Global Conformal Blocks on the Plane From Oscillator Representations

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Structure of the Defense

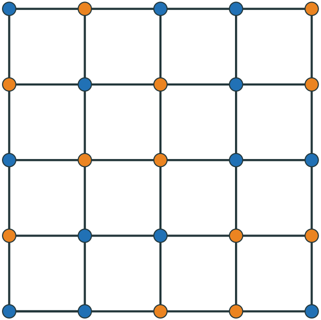
1. Motivation

Motivation

Applications of CFTs:

- critical statistical models (e. g. Ising model)
  Belavin, Polyakov, and Zamolodchikov, 1984

  Di Francesco, Mathieu, and Senechal, 1997



$\ell \to 0$

$$\langle \sigma(x_i)\sigma(x_j)\rangle \sim \frac{1}{|x_i - x_j|^{\eta}}$$

$$\langle \varepsilon(x_i)\varepsilon(x_j)\rangle \sim \frac{1}{|x_i - x_j|^{4-2/\nu}}$$

$\to$ critical behaviour characterized by critical exponents η, ν

Motivation

- string theory, holographic dualities
  Maldacena, 1998

quantum gravity in asymptotic AdS_{d+1} ⟷ (holographic dictionary) ⟷ conformal field theory on $\mathbb{R}^d$

bulk fields	$\longleftrightarrow$	local operators
spacetime geometries	$\longleftrightarrow$	quantum states
Witten diagrams	$\longleftrightarrow$	*conformal blocks*
Witten, 1998		Hijano et al., 2016

2. Aspects of Conformal Field Theories

Conformal Transformations

- setting: $(d=2)$-dimensional Euclidean space
- conformal transformations = diffeomorphisms $x \mapsto \tilde{x}(x)$ s. t.

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = \Omega^2(x) g_{\mu\nu}(x)$$

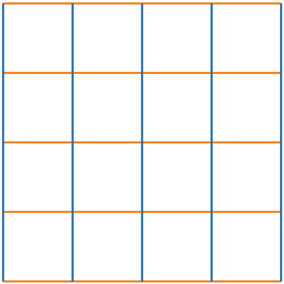
- infinitesimal transformation $\tilde{x}(x) = x + \varepsilon$ with $\varepsilon^\mu \ll 1$

$$\partial_0 \varepsilon_0 = \partial_1 \varepsilon_1 \qquad \text{and} \qquad \partial_0 \varepsilon_1 = -\partial_1 \varepsilon_0 \,,$$

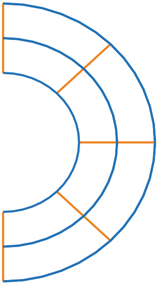
- Cauchy-Riemann equations $\longrightarrow$ complexified coordinates
- $x \mapsto \tilde{x}(x)$ conformal if the map $z \mapsto f(z) = z + \varepsilon(z)$ is holomorphic (in a domain)

Examples of Conformal Transformations in $d = 2$

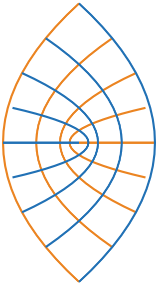
$f(z) = z$



$f(z) = \mathrm{e}^z$



$f(z) = z^2$



(Global) Conformal Algebra

- Virasoro algebra: *infinitely* many generators $L_n, n \in \mathbb{Z}$ with commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\,.$$

  $\rightarrow c$: central charge

- global (sub-)algebra $\mathfrak{sl}(2,\mathbb{C})$: generated by *finitely* many generators L_n, $n = -1, 0, 1$

Primary fields

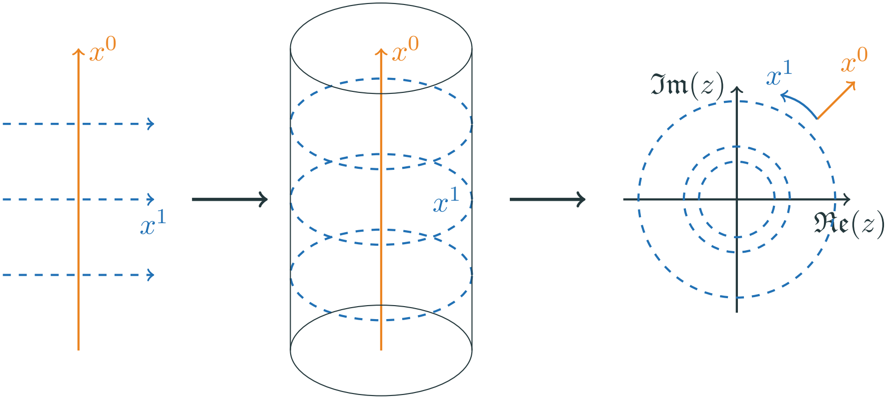
- primary fields $\phi(z,\bar{z})$ of *conformal dimensions* $(h,\bar{h})$ transform as

$$\phi(z,\bar{z}) \mapsto \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z}))\,,$$

under conformal transformations $z \mapsto f(z)$

Radial Quantization I: The Scheme

- canonical quantization → time direction?
- radial quantization procedure $z = \mathrm{e}^{x^0 + i x^1}$



- infinite past: $z = 0$; infinite future: $z \to \infty$

Radial Quantization II: Ordering and OPE

- radial-ordered product

$$\mathcal{R}\{\phi_1(z,\bar{z})\phi_2(w,\bar{w})\} = \begin{cases} \phi_1(z,\bar{z})\phi_2(w,\bar{w}) \text{ for } |z| > |w|\,, \\ \phi_2(w,\bar{w})\phi_1(z,\bar{z}) \text{ for } |z| < |w| \end{cases}$$

- operator product expansion (OPE):

$$\mathcal{R}\{\phi_i(z_i)\phi_j(z_j)\} = \sum_{\mathbb{h}} C_{ij\mathbb{h}} D(z_{ij}, \partial_j) \phi_{\mathbb{h}}(z_j)$$

3. Correlation Functions and Conformal Blocks

What restrictions arise from conformal symmetry?

Ward, 1950, Takahashi, 1957

- Ward identities ($j = -1, 0, 1$)

$$\sum_{k=1}^{n} \mathcal{L}_j^{(z_k,h_k)} \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = 0$$

imply *restrictions on correlation functions*

$$G^{(n)}_{h_1,\dots,h_n}(z_1 \dots, z_n) = \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle$$

Lower Point Correlation Functions ($n \leq 3$)

- Ward identities completely fixes lower point correlation functions

$$G^{(1)}(z_1) = 0 \quad \text{unless } \phi_1 \sim \mathbb{1},$$
$$G^{(2)}_{h_1 h_2}(z_1, z_2) = \frac{d_{h_1} \delta_{h_1,h_2}}{z_{12}^{2h_1}},$$
$$G^{(3)}_{h_1 h_2 h_3}(z_1, z_2, z_3) = \frac{C_{h_1 h_2 h_3}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}}$$

- structure constants d_{h_1} and $C_{h_1 h_2 h_3}$ *theory-dependent*!
- structure constants + conformal dimensions = conformal data

Higher Point Correlation Functions ($n \geq 4$)

Ward identities do not fix higher point correlators!

- there exist $n-3$ independent cross-ratios such as

$$\mathfrak{z} = \frac{z_{12} z_{34}}{z_{13} z_{24}}$$

- correlators factorize like

$$G^{(n)}_{h_1,\ldots,h_n}(z_1,\ldots,z_n) = \mathcal{L}^{(n)}_{h_1,\ldots,h_n}(z_1,\ldots,z_n)\, g^{(n)}(\mathfrak{z}_1,\ldots,\mathfrak{z}_{n-3}),$$

into leg factor $\mathcal{L}^{(n)}$ and bare correlator $g^{(n)}$

Conformal Block Expansion

- idea: apply OPE inside of correlation functions

$$\begin{aligned}
&G^{(4)}(z_1, z_2, z_3, z_4) \\
&= \langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle \\
&= \sum_{\mathbb{h}} \sum_{\mathbb{h}'} C_{12\mathbb{h}} C_{34\mathbb{h}'} D(z_{12}, \partial_2) D(z_{34}, \partial_4) \langle \phi_{\mathbb{h}}(z_2)\phi_{\mathbb{h}'}(z_4)\rangle \\
&= \sum_{\mathbb{h}} C_{12\mathbb{h}} C_{34\mathbb{h}} d_{\mathbb{h}} \cdot D(z_{12}, \partial_2) D(z_{34}, \partial_2) \frac{1}{z_{24}^{2\mathbb{h}}} \\
&= \sum_{\mathbb{h}} C_{1234}^{\mathbb{h}} \cdot G_{\mathbb{h}}^{(4)}
\end{aligned}$$

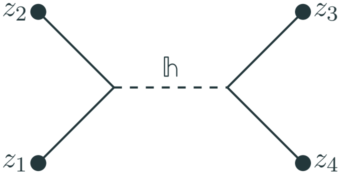
→ conformal four-point blocks $G_{\mathbb{h}}^{(4)}$ *fixed* by Casimir equation

$$\left(\mathcal{C}_2^{(3,4)} + \mathbb{h}(\mathbb{h} - 1)\right) G_{\mathbb{h}}^{(4)} = 0$$

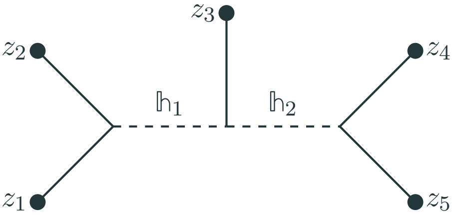
Examples of Higher Point Conformal Blocks

- diagrammatic language

$G^{(4)}_{\mathbb{h}} =$



$G^{(5)}_{\mathbb{h}_1,\mathbb{h}_2} =$



- different topologies (channels) for $n \geq 6$ points

4. Oscillator Representation Method

How to compute conformal blocks?

- shadow operator method

  Ferrara and Parisi, 1972, Ferrara et al., 1972
- bootstrap approach

  Polyakov, 1974, Ferrara, Grillo, and Gatto, 1973
- recurrence relation method

  Zamolodchikov, 1984
- oscillator representation method

  Beşken, Datta, and Kraus, 2020a

Global Highest-Weight Representations

- *highest-weight state* $|\mathbb{h}\rangle$ defined by

$$L_0 |\mathbb{h}\rangle = \mathbb{h} |\mathbb{h}\rangle \;, \qquad\qquad L_1 |\mathbb{h}\rangle = 0$$

- descendant states $|\mathbb{h}, n\rangle = (L_{-1})^n |\mathbb{h}\rangle$
- representation space = *Verma module*

$$\mathcal{V}_{\mathbb{h}} := \text{span}\{|\mathbb{h}, n\rangle \mid n \in \mathbb{N}\}$$

- conformal blocks as projections onto Verma modules

$$G^{(4)}_{\mathbb{h}} = \langle \phi_1(z_1)\phi_2(z_2) P_{\mathbb{h}} \phi_3(z_3)\phi_4(z_4)\rangle$$

Global Oscillator Representation

- representation space = weighted Bergman space

$$\mathcal{B}_{\mathbb{h}} := \left\{ f : \mathbb{D} \to \mathbb{C} | f \text{ holomorphic}, \int_{\mathbb{D}} [\mathrm{d}^2 u] \, |f(u)|^2 < \infty \right\}$$

  with weight

$$[\mathrm{d}^2 u] \equiv \frac{2\mathbb{h} - 1}{2\pi} \frac{\mathrm{d}^2 u}{(1 - u\bar{u})^{2-2\mathbb{h}}}$$

- inner product

$$(\psi, \chi) = \int [\mathrm{d}^2 u] \overline{\psi(u)} \chi(u)$$

Generators and Wavefunctions

- represent generators as differential operators

$$l_1 = \partial_u \,,$$
$$l_0 = u\partial_u + \mathbb{h} \,,$$
$$l_{-1} = u^2\partial_u + 2\mathbb{h}u$$

- assign wavefunctions: $|\psi\rangle \in \mathcal{V}_{\mathbb{h}} \mapsto \psi(u) = \langle u|\psi\rangle \in \mathcal{B}_{\mathbb{h}}$
- wavefunction of highest weight state $f_{\mathbb{h}} = \langle u|\mathbb{h}\rangle = 1$
- wavefunctions of descendent states

$$f_{\mathbb{h},n}(u) = \langle u|\mathbb{h}, n\rangle = n!\binom{-2\mathbb{h}}{n}(-u)^n$$

How to compute conformal blocks from oscillator representations?

- idea: rewrite projector as

$$P_{\mathbb{h}} = \sum_{m=0}^{\infty} \frac{|\mathbb{h}, m\rangle\langle\mathbb{h}, m|}{m!(2\mathbb{h})_m} \equiv \int [\mathrm{d}^2 u] |\bar{u}\rangle\langle u|$$

- four-point block

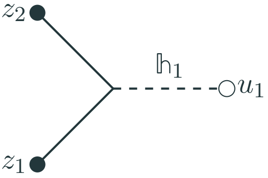
$$\begin{aligned} G^{(4)}_{\mathbb{h}} &= \langle \phi_1(z_1)\phi_2(z_2) P_{\mathbb{h}} \phi_3(z_3)\phi_4(z_4)\rangle \\ &= \int [\mathrm{d}^2 u] \underbrace{\langle 0|\, \phi_1(z_1)\phi_2(z_2)\, |\bar{u}\rangle}_{\chi_{\mathbb{h}}(z_1,z_2;\bar{u})} \underbrace{\langle u|\, \phi_3(z_3)\phi_4(z_4)\, |0\rangle}_{\psi_{\mathbb{h}}(z_3,z_4;u)} \end{aligned}$$

- five-point block

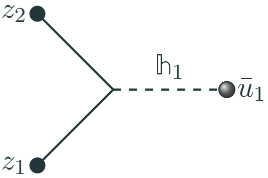
$$
\begin{aligned}
G^{(5)}_{\mathbb{h}_1,\mathbb{h}_2} &= \langle \phi_1(z_1)\phi_2(z_2)P_{\mathbb{h}_1}\phi_3(z_3)P_{\mathbb{h}_1}\phi_4(z_4)\phi_5(z_5)\rangle \\
&= \int [\mathrm{d}^2 u_1] \int [\mathrm{d}^2 u_2] \underbrace{\langle 0|\, \phi_1(z_1)\phi_2(z_2)\, |\bar{u}_1\rangle}_{\chi_{\mathbb{h}_1}(z_1,z_2;\bar{u}_1)} \\
&\quad \cdot \underbrace{\langle u_1|\, \phi_3(z_3)\, |\bar{u}_2\rangle}_{\Omega_{\mathbb{h}_1,\mathbb{h}_2}(z_3;u_1,\bar{u}_2)} \underbrace{\langle u_2|\, \phi_3(z_3)\phi_4(z_4)\, |0\rangle}_{\psi_{\mathbb{h}_2}(z_3,z_4;u_2)}
\end{aligned}
$$

- oscillator wavefunctions:

$$\psi_{\mathbb{h}_1}(z_1, z_2; u_1) = \frac{z_{12}^{\mathbb{h}_1 - h_1 - h_2}}{(1 - z_1 u_1)^{\mathbb{h}_1 - h_2 + h_1} (1 - z_2 u_1)^{\mathbb{h}_1 - h_1 + h_2}}$$

$=$ 

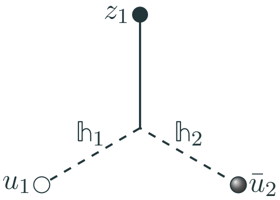
$$\chi_{\mathbb{h}_1}(z_1, z_2; \bar{u}_1) = \frac{z_{12}^{\mathbb{h}_1 - h_1 - h_2}}{(\bar{u}_1 - z_1)^{\mathbb{h}_1 - h_2 + h_1} (\bar{u}_1 - z_2)^{\mathbb{h}_1 - h_1 + h_2}}$$

$=$ 

- matrix element

$$\Omega_{\mathbb{h}_1,\mathbb{h}_2}(z_1;u_1,\bar{u}_2)=\frac{(z_1-\bar{u}_2)^{\mathbb{h}_1-\mathbb{h}_2-h_1}}{(1-z_1u_1)^{\mathbb{h}_1-\mathbb{h}_2+h_1}(1-u_1\bar{u}_2)^{\mathbb{h}_1+\mathbb{h}_2-h_1}}$$

$$=$$



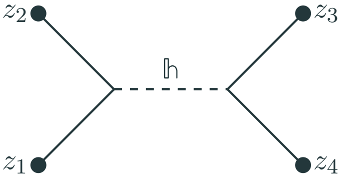
- Ω-ψ-connection

$$\Omega_{\mathbb{h}_1,\mathbb{h}_2}(z;u_1,\bar{u}_2)=\psi_{\mathbb{h}_1}(z,\bar{u}_2;u_1)$$

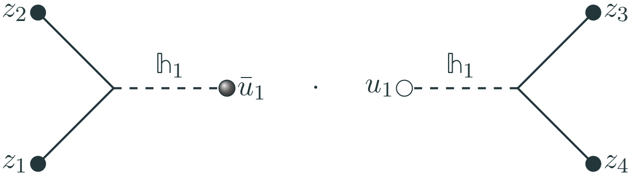
Computation of Blocks and Gluing Rules

- Computation of the four-point block:
  Beşken, Datta, and Kraus, 2020a

$$G_{\mathbb{h}}^{(4)} \quad =$$



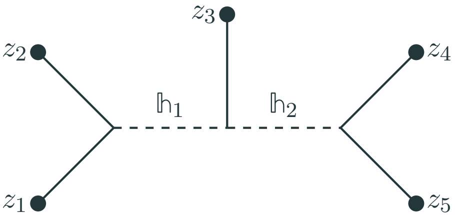
$$= \quad \int [\mathrm{d}^2 u_1]$$



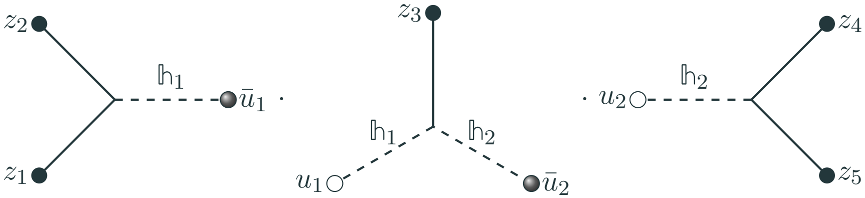
$$= \mathcal{L}^{(4)} \ \mathfrak{z}^{\mathbb{h}} \ {}_2F_1(\mathbb{h} + h_2 - h_1, \mathbb{h} - h_3 + h_4; 2\mathbb{h}; \mathfrak{z})$$

- computation of five-point block:

$$G^{(5)}_{\mathbb{h}_1,\mathbb{h}_2} =$$

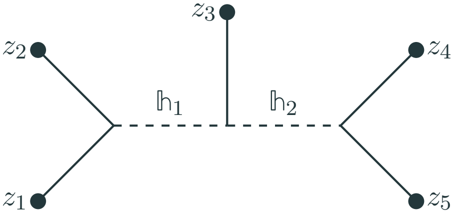


$$= \int [\mathrm{d}^2 u_1] \int [\mathrm{d}^2 u_2]$$

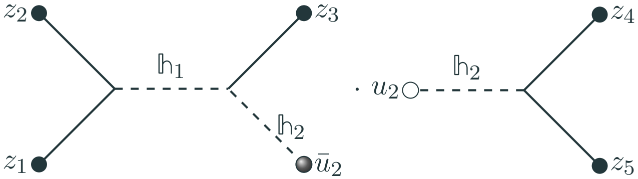


- computation of five-point block (alternative):

$G^{(5)}_{\mathbb{h}_1,\mathbb{h}_2} =$



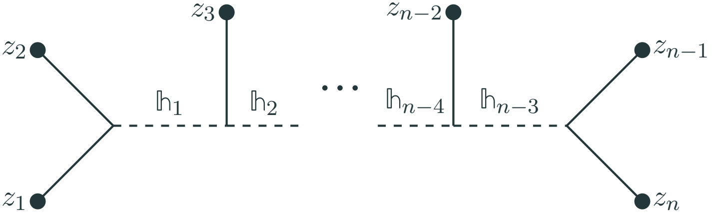
$= \int [\mathrm{d}^2 u_2]$



- computation of the n-point comb block:

$$G^{h_1,\dots,h_n}_{\mathbb{h}_1,\dots,\mathbb{h}_{n-3}}(z_1,\dots,z_n)$$

=



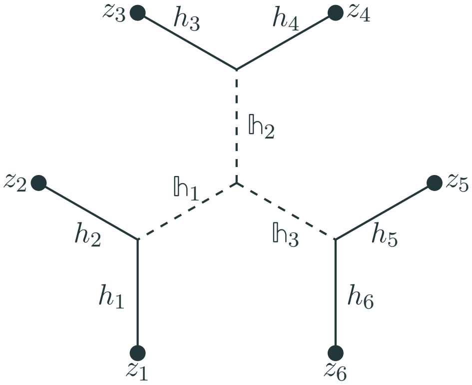
$$= \mathcal{L}^{(n)} \prod_{i=1}^{n-3} \mathfrak{z}_i^{\mathbb{h}_i}\, F_K(\dots;\mathfrak{z}_1,\dots,\mathfrak{z}_{n-3})$$

→ matches Rosenhaus, 2019

Can we obtain different channels from oscillator representation?

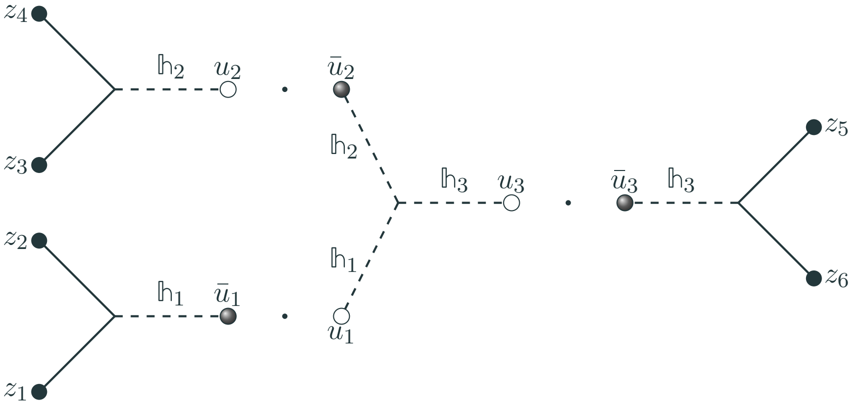
- six-point star channel

$S^{h_1,\ldots,h_6}_{\mathbb{h}_1,\mathbb{h}_2,\mathbb{h}_3} =$



- idea: introduce new oscillator diagram

$$\Xi_{\mathbb{h}_1,\mathbb{h}_3,\mathbb{h}_2} = (u_1-u_3)^{\mathbb{h}_2-\mathbb{h}_1-\mathbb{h}_3}(1-u_1\bar{u}_2)^{\mathbb{h}_3-\mathbb{h}_1-\mathbb{h}_2}(1-u_3\bar{u}_2)^{\mathbb{h}_1-\mathbb{h}_2-\mathbb{h}_3}$$



Results for Star Channel Blocks

- construction works for $\mathbb{h}_2 \geq \mathbb{h}_1 + \mathbb{h}_3$
- symmetric star channel block with $\mathbb{h}_2 = \mathbb{h}_1 + \mathbb{h}_3$

$$
\begin{aligned}
&S^{h_x,h_y,h_z}_{\mathbb{h}_1,\mathbb{h}_1+\mathbb{h}_3,\mathbb{h}_3} \\
=&\mathcal{L}^{(6)} \mathfrak{z}^{\mathbb{h}_1} (\mathfrak{v}-\mathfrak{u})^{\mathbb{h}_3} F_D^{(3)}(\mathbb{h}_1+\mathbb{h}_3, \mathbb{h}_1, \mathbb{h}_3, \mathbb{h}_3, 2(\mathbb{h}_1+\mathbb{h}_3); \mathfrak{z}, \mathfrak{u}, \mathfrak{v})
\end{aligned}
$$

  with cross-ratios $\mathfrak{z}, \mathfrak{u}$ and $\mathfrak{v}$

- no generalization for $\mathbb{h}_2 < \mathbb{h}_1 + \mathbb{h}_3$ so far

5. Conclusion and Outlook

Conformal blocks play a crucial role in solving CFTs.

- universal building blocks of correlation functions
- fixed by symmetry
- weighted by conformal data

Conclusion and Take-Home-Message

Oscillator representations provide an efficient method for the computation of conformal blocks.

- construction rules formulated as diagrammatic language
- iterative computation of n-point comb block
- extension of the method $\rightarrow$ computation of star channel blocks under the assumption $\mathbb{h}_2 \geq \mathbb{h}_1 + \mathbb{h}_3$

Further goals:

- remaining star channel blocks
- conformal blocks on Riemann surfaces of higher genus
  Hollweck, 2022
- Virasoro blocks from oscillator representations
  Beşken, Datta, and Kraus, 2020a
  Beşken, Datta, and Kraus, 2020b
- generalization of the method to higher dimensions

Thank you for your attention!

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7. Backup Slides

More on Casimir Equations

- second-order Casimir of $\mathfrak{sl}(2, \mathbb{C})$

$$C_2 = -L_0^2 + \frac{1}{2}\{L_{-1}, L_1\}$$

- eigenvalue equation with projector

$$C_2 P_{\mathbb{h}} = P_{\mathbb{h}} C_2 = \mathbb{h}(1 - \mathbb{h}) P_{\mathbb{h}} \,. \tag{1}$$

- multi-point Casimir operator

$$\mathcal{C}_2^{(i_1,\ldots,i_m)} = -\left(\mathcal{L}_0^{(i_1)} + \cdots + \mathcal{L}_0^{(i_m)}\right)^2 + \frac{1}{2}\left\{\mathcal{L}_{-1}^{(i_1)} + \cdots + \mathcal{L}_{-1}^{(i_m)}, \mathcal{L}_1^{(i_1)} + \cdots + \mathcal{L}_1^{(i_m)}\right\}$$

Six-Point Casimir Equations

- comb channel blocks

$$\left(\mathcal{C}_2^{(3,4,5,6)} + \mathbb{h}_1(\mathbb{h}_1 - 1)\right) G^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0\,,$$
$$\left(\mathcal{C}_2^{(4,5,6)} + \mathbb{h}_2(\mathbb{h}_2 - 1)\right) G^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0\,,$$
$$\left(\mathcal{C}_2^{(5,6)} + \mathbb{h}_3(\mathbb{h}_3 - 1)\right) G^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0$$

- star channel blocks

$$\left(\mathcal{C}_2^{(1,2)} + \mathbb{h}_1(\mathbb{h}_1 - 1)\right) S^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0\,,$$
$$\left(\mathcal{C}_2^{(3,4)} + \mathbb{h}_2(\mathbb{h}_2 - 1)\right) S^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0\,,$$
$$\left(\mathcal{C}_2^{(5,6)} + \mathbb{h}_3(\mathbb{h}_3 - 1)\right) S^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0$$

Casimir Equations for Comb Blocks

- Casimir equations for general n-point comb block

$$\begin{aligned}
\left(\mathcal{C}_2^{(3,4,5,\ldots,n)} + \mathbb{h}_1(\mathbb{h}_1 - 1)\right) G^{(n)}_{\mathbb{h}_1\ldots\mathbb{h}_{n-3}}(z_1,\ldots,z_n) &= 0\,,\\
\left(\mathcal{C}_2^{(4,5,\ldots,n)} + \mathbb{h}_2(\mathbb{h}_2 - 1)\right) G^{(n)}_{\mathbb{h}_1\ldots\mathbb{h}_{n-3}}(z_1,\ldots,z_n) &= 0\,,\\
&\vdots\\
\left(\mathcal{C}_2^{(n-1,n)} + \mathbb{h}_{n-3}(\mathbb{h}_{n-3} - 1)\right) G^{(n)}_{\mathbb{h}_1\ldots\mathbb{h}_{n-3}}(z_1,\ldots,z_6) &= 0\,.
\end{aligned}$$

Remarks on Hypergeometric Functions

- (ordinary) hypergeometric function as power series

$$ {}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} $$

$\rightarrow$ solution to the differential equation

$$ \left[z(1-z)\partial_z^2 + [c - (a+b+1)z]\partial_z - ab \right] F(a, b, c; z) = 0 $$

- generalized hypergeometric function

$$ {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!} $$

Comb Function

- definition as power series

$$F_K \left[\begin{matrix} a_1, b_1, \ldots, b_{n-4}, a_2 \\ c_1, \ldots, c_{n-3} \end{matrix} ; z_1, \ldots z_{n-3} \right]$$
$$= \sum_{k_1,\ldots,k_{n-3}}^{\infty} \frac{(a_1)_{k_1} (b_1)_{k_1+k_2} (b_2)_{k_2+k_3} \ldots (b_{n-4})_{k_{n-4}+k_{n-3}} (a_2)_{k_{n-3}}}{(c_1)_{k_1} \ldots (c_{n-3})_{k_{n-3}}} \cdot \frac{z_1^{k_1}}{k_1!} \ldots \frac{z_{n-3}^{k_{n-3}}}{k_{n-3}!}$$

- satisfies splitting equations

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Lauricella Hypergeometric Functions

- definition as power series

$$F_D^{(3)}(a, b_1, b_2, b_3, c; z_1, z_2, z_3) = \sum_{j_1, j_2, j_3 = 0}^{\infty} \frac{(a)_{j_1+j_2+j_3}(b_1)_{j_1}(b_2)_{j_2}(b_3)_{j_3}}{(c)_{j_1+j_2+j_3}} \frac{z_1^{j_1}}{j_1!} \frac{z_2^{j_2}}{j_2!} \frac{z_3^{j_3}}{j_3!}$$

- integral representation

$$F_D^{(3)}(a, b_1, b_2, b_3, c; z_1, z_2, z_3) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \cdot \int_0^1 \mathrm{d}t \, t^{a-1}(1-t)^{c-a-1}(1-z_1 t)^{-b_1}(1-z_2 t)^{-b_2}(1-z_3 t)^{-b_3}$$