Magnetic catalysis in the (2+1)-dimensional Gross-Neveu model

(handout version)

Michael Mandl
with J. Lenz and A. Wipf

Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena

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Handout version*

This handout is a slightly modified version of the talk given at the Physik-Combo. Some additional comments were added in order to give context to the slides shown.
Slides marked by an asterisk (*) were not part of the original talk.

Gross-Neveu (GN) model

$$\mathcal{L} = i\bar{\psi}\left(\not{\partial} + \mu\gamma_0 + ie\not{A}\right)\psi + \frac{g^2}{2N_f}(\bar{\psi}\psi)^2$$

- N_f flavors
- no mass term
- chemical potential μ
- external field A_μ

Gross-Neveu (GN) model

$$\mathcal{L} = i\bar{\psi}\left(\partial\!\!\!/ + \mu\gamma_0 + ieA\!\!\!/\right)\psi + \frac{g^2}{2N_f}(\bar{\psi}\psi)^2$$

or, equivalently,

$$\mathcal{L} = i\bar{\psi}\left(\partial\!\!\!/ + \sigma + \mu\gamma_0 + ieA\!\!\!/\right)\psi + \frac{N_f}{2g^2}\sigma^2$$

Ward identitiy	discrete chiral symmetry
$\langle\bar{\psi}\psi\rangle = \frac{iN_f}{g^2}\langle\sigma\rangle$	$\psi \to i\gamma_5\psi,\ \ \bar{\psi} \to i\bar{\psi}\gamma_5,\ \ \sigma \to -\sigma$

Gross-Neveu (GN) model*

One commonly gets rid of the $(\bar{\psi}\psi)^2$ term by introducing the auxiliary scalar field σ into the Lagrangian. These two Lagrangians are equivalent, as can be seen using the equations of motion for σ.
Notice that the GN model has a discrete $\mathbb{Z}_2$ symmetry.

Motivation & Goals

Why GN model?

- **Toy model for QCD**
- **Solid State Physics**
- ...

Why magnetic field?

- **Heavy-ion collisions**
- **Neutron stars**
- **Early universe**

In this talk

- Study influence of magnetic field on GN (T, μ) phase diagram in $2+1$ dimensions using Lattice Field Theory.
- Compare with mean-field results.

Motivation & Goals*

Variants of the Gross-Neveu model have been used successfully as toy models for QCD due to some intersting features they share with QCD, such as chiral symmetry and its spontaneous breakdown and in low dimensions renormalizability and asymptotic freedom.
In solid state physics they are used to describe planar and one-dimensional materials such as graphene, high-T_c superconductors or polymers.

Very strong magnetic fields are present in heavy-ion collisions, neutron stars and likely also were at the early stages of the universe. Thus it is important to understand their potential influence on the structure of matter.

Landau levels for fermions

The energy levels of non-interacting fermions in a magnetic field are quantized $\rightarrow$ **Landau levels**:

$$E_\ell = \sqrt{m^2 + p_z^2 + 2eB\left(\ell + \frac{1}{2} \pm \frac{1}{2}\right)}\,, \quad \ell \in \mathbb{N}_0$$

Landau levels for fermions*

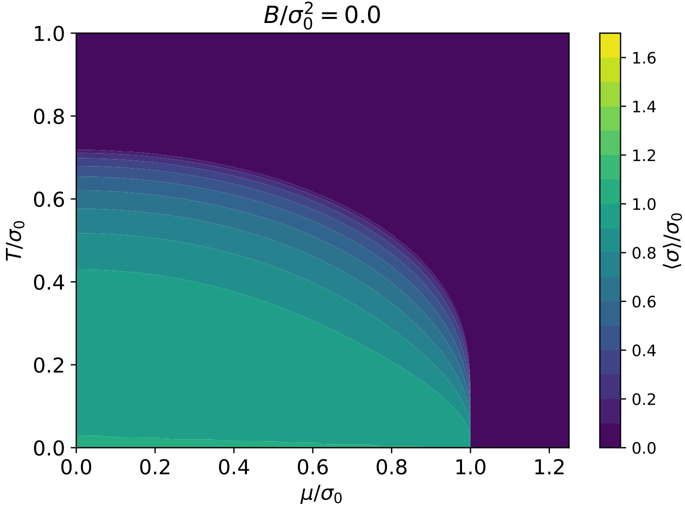
If one solves the Dirac equation for free fermions in $3+1$ dimensions, with a constant homogeneous magnetic field B pointing in z-direction, one finds the quantized energy levels shown on the previous slide. Here, ℓ labels the so-called Landau levels and the $\pm\frac{1}{2}$ distinguishes between spin up and down.
Notice that, as compared to the $B=0$ case, the momenta p_x and p_y do not enter the energy directly anymore. Notice that the lowest Landau level ($\ell=0$ and negative spin contribution) is independent of B and looks like the dispersion relation in $1+1$ dimensions.
Since we will from now on consider a $2+1$ dimensional spacetime the p_z-contribution is absent in our setup and the discreteness will be even more apparent in the following.

Large-N_f mean-field results*

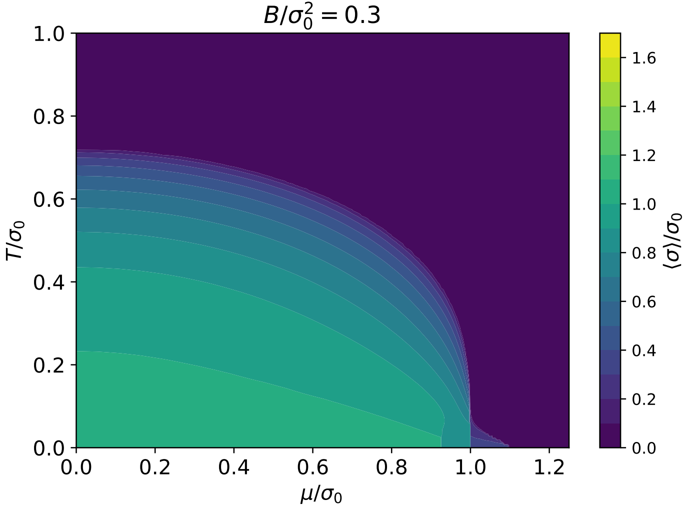
Going to the limit of an infinite flavor number, $N_\mathrm{f} \to \infty$, often gives a qualitatively correct picture of the phase structure of Four-Fermi theories even at finite N_f. In this case, computing the path integral reduces to a simple minimization problem (see the Appendix).

On the next few slides we show large-N_f phase diagrams in the (T, μ) plane of the $(2+1)$-dimensional GN model for various values of the magnetic field, which we assume constant, homogenous and orthogonal to the spatial plane. We assume σ to be homogeneous in space and time.
Lastly, we show phase diagrams in the (T, B) plane at $\mu = 0$ and in the (μ, B) plane at $T = 0$, respectively.

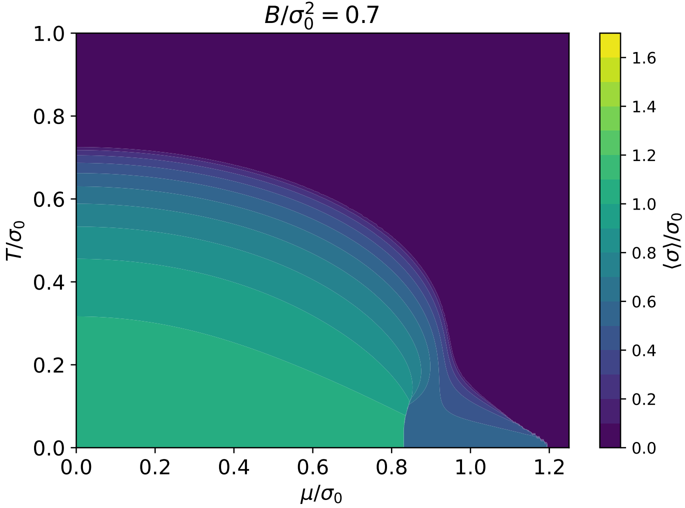
Large-N_f mean-field results



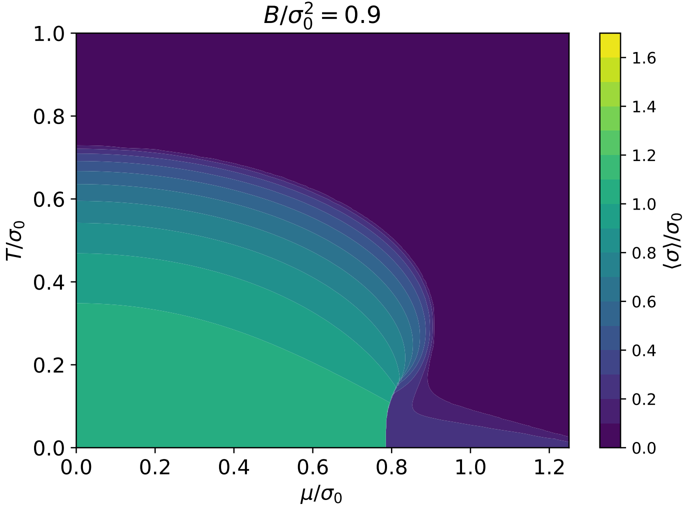
Large-N_f mean-field results



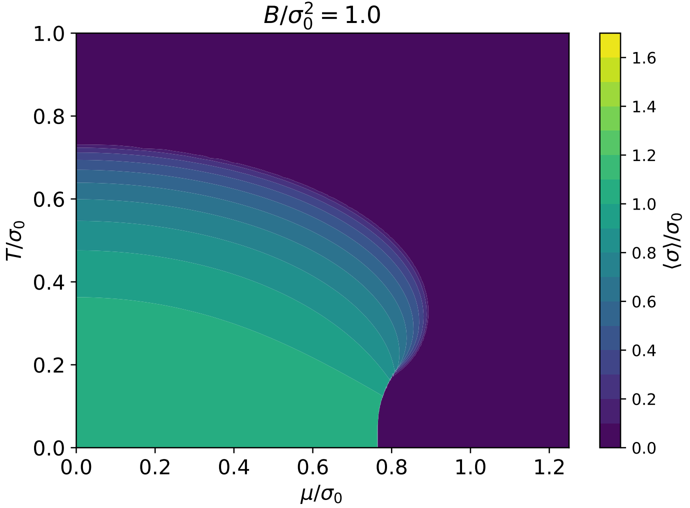
Large-N_f mean-field results



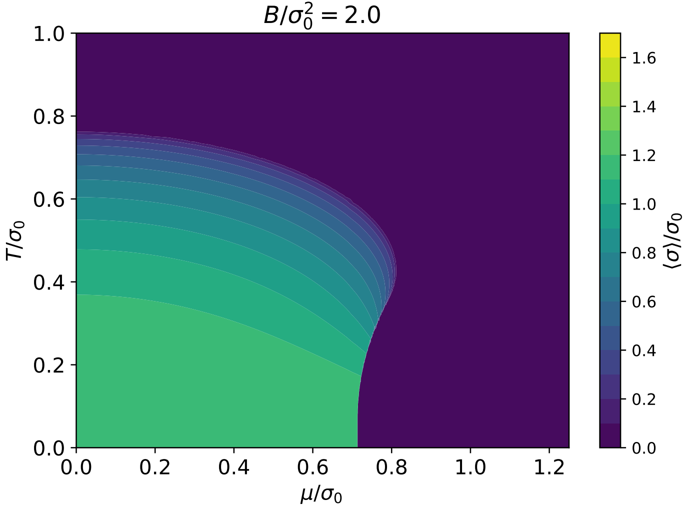
Large-N_f mean-field results



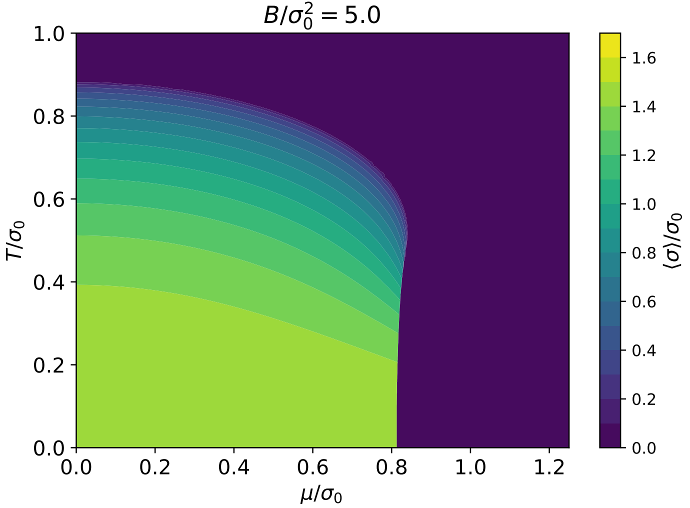
Large-N_f mean-field results



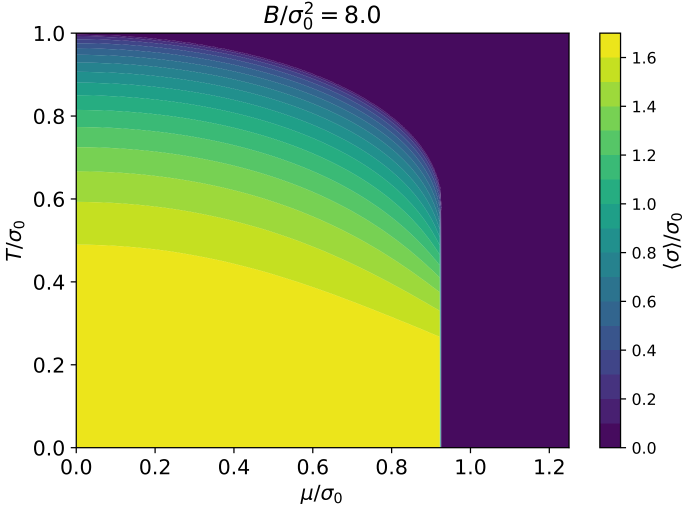
Large-N_f mean-field results



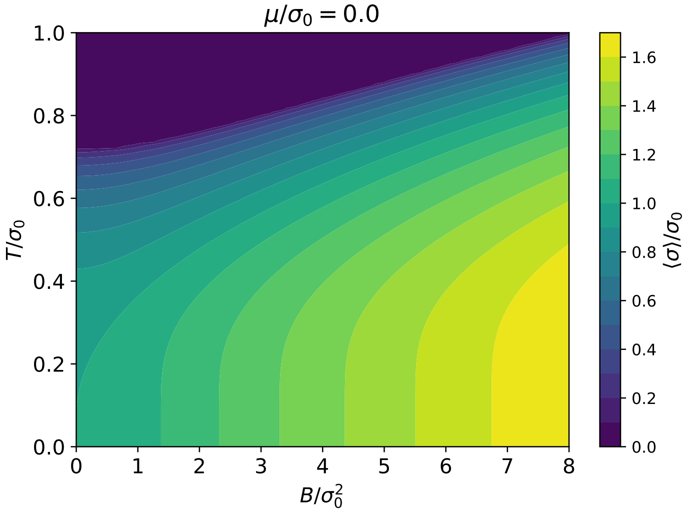
Large-N_f mean-field results



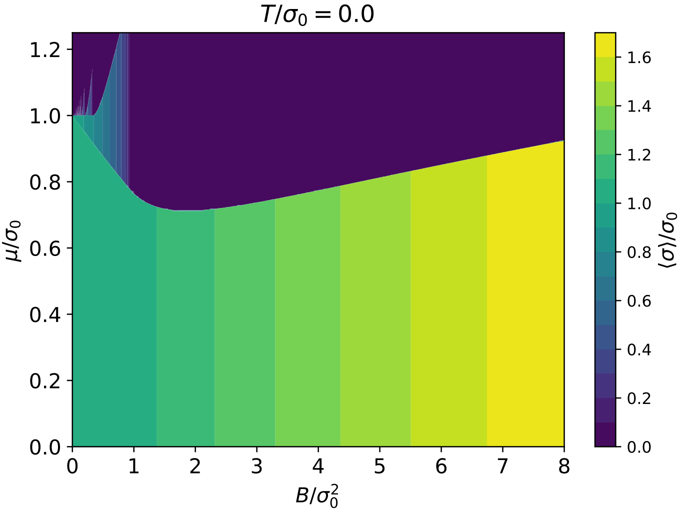
Large-N_f mean-field results



Large-N_f mean-field results



Large-N_f mean-field results



Large-N_f mean-field results*

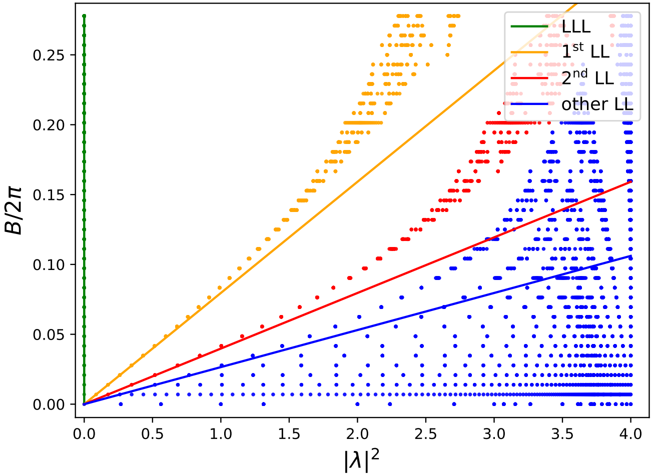
We observe:

- At $\mu = 0$:
  - $\frac{\partial\langle\sigma\rangle}{\partial B} > 0$, i.e. magnetic catalysis.
  - The critical temperature T_c grows with B.
- At $T = 0$:
  - $\frac{\partial\langle\sigma\rangle}{\partial B} \lessgtr 0$, i.e. both magnetic catalysis and inverse magnetic catalysis.
  - Multiple phase transitions for small B.
  - The critical chemical potential μ_c is non-monotonic in B.
- At large B:
  - $\frac{\partial\langle\sigma\rangle}{\partial B} > 0$, i.e. magnetic catalysis everywhere.
  - T_c and μ_c grow with B.

Landau levels on the lattice*

The choice of which lattice discretization to use for fermions is a non-trivial one. Ideally one would like to keep as many properties of the continuum theory as possible on the lattice. One lattice Dirac operator with the desired properties is the so-called overlap operator, which we use in our simulations.
To see that this discretization indeed preserves the very important Landau levels, the next slide shows the (absolute-squared) eigenvalues of the lattice operator (for free fermions in 2D) as a function of the magnetic field and compares with the known continuum result. The coloring represents the continuum degeneracies.
One observes that for small B the lattice and continuum results agree very well but they start to diverge for stronger magnetic fields due to discretization effects. A similar fractal structure is known from condensed-matter physics as the Hofstadter butterfly.

Landau levels on the lattice



Lattice results*

To investigate how much of the large-N_f phase structure survives when going beyond mean-field, we have performed lattice simulations along the T and μ axes of the phase diagram at finite magnetic field.

We show on the next few slides the behavior of the chiral condensate[1] $\langle|\sigma|\rangle$ in various scenarios (for our lattice setup see the appendix):

(6) ... T-dependence at $B = 0 = \mu$.

(7) ... μ-dependence at $B = 0 \approx T$.

(8) ... B-dependence at $\mu = 0 \approx T$.

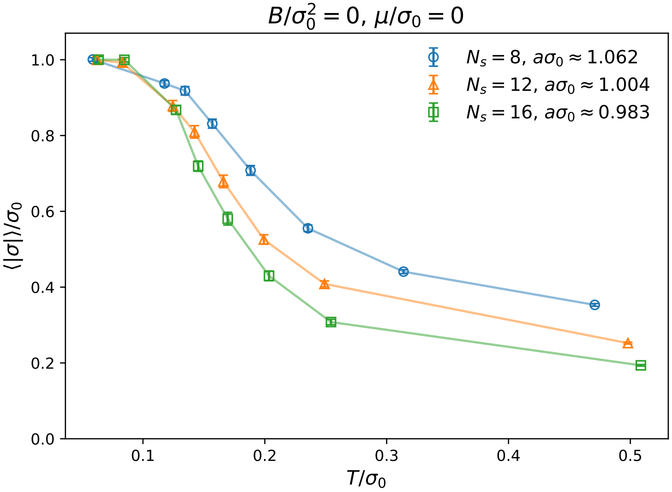
(9) ... B-dependence at various T and $\mu = 0$ for one lattice size.

(10) ... μ-dependence at various B and $T \approx 0$ for one lattice size.

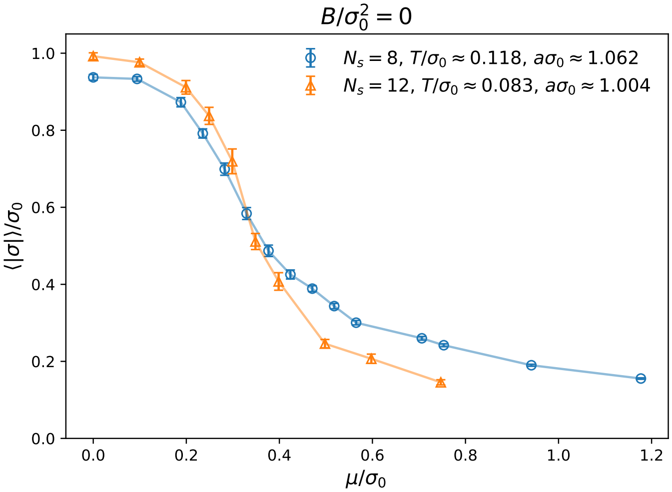
There, σ_0 denotes $\langle|\sigma|\rangle$ obtained at $\mu = 0 \approx T$, N_s is the number of lattice points in the spatial direction and a is the lattice spacing.

[1]We cannot use $\langle\sigma\rangle$ as the order parameter since it will always average to zero.

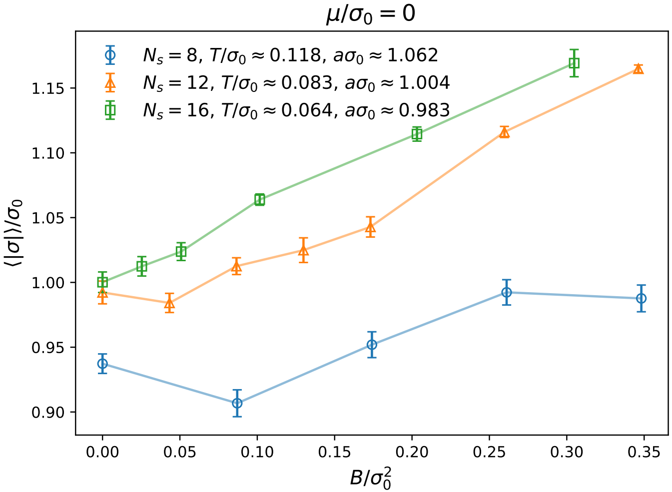
Lattice results



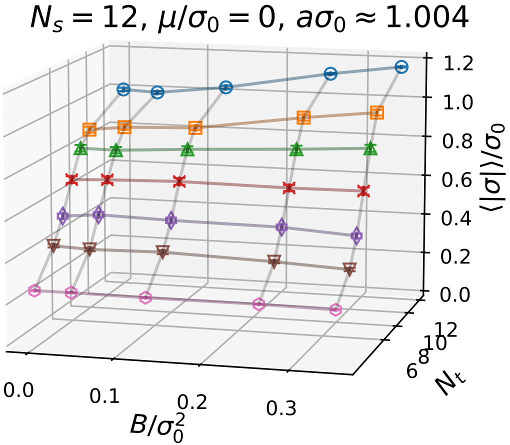
Lattice results



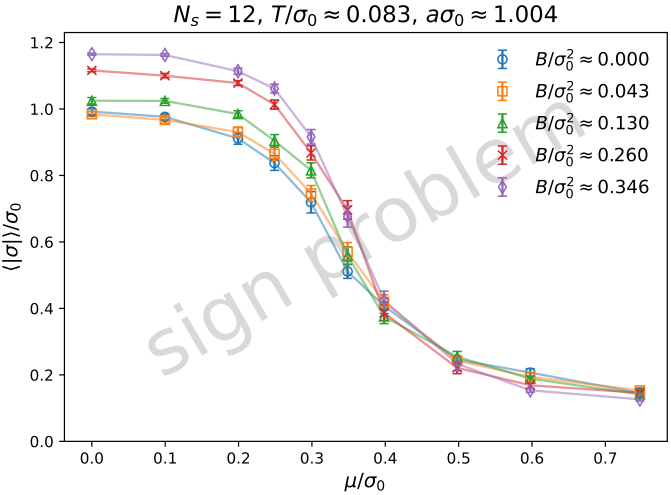
Lattice results



Lattice results



Lattice results



Lattice results*

Notice that due to taking the absolute value $|\sigma|$ we cannot observe a vanishing order parameter beyond the ”phase transition”.

Our results indicate that in the infinite-volume limit a proper phase transition is recovered. The $B = 0$ results are furthermore qualitatively consistent with existing literature.

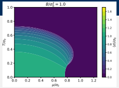
On slide (8) we see that the non-monotonic behavior at small B is a finite-size effect.

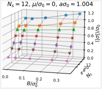
Apart from uncertainties due to the complex-action problem at $\mu \neq 0 \neq B$ our results suggest that there is magnetic catalysis everywhere below the phase transition.

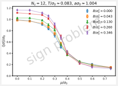
Summary & Outlook

Summary

- Magnetic catalysis and inverse magnetic catalysis in large-N_f.







- Only magnetic catalysis (?) for $N_f = 1$.
- Contradicts OPT calculations [J.-L. Kneur, M. B. Pinto, R. O. Ramos; Phys. Rev D **88** (2013)].

Outlook

- Inhomogeneous phases?
- Towards QCD: More sophisticated models.

Summary & Outlook*

It is known that magnetic fields can induce the presence of inhomogeneous phases in Four-Fermi theories. We wish to investigate this on the lattice in the future.

Furthermore, in order to make predictions for QCD, we would like to study more sophisticated models that more closely resemble QCD, both in 3D and in 4D.

Contact*

For questions/discussion please do not hesitate to contact the author of this talk via
michael.mandl@uni-jena.de

Backup

Reducible representation of γ_μ

To allow for a notion of chirality in $(2+1)$ dimensions, we combine the two irreducible representations of the Dirac algebra into a reducible one:

$$\gamma_0 = \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix} , \quad \gamma_1 = \begin{pmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix} , \quad \gamma_2 = \begin{pmatrix} \tau_1 & 0 \\ 0 & -\tau_1 \end{pmatrix} ,$$

where τ_μ are the usual Pauli matrices.

There are now two γ matrices which anti-commute with all the others:

$$\gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 0 & i\mathbb{1}_2 \\ -i\mathbb{1}_2 & 0 \end{pmatrix} .$$

Chiral symmetry in the continuum

The 1-flavor massless $(2+1)$-dimensional GN model in a reducible representation of the Dirac algebra has the following symmetries:

$$U_{\mathbb{1}}(1) : \psi \to e^{i\alpha}\psi \;,$$
$$U_{\gamma_{45}}(1) : \psi \to e^{i\alpha\gamma_{45}}\psi \;, \quad \gamma_{45} = i\gamma_4\gamma_5 \;,$$
$$\mathbb{Z}_2 : \psi \to i\gamma_5\psi.$$

The $\mathbb{Z}_2$ symmetry generated by γ_4 is not independent.

A mass term induces the breaking pattern

$$U_{\mathbb{1}}(1) \times U_{\gamma_{45}}(1) \times \mathbb{Z}_2 \to U_{\mathbb{1}}(1) \times U_{\gamma_{45}}(1) \;.$$

Chiral symmetry on the lattice

On the lattice the symmetries look as follows:

$$U_{\mathbb{1}}(1) : \psi \to e^{i\alpha}\psi \;,$$
$$U_{\gamma_{45}}(1) : \psi \to e^{i\alpha\gamma_{45}}\psi \;,$$
$$\mathbb{Z}_2 : \psi \to i\gamma_5(1 - D_{\text{ov}})\psi \;, \quad \bar{\psi} \to i\bar{\psi}\gamma_5 \;,$$

where D_{ov} is the massless overlap operator, in our case.

There is another $\mathbb{Z}_2$ generated by γ_4, which is, again, independent.

A mass term $\bar{\psi}\left(1 - \frac{D_{\text{ov}}}{2}\right)\psi$ again breaks the $\mathbb{Z}_2$ symmetry, but leaves both $U(1)$'s intact.

The Large-N_f limit

$$\mathcal{L} = i\bar{\psi}\mathbb{1}_{N_\mathrm{f}}\left(\partial\!\!\!/ + \sigma + \mu\gamma_0 + ieA\!\!\!/\right)\psi + \frac{N_f}{2g^2}\sigma^2 \ .$$

In the limit $N_\mathrm{f} \to \infty$, $\langle\sigma\rangle$ is given by the minimum of

$$S_\mathrm{eff}[\sigma] = -\ln\det(D[\sigma]) + \frac{1}{2g^2}\int d^3x\,\sigma^2(x) \ ,$$

$$D[\sigma] = \partial\!\!\!/ + \sigma + \mu\gamma_0 + ieA\!\!\!/ \ .$$

Overlap operator in the GN model

We use Neuberger's overlap operator [H. Neuberger; Phys. Lett B **417** (1998)]

$$D_{\mathrm{ov}} = \mathbb{1} + A/\sqrt{A^\dagger A}\,, \quad A = D_{\mathrm{W}} - \mathbb{1}\ ,$$

where D_W is the standard Wilson operator. The full operator, including σ and μ, reads [R. V. Gavai, S. Sharma; Phys. Lett B **716** (2012)]

$$D_{\mathrm{full}} = \left(1 - \frac{\sigma + \mu\gamma_0}{2}\right) D_{\mathrm{ov}} + \sigma + \mu\gamma_0\ .$$

With the Ginsparg-Wilson chiral condensate $\Sigma_{\mathrm{GW}} = \left\langle \bar{\psi} \left(\mathbb{1} - \frac{D_{\mathrm{ov}}}{2}\right) \psi \right\rangle$ we have a Ward identity in analogy to the continuum theory:

$$\langle \sigma \rangle = \Sigma_{\mathrm{GW}}\ .$$

The observable

In order to avoid cancellation of contributions from the two minima of the effective action ($\pm\sigma$) in the broken phase we measure the absolute value

$$\langle|\sigma|\rangle = \langle|\sum_{x\in\Lambda}\sigma(x)|\rangle \ ,$$

where the sum runs over the whole lattice.

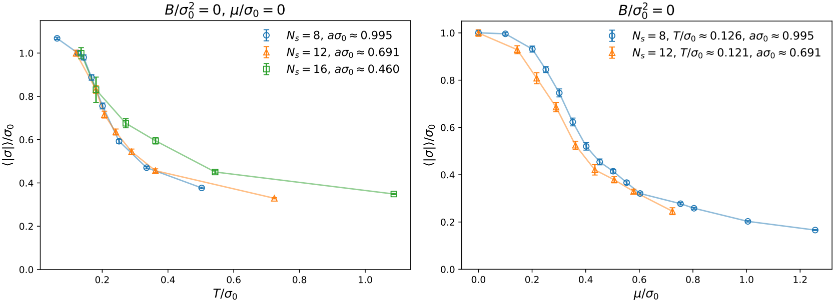
As a caveat, this definition makes it harder to determine a phase transition, as we cannot measure $\langle|\sigma|\rangle = 0$.

Lattice results

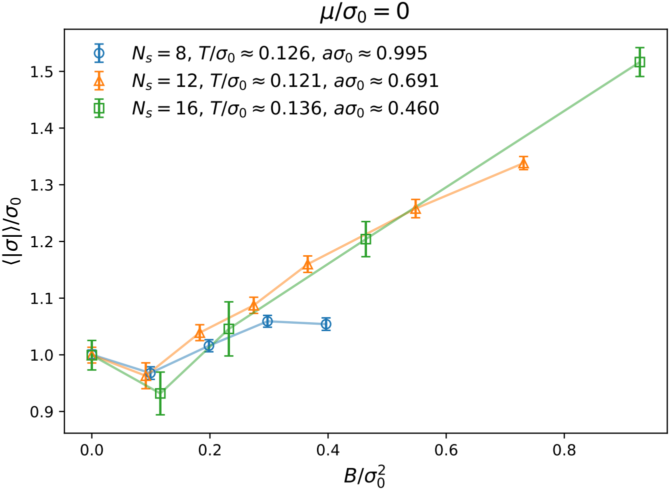
Simulation details:

- $N_{\mathrm{f}} = 1$ overlap fermions.
- Measure $|\sigma|$ to avoid cancellations.
- 8^3, 12^3 and 16^3 lattices with different lattice spacings.
- Change temperature by varying N_t.
- RHMC algorithm.
- Complex-action problem for $\mu \neq 0 \neq B$, probably mild.
  For now: phase quenching, $\det(D) \to |\det(D)|$.

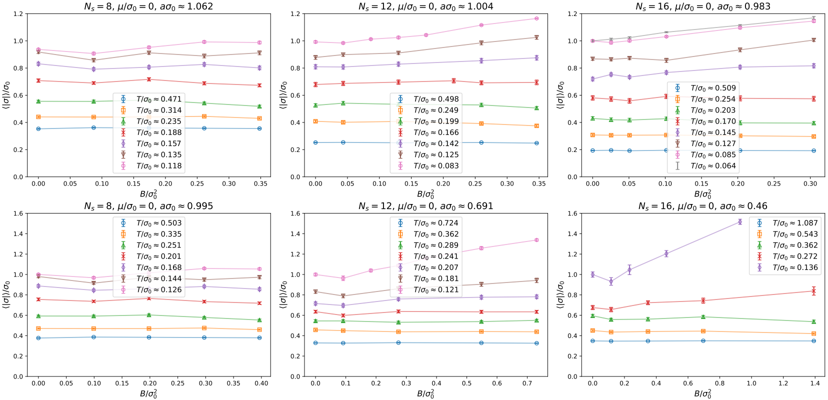
Lattice results at $B = 0$



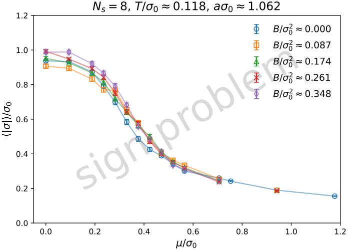
Lattice results at $B \neq 0$

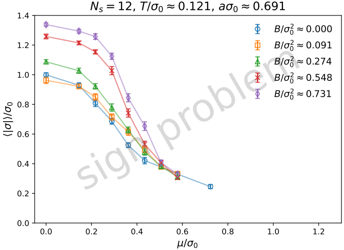


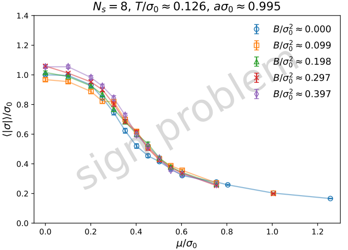
Lattice results at $\mu = 0$

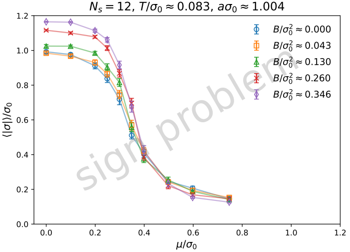


Lattice results at $T \approx 0$









The complex-action problem

Using charge conjugation

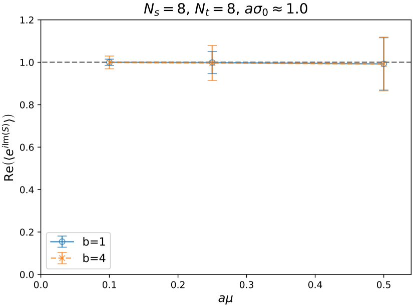
$$\mathcal{C}^{-1}\gamma_\mu\mathcal{C} = \gamma_\mu^T$$

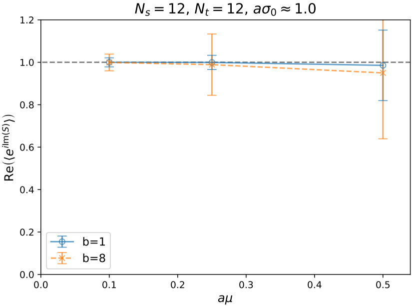
one can show that the full overlap operator at finite B and μ has the following properties:

$$\det[D_{\text{full}}(B,\mu)]^* = \det[D_{\text{full}}(-B,\mu)] = \det[D_{\text{full}}(B,-\mu)] \ ,$$

i.e. there can only be a complex-action problem if $B \neq 0 \neq \mu$.

The complex-action problem





Index theorem

An exact index theorem in our case would read

$$I := \text{Index}[D_{\text{ov}}] = \frac{1}{2}\text{tr}\left[\gamma_5 D_{\text{ov}}\right] = b\ ,$$

where b is proportional to the magnetic flux:

$$B = \frac{2\pi}{V} b\ .$$

Due to the vanishing theorem $|I|$ is equal to the total number of zero modes of D_{ov}. As the next slide shows the index theorem can be violated for large b and/or Wilson mass parameter m far from 1. λ_W is the smallest eigenvalue of the Wilson kernel used for the overlap and $||\text{GW}||$ measures a possible violation of the Ginsparg-Wilson equation.

Index theorem