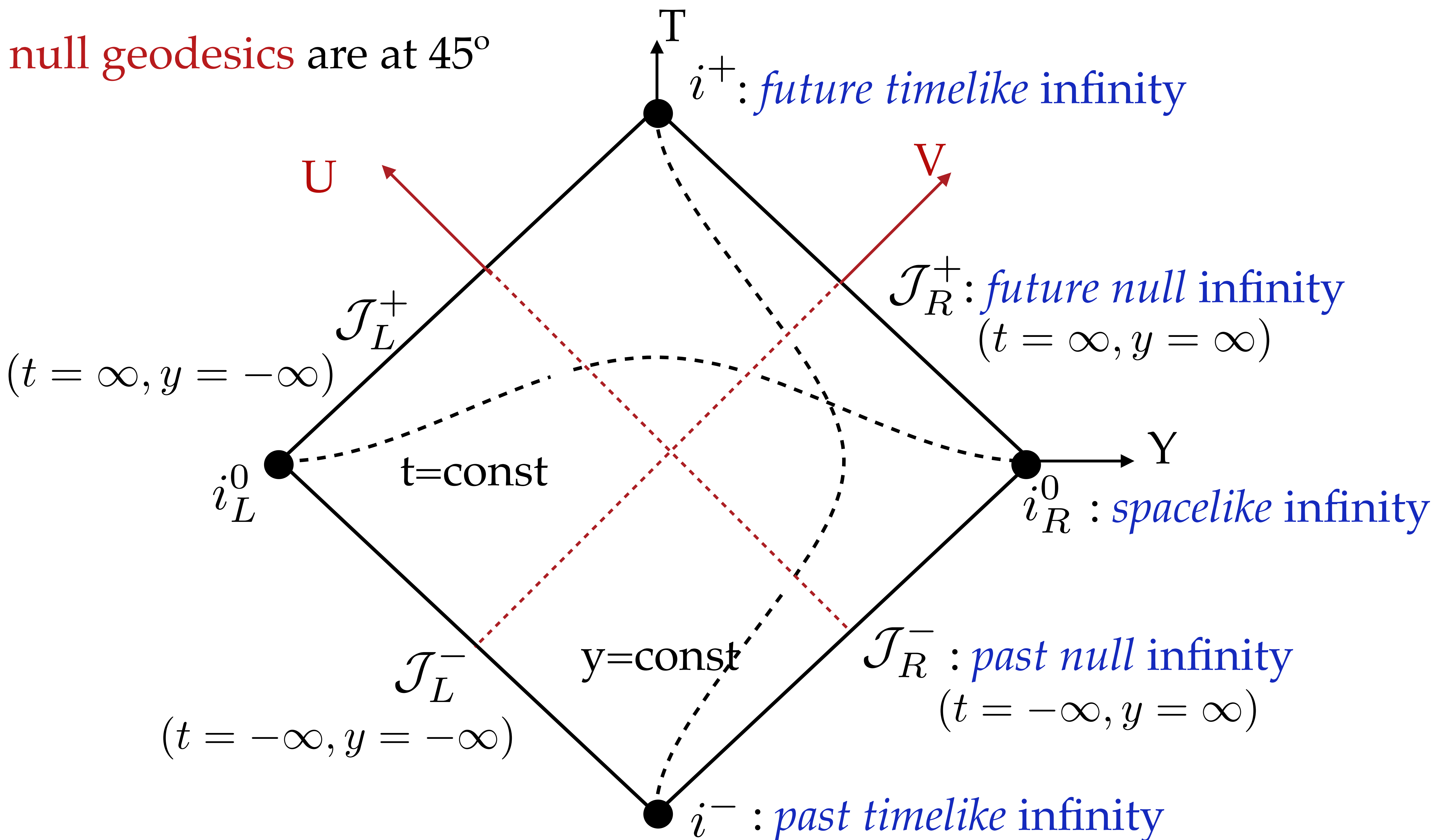
null geodesics (light) are along u=const or v=const

Penrose Diagram

Compactify $U \equiv \arctan u,\ V \equiv \arctan v \ \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$T \equiv U + V,\ Y \equiv V - U \ \in (-\pi, \pi)$$

null geodesics are at 45°



Rindler Observers

- Rindler obs.: obs. with uniform acceleration $a^2 \equiv a^\alpha a_\alpha = const$ and proper time τ . They have

$$t(\tau) = \frac{\sinh(a\tau)}{a}, \quad y(\tau) = \frac{\cosh(a\tau)}{a} \quad \rightarrow y^2 - t^2 = a^{-2}$$

- Coordinate transf. to Rindler coords. $\{\eta, \zeta\}$:

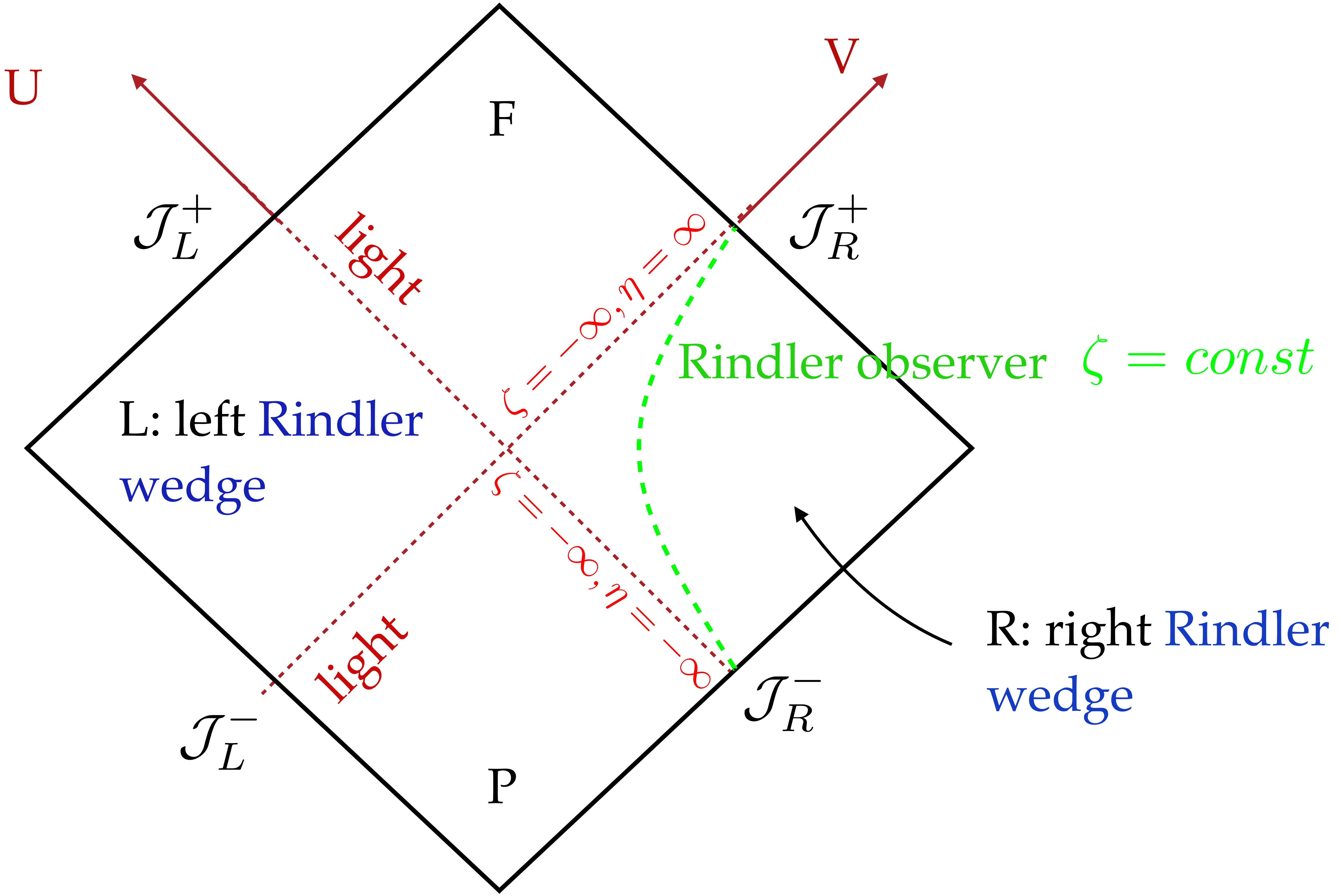
$$\alpha t = e^{\alpha\zeta} \sinh(\alpha\eta), \;\; \alpha y = e^{\alpha\zeta} \cosh(\alpha\eta) \qquad \alpha > 0$$

$$\rightarrow ds^2 = e^{2\alpha\zeta} \left(-d\eta^2 + d\zeta^2\right)$$

↑ conformal factor

Rindler obs. are at $\zeta = const$ and measure proper time $\tau = e^{\alpha\zeta}\eta$

- Range $\eta, \zeta \in (-\infty, \infty)$ only cover $y > |t|$: the right Rindler wedge. Similar transformations can be used to cover the whole s-t



$\zeta = -\infty$ are Killing horizons (the Killing vect. ∂_η is null there)

- No events in $L \cup F$ can be observed by Rindler obs. in R
- No events in $R \cup F$ can be observed by Rindler obs. in L
- L & R are causally disconnected

Wave Eq.

K-G eq. for a scalar field in the inertial frame: $\left(-\partial_t^2 + \partial_y^2\right)\phi = 0 \longrightarrow$

$$\phi_k = \frac{e^{-i\omega t + iky}}{\sqrt{4\pi\omega}} = \frac{1}{\sqrt{4\pi\omega}} \begin{cases} e^{-i\omega u}, & k > 0 \\ e^{-i\omega v}, & k < 0 \end{cases}$$

$$\forall k \in \mathbb{R}, \quad \omega \equiv |k| > 0$$

These modes:

- are pos. freq. wrt ∂_t: $\partial_t \phi_k = -i\omega\phi_k$

$$\downarrow$$

$$(\phi_k, \phi_{k'}) = \delta(k - k') \quad \text{(pos. norm)}$$

- together with their c.c., form a complete set in the whole s-t:

$$\hat{\phi} = \int_{\mathbb{R}} dk \, \left(\hat{a}_k \phi_k + \hat{a}_k^\dagger \phi_k^*\right)$$

K-G eq. in the Rindler frame: $\left(-\partial_\eta^2 + \partial_\zeta^2\right)\phi = 0$

$$\phi_{\bar{k}}^R = \frac{e^{-i\bar{\omega}\eta + i\bar{k}\zeta}}{\sqrt{4\pi\bar{\omega}}} = \frac{1}{\sqrt{4\pi\bar{\omega}}} \begin{cases} (-\alpha u)^{i\bar{k}/\alpha}, & \bar{k} > 0 \\ (\alpha v)^{i\bar{k}/\alpha}, & \bar{k} < 0 \end{cases} \quad \text{in R}$$

$$\bar{\phi}_{\bar{k}}^R = 0 \quad \text{in L} \qquad \forall \bar{k} \in \mathbb{R}, \quad \bar{\omega} \equiv |\bar{k}| > 0$$

These modes:

- are pos. freq. modes wrt ∂_η: $\partial_\eta \bar{\phi}_{\bar{k}}^R = -i\bar{\omega}\bar{\phi}_{\bar{k}}^R$

$$\downarrow$$

$$\left(\bar{\phi}_{\bar{k}}^R, \bar{\phi}_{\bar{k}'}^R\right) = \delta(\bar{k} - \bar{k}')$$

- with their c.c., form a complete set in R only -> define the mode slns.:

$$\bar{\phi}_{\bar{k}}^L = \frac{1}{\sqrt{4\pi\bar{\omega}}} \begin{cases} (-\alpha v)^{i\bar{k}/\alpha}, & \bar{k} > 0 \\ (\alpha u)^{i\bar{k}/\alpha}, & \bar{k} < 0 \end{cases} \quad \text{in L}$$

$$\bar{\phi}_{\bar{k}}^L = 0 \quad \text{in R}$$

Quantum States

- The two sets of modes together form a complete set in the whole s-t:

$$\hat{\phi} = \int_{\mathbb{R}} d\bar{k} \ \left(\hat{b}^R_{\bar{k}} \bar{\phi}^R_{\bar{k}} + \hat{b}^L_{\bar{k}} \bar{\phi}^L_{\bar{k}} + \text{h.c.} \right)$$

- Vacuum states:

  - Minkowski vacuum: $\hat{a}_k \mid M\rangle = 0, \ \forall k \in \mathbb{R}$

  - Rindler vacuum: $\hat{b}^R_{\bar{k}} \mid R\rangle = \hat{b}^L_{\bar{k}} \mid R\rangle = 0, \ \forall \bar{k} \in \mathbb{R}$

- How many Rindler particles does the Minkowski vacuum contain? (Unruh'76)

- The pos. freq. (wrt ∂_t) Minkowski modes

$$\phi_k = \frac{1}{\sqrt{4\pi\omega}} \begin{cases} e^{-i\omega u}, & k > 0 \\ e^{-i\omega v}, & k < 0 \end{cases}$$

and any linear combinations of them (ie, trivial Bogolyubov transf. $\beta_{kk'} = 0$ are analytic and bounded in the lower half of the complex u- and v-planes

But this property is not satisfied if any negat. freq. mode ϕ_k^* is included -> characterization of Minkowski pos.freq. modes: they are analytic and bounded in the lower u- and v-planes

- Rindler modes for $\bar{k} > 0$: $\phi_{\bar{k}}^R \propto \begin{cases} (-\alpha u)^{i\bar{k}/\alpha}, & u < 0 \text{ (eg, in R)} \\ 0, & u > 0 \text{ (eg, in L)} \end{cases}$

  are not analytic in the lower u-plane

- But it can be shown that the new modes

$$u^R_{\bar{k}} \equiv \frac{e^{\pi\bar{\omega}/(2\alpha)}}{\sqrt{2\sinh\left(\pi\bar{\omega}/\alpha\right)}}\left[\bar{\phi}^R_{\bar{k}} + e^{-\pi\bar{\omega}/\alpha}\bar{\phi}^{L^*}_{-\bar{k}}\right] \qquad \forall \bar{k} \in \mathbb{R}$$

are analytic and bounded in the lower half of the u- and v-planes

- For $\bar{k} > 0$ they are concentrated in R, so also construct

$$u^L_{\bar{k}} \equiv \frac{e^{-\pi\bar{\omega}/(2\alpha)}}{\sqrt{2\sinh\left(\pi\bar{\omega}/\alpha\right)}}\left[\bar{\phi}^{R^*}_{-\bar{k}} + e^{\pi\bar{\omega}/\alpha}\bar{\phi}^{L}_{\bar{k}}\right] \qquad \forall \bar{k} \in \mathbb{R}$$

which are also analytic and bounded in the lower half of the u- and v-planes, and, for $\bar{k} > 0$, they are concentrated in L

So $u^R_{\bar{k}}$ and $u^L_{\bar{k}}$ are *pos. freq. Minkowski modes*

- We can expand $\hat{\phi} = \int_{\mathbb{R}} d\bar{k} \ \left(\hat{a}^R_{\bar{k}} u^R_{\bar{k}} + \hat{a}^L_{\bar{k}} u^L_{\bar{k}} + \text{h.c.}\right)$

  Then $\hat{a}^R_{\bar{k}} \mid M\rangle = \hat{a}^L_{\bar{k}} \mid M\rangle = 0, \ \forall \bar{k} \in \mathbb{R}$

- We have explicitly constructed the Bogolyubov transf.:

$$b^R_{\bar{k}} = \left(\phi, \bar{\phi}^R_{\bar{k}}\right)$$

$$\downarrow$$

$$\hat{b}^R_{\bar{k}} = \frac{1}{\sqrt{2\sinh\left(\pi\bar{\omega}/\alpha\right)}} \left[e^{\pi\bar{\omega}/(2\alpha)} \hat{a}^R_{\bar{k}} + e^{-\pi\bar{\omega}/(2\alpha)} \hat{a}^{L\dagger}_{-\bar{k}}\right]$$

- Thus, the num. of Rindler particles contained in Minkowski vac. is

$$\langle M \mid \hat{b}^{R\dagger}_{\bar{k}} \hat{b}^R_{\bar{k}} \mid M\rangle = \frac{1}{e^{2\pi\bar{\omega}/\alpha} - 1}$$

  This is a thermal Planck spectrum of Rindler particles with temperature $T_0 = \dfrac{\alpha}{2\pi k_B}$

Unruh Effect

If the field $\hat{\phi}$ is in the *Minkowski vacuum* $|\, M \rangle$

- an inertial observer detects no particles

- a Rindler observer detects a thermal bath of (Rindler) particles at the Unruh temperature

$$T = e^{-\alpha\zeta} T_0$$

↑ conformal factor

This Unruh effect is not currently measurable, eg,

for $T = 1K < T_{CMB} \approx 3K$ we need $a \sim 10^{20} m/s^2$

currently unachievable

- The *Rindler vacuum* is seen as empty by Rindler observers

- It can be shown that the 'renormalized' $\langle R \,|: \hat{T}_{\mu\nu} :|\, R \rangle$ diverges at the Killing horizons (in the regular inertial frame)

  Therefore, the Rindler vacuum is an unphysical state

What semiclassical effects do we get if the space-time is curved?…

Hilbert Space

The Minkowski vacuum state is defined via $\hat{a}_{\vec{k}} \mid M\rangle = 0, \ \forall \vec{k} \in \mathbb{R}^3$

$\longrightarrow$ $\mathcal{H}_1 \equiv \left\{ \hat{a}^\dagger_{\vec{k}} \mid M\rangle, \forall \vec{k} \in \mathbb{R}^3 \right\}$ is the 1-particle Hilbert sp.

$\longrightarrow$ $\ldots$

The Hilbert sp. of the QFT (Fock space) is

$$\mathcal{H} = \mathbb{C} \oplus \mathcal{H}_1 \oplus (\mathcal{H}_1 \otimes \mathcal{H}_1)_{\rm sym} \oplus (\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1)_{\rm sym} \oplus \ldots$$