Modelling Gravitational Waves

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Outline

Part I

- Linearized general relativity and gravitational waves
- Quadrupole formula

Part II

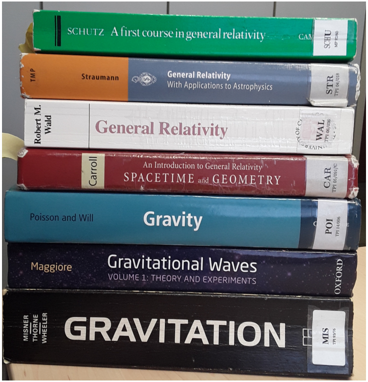
- Newtonian inspiral: the chirp
- Post-Newtonian theory

Part III

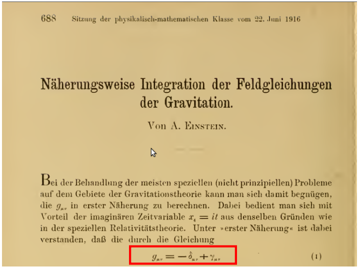
- Perturbation theory

Part IV

- Effective-one-body theory
- Synergies



Gravitational waves: Einstein 1916



688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$\boxed{g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}} \qquad (1)$$

Summary:

- Perturbations of spacetime with speed $= c$, sourced by accelerating masses (non-spherical).
- Wave equation: $\Box \bar{h}_{\mu\nu} = 0$.
- Plane-wave solution: $\bar{h}_{\mu\nu} = \Re\left[A_{\mu\nu}\,\mathrm{e}^{\pm i\omega(t - z/c)}\right]$
- Only 2 DoF $\Longrightarrow$ 2 polarizations (transverse)

Gravitational waves: linearized gravity I

Assume:

$\exists$ a global inertial coordinate system in which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{with} \quad |h_{\mu\nu}| \ll 1$$

- $\eta_{\mu\nu} = \text{diag}[-1, 1, 1, 1]$ is the **flat background metric**.
- Gravitational field generated by the source $T_{\mu\nu}$ does not back-react on itself: $\partial_\mu T^{\mu\nu} = 0$.
- $h_{\mu\nu}$ is Lorentz covariant

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} = \underbrace{\eta_{\mu'\nu'}}_{=\eta_{\mu\nu}} + \underbrace{\Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} h_{\mu\nu}}_{\text{(0,2) tensor}}$$

- Gauge transformations: $x^\mu \to x'^\mu = x^\mu + \xi^\mu \quad \text{with} \quad |\partial_\mu \xi_\nu| \lesssim |h_{\mu\nu}|$

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$h \to h - \mathcal{L}_\xi \eta$$

Gravitational waves: linearized gravity II

Solve the vacuum ($r \gg M$) Einstein field equation to $\mathcal{O}(|h_{\mu\nu}|)$:

- Zeroth-order (background) terms: $G^0_{\mu\nu}[\eta] = 0 \Rightarrow 0 = 0$
- First-order terms: $G^1_{\mu\nu}[h] = 0$
- Rewrite in terms of trace-reversed metric $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$
- Pick a gauge: $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ implies

$$\partial^\nu \bar{h}_{\mu\nu} \to (\partial^\nu \bar{h}_{\mu\nu})' = \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu$$

Let ξ_μ be such that it solves $\Box \xi_\mu = f_\mu \Longrightarrow \boxed{(\partial^\nu \bar{h}_{\mu\nu})' = 0}$ (Lorenz)[1]

$[f_\mu \equiv \partial^\nu \bar{h}_{\mu\nu}.\ \xi_\mu = \int d^4y\, G(x-y) f_\mu(x)$ ($\Box$ is invertible)]

When the dust settles we get a wave equation!

$$\boxed{\Box \bar{h}_{\mu\nu} = 0} \qquad \text{(sourced version: } -16\pi T_{\mu\nu})$$

Plane-wave solutions $\bar{h}_{\mu\nu} = \Re\left[A_{\mu\nu}\, \mathrm{e}^{ik_\mu x^\mu}\right]$, where $k^\mu = (\omega/c, \vec{k})^T$.

[1] Also known as Hilbert gauge or Lorentz gauge (see Maggiore pg. 8, footnote 4)

Gravitational waves: DoF and TT gauge

$A_{\mu\nu}$ is the **polarization** tensor.

- $\{\bar{h}_{\mu\nu}, A_{\mu\nu}\}$: 4×4, symmetric $\Rightarrow \frac{4\times5}{2} = 10$ d.o.f.
- Lorenz gauge: $\partial_\mu \bar{h}^{\mu\nu} = A_{\mu\nu} k^\nu = 0 \Rightarrow 10 - 4 =$ 6 d.o.f.
- Residual gauge freedom: consider $x^\mu \to x''^\mu = x^\mu + \chi^\mu : \Box \chi^\mu = 0$
  Then, $\partial^\nu \bar{h}_{\mu\nu} = 0 \to \partial^\nu \bar{h}_{\mu\nu} - \Box \chi_\mu = 0$
  and $\Box(\bar{h}_{\mu\nu} - \chi_{\mu\nu}) = 0$, where $\chi_{\mu\nu} = \partial_\mu \chi_\nu + \partial_\nu \chi_\mu - \eta_{\mu\nu} \partial_\alpha \chi^\alpha$
  χ^μ: 4 conditions $\Rightarrow 6 - 4 = 2$ d.o.f.

Pick χ^μ such that

$$\begin{cases} h_{0\mu} = 0, & \text{transverse}^2 \\ h^\mu{}_\mu = h^i{}_i = 0, & \text{traceless} \end{cases}$$

TT gauge is a vacuum gauge!

W/o loss of generality: $k^\mu = (k, 0, 0, k)^T$ then

$$A_{11} = -A_{22} \neq 0 \text{ and } A_{12} = A_{21} \neq 0$$

$\Rightarrow$ 2 POLARIZATIONS: $h_+ \equiv A_{11}, h_\times \equiv A_{12}$

Consistent with ± 2 helicities of a massless spin 2 boson.

[2] See Maggiore pg. 8 as to why we list 5, not 4 conditions.

Gravitational waves: effects on test masses

Geodesic deviation

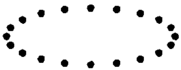
Two geodesics separated by ξ^μ: $\frac{d^2\xi^i}{d\tau^2} = -R^i{}_{0j0}\xi^j \left(\frac{dx^0}{d\tau}\right)^2$

$$\boxed{\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{\mathsf{TT}}\xi^j} \qquad \text{(relative acceleration)}$$

GW along $\mathbf{z}$: $h_{ij}^{\mathsf{TT}} = \begin{pmatrix} h_+ & 0 \\ 0 & -h_+ \end{pmatrix} \sin\left(\omega\left(t - \frac{z}{c}\right)\right)$

Under the GW: $(x_0, y_0) \to (x_0 + \delta x(t), y_0 + \delta y(t))$

$$(\delta\ddot{x}, \delta\ddot{y})^T = \frac{h_+}{2}(-x_0, y_0)^T \omega^2 \sin\omega t$$



$$(\delta x(t), \delta y(t))^T = \frac{h_+}{2}(x_0, -y_0)^T \sin\omega t$$

Likewise, $(\delta x(t), \delta y(t))^T = \frac{h_\times}{2}(y_0, x_0)^T \sin\omega t$

Gravitational waves: effects on test masses

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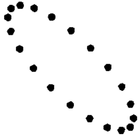

$$(\delta\ddot{x}, \delta\ddot{y})^T = \frac{h_+}{2}(-x_0, y_0)^T\,\omega^2 \sin\omega t$$

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Making gravitational waves

Sourced linearized field equation: $\Box \bar{h}_{\mu\nu} = -16\pi \frac{G}{c^4} T_{\mu\nu}$.
Formal solution in terms of retarded Green's function:

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4\frac{G}{c^4}\int \frac{T_{\mu\nu}(t_R,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \,, \quad \text{with} \quad t_R \equiv t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$$

(i) Large-distance $(r \gg M) \Longrightarrow |\mathbf{x}-\mathbf{x}'| \simeq r - \mathbf{x}'\cdot\mathbf{n}$, $\mathbf{n} \equiv \mathbf{x}/r, r \equiv \sqrt{x_i x^i}$
(ii) Slow-motion $(\lambda \gg |\mathbf{x}'|) \Longrightarrow \omega\frac{|\mathbf{x}'|}{c} \ll 1$ then

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4 r}\int T_{\mu\nu}(t - \tfrac{r}{c},\mathbf{x}')\, d^3x' + \mathcal{O}(r^{-2}) \qquad \text{(you do)}$$

Conservation law: $\partial_\mu T^{\mu\nu} = 0 \Longrightarrow \partial_t^2 T_{00} = \partial_k\partial_l T_{kl}$ (you do)

$\Longrightarrow \frac{d^2}{dt^2}\int T_{00} x^i x^j d^3x = 2\int T_{ij} d^3x$ (you do)

Slow-motion: $T_{00} \simeq \rho c^2$ so $4\int T_{ij} d^3x' = 2\frac{d^2}{dt^2}\int \rho\, x'^i x'^j d^3x' \equiv 2\partial_t^2 I_{ij}$.
Thus

$$\boxed{\bar{h}_{ij}(t,\mathbf{x}) = \frac{2G}{c^4 r}\ddot{I}_{ij}(t - \tfrac{r}{c})}$$

Quadrupole formula

Given any symmetric tensor S_{ij}, we can project out its TT part via

$$S_{ij}^{TT} = \Lambda_{ij,kl} S_{kl},$$

where $\Lambda_{ij,kl} \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}, \quad P_{ij} = \delta_{ij} - n_i n_j$ and $n_i = k_i/k$.

Define $Q_{ij} \equiv I_{ij} - \frac{1}{3}\delta_{ij} I_{kk} \Longrightarrow \Lambda_{ij,kl} I^{lk} = \Lambda_{ij,kl} Q^{kl}$ (you do)

Thus

$$\boxed{h_{ij}^{TT}(t, \mathbf{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\ddot{Q}_{kl}(t - \tfrac{r}{c})}$$

Classic example: Binary in circular orbit $\rho = m_1 \delta^3(\mathbf{x} - \mathbf{x}_1) + m_2 \delta^3(\mathbf{x} - \mathbf{x}_2)$

CM frame: $M = m_1 + m_2, \mu = m_1 m_2/M, \mathbf{R} \equiv |\mathbf{x}_1 - \mathbf{x}_2|, \Omega = (GM/R^3)^{1/2}$

$$h_+(t, \theta, \phi) = \frac{1}{r}\frac{G}{c^4}\mu R^2 (2\Omega)^2 \left(\tfrac{1+\cos^2\theta}{2}\right) \cos(2\Omega t_R + \phi),$$

$$h_\times(t, \theta, \phi) = \frac{1}{r}\frac{G}{c^4}\mu R^2 (2\Omega)^2 \cos\theta \, \sin(\underbrace{2\Omega}_{\omega} t_R + \phi)$$ (you do)

[See Maggiore Sec. 3.3 and problem 3.2]

Energy carried by gravitational waves I

Remark:[3] No definition of local energy density in GR (can't separate background from dynamics). **However**

- Notion of energy exists for an isolated system, far away.
- Energy must be quadratic in $|h_{\mu\nu}|$, come from a stress-energy tensor

Focus on small deviations from flat spacetime: $g = \eta + h^{(1)} + h^{(2)}$

Vacuum field equation: $G^{(0)}[\eta] = 0$ (background, trivial) then

$$0 = \underbrace{G^{(1)}[h^{(1)}]}_{=0,\text{ linear term}} + \underbrace{G^{(2)}[h^{(1)}]}_{\text{nonzero}} + \underbrace{G^{(1)}[h^{(2)}]}_{h^{(2)}:=-2^{\text{nd}}\text{term}}$$

Define $t_{\mu\nu} \equiv -\frac{1}{8\pi} G^{(2)}_{\mu\nu}[h^{(1)}]$ (i) symmetric, (ii) $\partial_\mu t^{\mu\nu} = 0$, (iii) $\sim |h_{\mu\nu}|^2$.

NOT $(*)$ gauge invariant, $(**)$ unique, $(***)$ a tensor in full GR.

BUT $\boxed{E \equiv \int_\Sigma d^3x\, t_{00}}$ (GW energy) is *unique* and gauge invariant

$\boxed{\Delta E = -\int_S t_{i0}\, dS^i}$ is the total radiated energy in GWs!

[3] See Carroll Ch. 7.6, Wald pg. 84-86, MTW Chs. 35.7, 35.13, and Schutz pg. 239

Energy carried by gravitational waves II

Using a suitable average ("shortwave formalism" Isaacson 1968[4])

$$\boxed{t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{ij}^{TT}\,\partial_\nu h^{ij}_{TT}\right\rangle}$$

$\partial_\mu t^{\mu\nu} = 0 \Longrightarrow 0 = \int_\Sigma (\partial_0 t^{00} + \partial_i t^{0i}) = -\dot{E} + \oint_{\partial\Sigma} t^{0i} n_i = -\dot{E} + \oint_{S^2} t^{0r} n_r$

Thus, $\dot{E} \propto \oint_{S^2} r^2 \langle \partial^0 h_{ij}^{TT} \partial_r h^{ij}_{TT}\rangle = -(r^2/c)\oint_{S^2}\langle \partial_t h_{ij}^{TT}\partial_t h^{ij}_{TT}\rangle$

Using $h_{ij}^{TT} h^{ij}_{TT} = h_{ij}h^{ij} - 2h_i^j h^{ik} n_j n_k + \frac{1}{2} h^{ij} h^{kl} n_i n_j n_k n_l$ in $\oint_{S^2} d\Omega$

Famous Einstein quadrupole formula

$$\boxed{\frac{dE}{dt} = \frac{G}{5c^5}\left\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\right\rangle} \tag{1}$$

Radiated orbital angular momentum(See Maggiore Ch. 3.3.3)

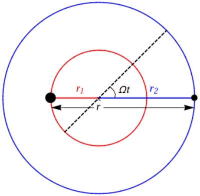
$$\frac{dL^i}{dt} = \frac{2G}{15c^5}\epsilon^{ijk}\left\langle \ddot{Q}_{jl}\dddot{Q}_{kl}\right\rangle$$

[4] Also see Wheeler 1964, Brill & Hartle 1964, Choquet-Bruhat 1969, and MacCallum & Taub 1973.

Compact Binary Inspirals I

Newtonian inspiral driven by quadrupole GWs

Binary systems: 2 points masses in a circular orbit ($Q^{ij} = \mu\, x^i x^j$)



f: GW frequency $\Longrightarrow \;\; \omega \equiv 2\pi f = 2\Omega$.

Chirp mass $\mathcal{M}_c \equiv \mu^{3/5} M^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$

Kepler's third law: $\Omega^2 = GMr^{-3} = \omega^2/4$

$$\boxed{\dot{E} = \frac{32}{5}\frac{c^5}{G}\left(\frac{G\mathcal{M}_c\,\omega}{2c^3}\right)^{10/3}}$$

(you do)

Assumption: Quasi-circular orbits: $e \ll 1$ and $\frac{|\dot{r}|}{r\Omega} < 10^{-3}$

Key idea: energy balance between $\dot{E}$ and $\dot{E}_b$ (binding energy)

where $E_b = -\frac{Gm_1m_2}{2r} = -\left(\frac{G^2\mathcal{M}_c^5\omega^2}{32}\right)^{1/3}$

$$\dot{E}(f) = -\dot{E}_b(f) \Longrightarrow \boxed{\dot{f} = \frac{96}{5}\pi^{8/3}\frac{(G\mathcal{M}_c)}{c^5}^{5/3} f^{11/3}}$$

$$\boxed{\dot{f} \propto f^{11/3} \qquad (2)}$$

Compact Binary Inspirals II

Integrate $\dot{f} \sim f^{11/3} \quad \Longrightarrow \quad \int dt \sim \int f^{-11/3} df \quad \Rightarrow \quad t \sim f^{-8/3}$

Fix integration constant $\tau = t_{\text{coal}} - t > 0$.

Introduce inspiral time, i.e, time to coalescence (merger)

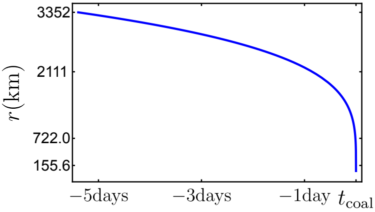
$$\boxed{\tau_{\text{insp}}(f) \simeq 16.72\,\text{minutes} \left(\frac{1.219 M_\odot}{M_c}\right)^{5/3} \left(\frac{10\,\text{Hz}}{f}\right)^{8/3} \qquad (3)}$$

Number of GW cycles to coalescence: $\int \frac{f}{\dot{f}} df \sim f^{-5/3}$

$$\boxed{\mathcal{N}_{\text{cyc}}(f) \approx 1.605 \times 10^4 \left(\frac{1.219 M_\odot}{M_c}\right)^{5/3} \left(\frac{10\,\text{Hz}}{f}\right)^{5/3} \qquad (4)}$$

Binary separation

$$\frac{\dot{r}}{r} \sim \frac{\dot{f}}{f} \Longrightarrow r(\tau) = r_i \left(\frac{\tau}{\tau_i}\right)^{1/4}$$



Gravitational waves from inspirals

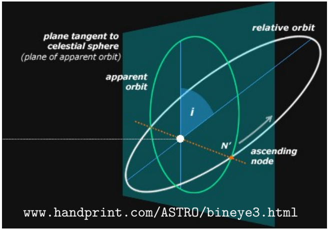
Recall $h_+, h_\times$ from the **circular binary** example $(r \to D, R \to r)$

$$h_+(t) = \frac{1}{D}\frac{G}{c^4}\mu r^2 (2\Omega)^2 \left(\tfrac{1+\cos^2\theta}{2}\right) \cos(2\Omega t_R + \phi),$$

$$\equiv h_c(t) \left(\tfrac{1+\cos^2\iota}{2}\right) \cos[\Phi(t)],$$

$$h_\times(t) = \frac{1}{D}\frac{G}{c^4}\mu r^2 (2\Omega)^2 \cos\theta \, \sin(2\Omega t_R + \phi)$$

$$\equiv h_c(t) \, \cos\iota \, \sin[\Phi(t)]$$



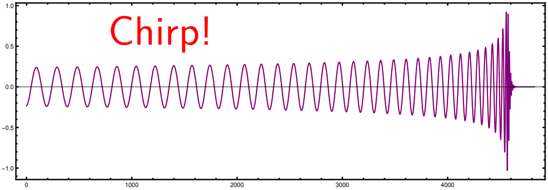
$\iota =$ orbital inclination

$$h_c(t) = \frac{4}{D}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f(t)}{c}\right)^{2/3} \approx 7.5 \times 10^{-24} \left(\tfrac{100\,\mathrm{Mpc}}{D}\right)\left(\frac{M_c}{1.219 M_\odot}\right)^{5/3}\left(\tfrac{f}{10\,\mathrm{Hz}}\right)^{2/3}$$

$$\Phi(t) = \int_{t_i}^{t} dt' \omega(t') + \mathrm{PN}$$

$$\frac{df}{dt} \sim f^{11/3} \sim \tau^{-11/8}$$

$$\frac{dh_c}{dt} \sim \frac{dh_c}{df}\dot{f} \sim f^{10/3} \sim \tau^{-5/4}$$



Post-Newtonian theory

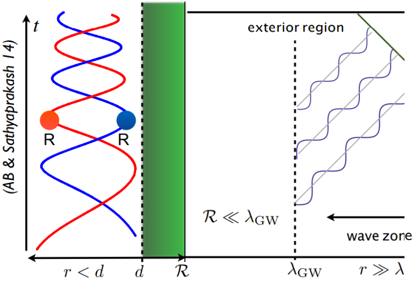
Note: $\Phi(t) = \int_{t_i}^{t} dt'\omega(t') + \text{PN}$

- Einstein 1916
- Droste & de Sitter 1916, Droste & Lorentz 1917.

Key idea: Weak-field ($\frac{GM}{c^2 r} \ll 1$)/slow-motion ($\frac{v}{c} \ll 1$) expansion

Three zones

- Near zone: $d < r \ll \lambda$
- Intermediate zone: $d < r \ll \lambda$
- Far (wave) zone: $r \gg \lambda$



Nomenclature: LO is N, NLO is 1PN.

E.g., quadrupole formula: $\dot{E} \sim c^{-5}$

NLO correction ("1PN"): $\sim c^{-7}$

Refer to

- L. Blanchet, Living Reviews in Relativity, arXiv:1310.1528[gr-qc].
- Poisson & Will, Gravity.
- Maggiore Chapter 5, Straumann Chapter 5.

Post-Newtonian expansion

Define $\epsilon \equiv \frac{v}{c} \sim \left(\frac{GM}{c^2 r}\right)^{1/2}$

Expand the metric

$$g_{00} = -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + \mathcal{O}(\epsilon^6),$$
$$g_{0i} = {}^{(3)}g_{0i} + \mathcal{O}(\epsilon^5),$$
$$g_{ij} = \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \mathcal{O}(\epsilon^6).$$

Pick a gauge: $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$ (de Donder).

Expanded field equation ($\Box = -c^{-2}\partial_t^2 - \nabla^2 \approx \nabla^2$, Weinberg 1972, Ch. 9.1)

$$\nabla^2[{}^{(2)}g_{00}] = -\frac{8\pi G}{c^4}{}^{(0)}T^{00} \qquad \text{(0PN)},$$
$$\nabla^2[{}^{(2)}g_{ij}] = -\frac{8\pi G}{c^4}\delta_{ij}{}^{(0)}T^{00} \qquad \text{(1PN)},$$
$$\nabla^2[{}^{(3)}g_{0i}] = \frac{16\pi G}{c^4}{}^{(1)}T^{0i} \qquad \text{(1PN)},$$
$$\nabla^2[{}^{(4)}g_{00}] = \ldots \qquad \text{(1PN)}$$

1PN equations

Introduce $\;{}^{(2)}g_{00} = -2\phi,\; {}^{(2)}g_{ij} = -2\delta_{ij}\phi,\; {}^{(3)}g_{0i} = \zeta_i,\;$ we have

$$\phi = -\frac{G}{c^4}\int d^3x' \frac{{}^{(0)}T^{00}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \quad \zeta_i = -\frac{4G}{c^4}\int d^3x' \frac{{}^{(1)}T^{0i}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

Gauge condition $\Longrightarrow 4\partial_0\phi + \nabla\cdot\zeta = 0.$

$\nabla^2\left[{}^{(4)}g_{00}\right] = \ldots \Longrightarrow \nabla^2\psi = \partial_0^2\phi + \frac{4\pi G}{c^4}\left[{}^{(2)}T^{00} + {}^{(2)}T^{ii}\right]$

$$\begin{cases} g_{00} &= -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + \mathcal{O}(c^{-6}), \\ g_{0i} &= -\frac{4}{c^3}V_i + \mathcal{O}(c^{-5}), \\ g_{ij} &= \delta_{ij}\left(1 + \frac{2V}{c^2}\right) + \mathcal{O}(c^{-4}). \end{cases}$$

1PN

where

$$V = \frac{G}{c^2}\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[T^{00}(t_R,\mathbf{x}') + T^{ii}(t_R,\mathbf{x}')\right],$$

$$V_i = \frac{G}{c^2}\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}T^{0i}(t_R,\mathbf{x}')^5$$

[5] We "promoted" t to retarded time t_R. In near zone, $t \simeq t_R$. Additionally, $\sigma(t_R) = \sigma(t) - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\partial_t\sigma + \mathcal{O}(c^{-2})$

1PN equations of motion I

N-body system (point particles)

$$T^{\mu\nu} = \underbrace{\frac{1}{\sqrt{-g}}}_{\text{PN expand}} \sum_A m_A \frac{d\tau_A}{dt} u_A^\mu u_A^\nu \, \delta^3(\mathbf{x} - \mathbf{x}_A(t))$$

$$^{(0)}T^{00} = \sum_A m_A c^2 \delta^3(\mathbf{x} - \mathbf{x}_A(t)) \qquad \text{(rest mass)},$$

$$^{(2)}T^{00} = \sum_A m_A \left(\tfrac{1}{2} v_A^2 + \phi c^2\right) \delta^3(\mathbf{x} - \mathbf{x}_A(t)) \qquad \text{(0PN } E_{\text{tot}}\text{)},$$

$$^{(1)}T^{0i} = c \sum_A m_A v_A^i \delta^3(\mathbf{x} - \mathbf{x}_A(t)), \; ^{(2)}T^{ij} = \sum_A m_A v_A^i v_A^j \delta^3(\mathbf{x} - \mathbf{x}_A(t)) \qquad \text{(1PN)}$$

$$^{(0,2)}T^{\mu\nu} \implies {}^{(0,2,4)}g_{\mu\nu} \implies \mathcal{L} = \underbrace{-g_{00} - 2g_{0i}\frac{v^i}{c} - g_{ij}\frac{v^i v^j}{c^2}}_{\mathcal{L}_0 \, + \, \frac{1}{c^2}\mathcal{L}_2} \implies \text{EL Eqs}^6$$

$$\mathbf{a} = -\frac{GM}{r^2}\hat{\mathbf{n}} - \frac{GM}{c^2 r^2}\left[\left\{(1+3\nu)v^2 - \frac{3}{2}\nu\dot{r}^2 - 2(2+\nu)\frac{GM}{r}\right\}\hat{\mathbf{n}} - 2(2-\nu)\dot{r}\mathbf{v}\right]$$

where $\nu = m_1 m_2/M^2$, $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{a} = \dot{\mathbf{v}}$, $\hat{\mathbf{n}} = \mathbf{r}/r$, $\dot{r} = \hat{\mathbf{n}} \cdot \mathbf{v}$

[6] Einstein-Infeld-Hoffmann equations.

1PN equations of motion II

Energy $E = \mu\varepsilon$, with $\mu = m_1 m_2/M$

$$\varepsilon = \frac{1}{2}v^2 - \frac{GM}{r} + \frac{1}{c^2}\left[\frac{3}{8}(1-3\nu)v^4 + \frac{GM}{2r}\left\{(3+\nu)v^2 + \nu\dot{r}^2 + \frac{GM}{r}\right\}\right]$$

(i) $\dot{E} = 0$, (ii) $\lim_{M/r\to 0} \mu\varepsilon = [(\gamma_1 - 1)m_1 + (\gamma_2 - 1)m_2]c^2$ (you do)

Setting $\mathbf{r}_{\mathsf{CM}} = 0$ we have 1PN-accurate positions ($\delta m \equiv m_1 - m_2$)

$$\mathbf{x}_1 = \frac{m_2}{M}\mathbf{r} + \frac{\nu\delta m}{2c^2M^2}\left(v^2 - \frac{GM}{r}\right)\mathbf{r} + \mathcal{O}(c^{-4}),$$

$$\mathbf{x}_2 = -\frac{m_1}{M}\mathbf{r} + \frac{\nu\delta m}{2c^2M^2}\left(v^2 - \frac{GM}{r}\right)\mathbf{r} + \mathcal{O}(c^{-4}),$$

1 PN EoM suffice to give us

- Mercury's perihelion precession: $\delta\varphi = \frac{6\pi GM}{c^2 p}$ (Einstein Nov 1915).
- Geodetic (de Sitter) and Lense-Thirring precessions due to the Earth (gravito-electromagnetism).
- Deflection of light around the Sun.

Going beyond 1PN

Two issues with the PN expansion

- At some PN order, divergences appear in the multipolar expansion of the Poisson integral

$$\underbrace{[\Delta^{-1}f](\mathbf{x})}_{\text{inversion of } \nabla^2 f(\mathbf{x})} \equiv -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\,f(\mathbf{x}')$$

of the form $\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r^\ell} + \frac{(\mathbf{x}\cdot\mathbf{x}')^\ell}{r^3} + \ldots \sim \frac{(\mathbf{x}'\cdot\hat{\mathbf{n}})^\ell}{r^{\ell+1}} \to \infty$ if $|\mathbf{x}'| \gg r$

- PN expansion can NOT use BC at ∞, i.e., is ill-equipped to study the large-r region

$$\frac{1}{r}F_{\mu\nu}(t-r/c) = \frac{1}{r}F_{\mu\nu}(t) - \frac{1}{c}\dot{F}_{\mu\nu}(t) + \frac{r}{2c^2}\ddot{F}_{\mu\nu}(t) + \mathcal{O}(r^2) \to \infty \quad \text{as } r \to \infty$$

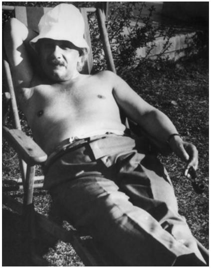
Mitigation:

Use PN in the near zone. Use post-Minkowski (**PM**) in the **far zone** then match using *Matched-asymptotic expansions* in the intermediate zone[7].

[7] This is known as the Blanchet-Damour approach. See Phil. Trans. Roy. Soc. Lond. A320 (1986) 379-430.

Beyond 1PN road map

The **Relaxed** Einstein equations



Define $\quad \mathsf{h}^{\mu\nu} \equiv \eta^{\mu\nu} - \sqrt{-g}\, g^{\mu\nu} \quad (= \bar{h}^{\mu\nu})$

de Donder gauge: $\quad \partial_\mu \mathsf{h}^{\mu\nu} = 0$

Stress-energy conservation[8] $\quad \partial_\mu \tau^{\mu\nu} = 0 \quad$ where

$$\tau^{\mu\nu} = -g\left[T^{\mu\nu} + \tau^{\mu\nu}_{\mathsf{LL}}\right] + (\partial \mathsf{h})^2 - \mathsf{h}\partial^2 \mathsf{h}$$

$\tau^{\mu\nu}_{\mathsf{LL}}$ is the Landau-Lifshitz energy-momentum pseudotensor.

The relaxed Einstein equations

$$\boxed{\Box \mathsf{h}^{\mu\nu} = -\frac{16\pi G}{c^4}\tau^{\mu\nu}}$$

Solution

$$\mathsf{h}^{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\,\tau^{\mu\nu}(t - |\mathbf{x}-\mathbf{x}'|/c, \mathbf{x}')$$

[8] Note, this comes from gauge condition $\oplus\ \nabla_\mu T^{\mu\nu} = 0$.

Beyond 1PN road map

The **post-Minkowskian** expansion outside the source

Valid for $d < r < \infty$ where $d \sim$ orbital radius.

Expand in powers of G for $|\mathsf{h}_{\mu\nu}| \ll 1$, iterate the RRE

$$\mathsf{h}^{\mu\nu} = \sum_{n=1}^{\infty} G^n \mathsf{h}_n^{\mu\nu},$$

$$\Box \mathsf{h}_{n+1}^{\mu\nu} = -\frac{16\pi G}{c^4}\,\tau^{\mu\nu}(h_n),$$

$$\mathsf{h}_{n+1}^{\mu\nu} = \frac{4G}{c^4}\int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\,\tau^{\mu\nu}(\mathsf{h}_n)(t-|\mathbf{x}-\mathbf{x}'|/c, \mathbf{x}')$$

Motion from $\partial_\mu \tau^{\mu\nu}(h_n) = 0$.

First iteration gives $\mathsf{h}_1^{\mu\nu} = \bar{h}^{\mu\nu}$ since $\tau_0^{\mu\nu} = T^{\mu\nu}$.

Higher iterations: $\Box \mathsf{h}_n^{\mu\nu} = \Lambda_n^{\mu\nu}[\mathsf{h}_1, \mathsf{h}_2, \ldots, \mathsf{h}_{n-1}]$[9]

Blanchet-Damour: iterate a finite multipole expansion of $\mathsf{h}_1^{\mu\nu}$ for finite PN order (multipolar post-Minkowskian expansion).

[9] See Maggiore pg. 254 for details.

Beyond 1PN road map

The multipolar post-Minkowskian expansion

Blanchet-Damour "regularization" of $r = 0$

Replace $$(\Box^{-1}_{\text{ret}} f)(t, \mathbf{x}) = -\frac{1}{4\pi}\int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|} f(t - |\mathbf{x}-\mathbf{x}'|.\mathbf{x}')$$

with $$\text{FP}_{B=0}\left[\Box^{-1}_{\text{ret}}\left(\tilde{r}^B f\right)\right](t, \mathbf{x}) = -\frac{1}{4\pi}\int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|} \tilde{r}^B f(t - |\mathbf{x}-\mathbf{x}'|.\mathbf{x}')$$

where $B \in \mathbb{C}$, $\Re(B) > k_{\text{max}} - 3$, and $\tilde{r} \equiv r/r_0$[10]

Thus, the solution at each PM order is given by

$$\mathsf{h}^{\mu\nu}_n = \underbrace{\text{FP}_{B=0}\left[\Box^{-1}_{\text{ret}}\left(\tilde{r}^B \Lambda^{\mu\nu}_n\right)\right]}_{\equiv \mathcal{FP}\Box^{-1}_{\text{ret}}\Lambda^{\mu\nu}_n \text{ [particular solution]}} + \underbrace{v^{\mu\nu}_n}_{\text{hom. sol.}}$$

Finite PN expansion

$$\bar{\mathsf{h}}^{\mu\nu} \equiv \sum_{m=2}^{N} \frac{1}{c^m}\,{}^{(m)}\mathsf{h}^{\mu\nu}, \quad \bar{\tau}^{\mu\nu} \equiv \sum_{m=-2}^{N} \frac{1}{c^m}\,{}^{(m)}\tau^{\mu\nu},$$

$$\bar{\mathsf{h}}^{\mu\nu} = \frac{16\pi G}{c^4}\mathcal{FP}\Box^{-1}_{\text{ret}}\bar{\tau}^{\mu\nu} + \bar{\mathsf{h}}^{\mu\nu}_{\text{hom}}$$

[10] See Blanchet LRR Sec. 2.3 and Maggiore Ch. 5.3.2 for details.

Beyond 1PN road map

Match the solutions

- $d < r < \infty$ **PM** regime
- $0 < r < \mathcal{R}$ PN regime

For $v/c \ll 1$, $\mathcal{R} \gg d \implies$ Matching region $d < r < \mathcal{R}$ (green band)

Re-Expand PN terms in $d/r < 1$ and **PM** terms in $\mathbf{v}/\mathbf{c}$

Recall, we match multipole expansions (Blanchet LRR Sec. 4.4 for details)
n-th PM term has the following PN expansion

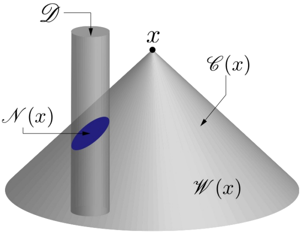
$$\mathrm{h}_n^{00} = \mathcal{O}(c^{-2n}), \quad \mathrm{h}_n^{0i} = \mathcal{O}(c^{-(2n+1)}), \quad \mathrm{h}_n^{ij} = \mathcal{O}(c^{-2n})\,.$$

E.g., 2PN order, i.e., $\mathcal{O}(c^{-4})$ correction to Newtonian metric $\Longleftarrow \mathrm{h}_1, \mathrm{h}_2, \mathrm{h}_3$

Equivalent formalism

by Will, Wiseman and Pati (DIRE)[11]

$$\int_{\mathcal{C}} d^3x' = \int_{\mathcal{N}} d^3x' + \int_{\mathcal{W}} d^3x'$$



[11] See Maggiore Ch. 5.4 for a brief intro and Poisson-Will for abundant details.

More PN!

We solved the wave-zone issues, but what about near zone?
Blanchet et al. $\bar{h} = \bar{h}(t, \mathbf{x}), \bar{\tau} = \bar{\tau}(t, \mathbf{x})$ [NO retardation]

$$\bar{\mathsf{h}}^{\mu\nu} = \frac{16\pi G}{c^4} \Box^{-1}_{\mathsf{inst}} \bar{\tau}^{\mu\nu} + \underbrace{\bar{\mathsf{h}}^{\mu\nu, RR}_{\mathsf{hom}}}_{\text{dissipative}}$$

where $\Box^{-1}_{\mathsf{inst}}[\bar{\tau}] \equiv \sum_{k=0}^{\infty} \left(\frac{\partial}{c\partial t}\right)^{2k} \Delta^{-k-1}[\bar{\tau}]$

Regularization of **point-particle infinities**: Hadamard and/or dimensional

$$\begin{aligned}
\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + 4\pi x^{3/2} \\
& + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right) \pi x^{5/2} \\
& + \Bigg[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_{\mathrm{E}} - \frac{856}{105}\ln(16\,x) \\
& \qquad + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\Bigg] x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right) \pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right) \Bigg\}.
\end{aligned}$$

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Regularization of **point-particle infinities**: Hadamard and/or dimensional

$$\begin{aligned}
\phi = -\frac{x^{-5/2}}{32\nu} \Bigg\{ & 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right) x - 10\pi x^{3/2} \\
& + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right) x^2 + \left(\frac{38645}{1344} - \frac{65}{16}\nu\right) \pi x^{5/2} \ln\left(\frac{x}{x_0}\right) \\
& + \left[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_{\text{E}} - \frac{856}{21}\ln(16\,x) \right. \\
& \quad \left. + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\right] x^3 \\
& + \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right) \pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right) \Bigg\},
\end{aligned}$$

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where $\Box^{-1}_{\text{inst}}[\bar{\tau}] \equiv \sum_{k=0}^{\infty} \left(\frac{\partial}{c\partial t}\right)^{2k} \Delta^{-k-1}[\bar{\tau}]$

Regularization of **point-particle**

$$\mathbf{x}_1 = \left[\frac{m_2}{M} + \nu \frac{\delta m}{M} \mathcal{P}\right] + \nu \frac{\delta m}{M} \mathcal{Q} \mathbf{v}$$

$$\mathbf{x}_2 = \left[-\frac{m_1}{M} + \nu \frac{\delta m}{M} \mathcal{P}\right] + \nu \frac{\delta m}{M} \mathcal{Q} \mathbf{v}$$

$$+\mathcal{O}(c^{-7})$$

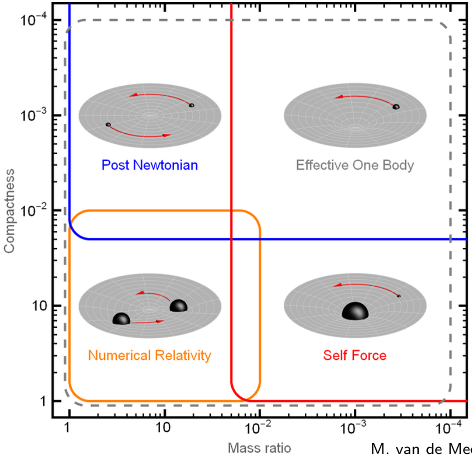
$$\begin{aligned}
\mathcal{P} = {} & \frac{1}{c^2}\left\{\frac{v^2}{2} - \frac{Gm}{2\,r}\right\} \\
& + \frac{1}{c^4}\left\{\frac{3\,v^4}{8} - \frac{3\,\nu\,v^4}{2} + \frac{Gm}{r}\left(-\frac{\dot{r}^2}{8} + \frac{3\,\dot{r}^2\,\nu}{4} + \frac{19\,v^2}{8} + \frac{3\,\nu\,v^2}{2}\right) + \frac{G^2m^2}{r^2}\left(\frac{7}{4} - \frac{\nu}{2}\right)\right\} \\
& + \frac{1}{c^6}\left\{\frac{5\,v^6}{16} - \frac{11\,\nu\,v^6}{4} + 6\,\nu^2\,v^6 \right. \\
& \qquad + \frac{Gm}{r}\left(\frac{\dot{r}^4}{16} - \frac{5\,\dot{r}^4\,\nu}{8} + \frac{21\,\dot{r}^4\,\nu^2}{16} - \frac{5\,\dot{r}^2\,v^2}{16} + \frac{21\,\dot{r}^2\,\nu\,v^2}{16} \right. \\
& \qquad\qquad \left. - \frac{11\,\dot{r}^2\,\nu^2\,v^2}{2} + \frac{53\,v^4}{16} - 7\,\nu\,v^4 - \frac{15\,\nu^2\,v^4}{2}\right) \\
& \qquad + \frac{G^2m^2}{r^2}\left(-\frac{7\,\dot{r}^2}{3} + \frac{73\,\dot{r}^2\,\nu}{8} + 4\,\dot{r}^2\,\nu^2 + \frac{101\,v^2}{12} - \frac{33\,\nu\,v^2}{8} + 3\,\nu^2\,v^2\right) \\
& \qquad \left. + \frac{G^3m^3}{r^3}\left(-\frac{14351}{1260} + \frac{\nu}{8} - \frac{\nu^2}{2} + \frac{22}{3}\ln\left(\frac{r}{r_0''}\right)\right)\right\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q} = {} & \frac{1}{c^4}\left\{-\frac{7\,Gm\,\dot{r}}{4}\right\} + \frac{1}{c^5}\left\{\frac{4\,Gm\,v^2}{5} - \frac{8\,G^2m^2}{5\,r}\right\} \\
& + \frac{1}{c^6}\left\{Gm\,\dot{r}\left(\frac{5\,\dot{r}^2}{12} - \frac{19\,\dot{r}^2\,\nu}{24} - \frac{15\,v^2}{8} + \frac{21\,\nu\,v^2}{4}\right) + \frac{G^2m^2\,\dot{r}}{r}\left(-\frac{235}{24} - \frac{21\,\nu}{4}\right)\right\}
\end{aligned}$$

Perturbation Theory

Perspective: **two-body problem** in general relativity

PN approach: weak-field (large separation), arbitrary mass ratio



M. van de Meent

Perturbation Theory in strong field

Specifically around black holes

Consider a black hole solution (Schwarzschild, Kerr, etc.)

- Vacuum so $\mathcal{L}[\mathring{g}_{\mu\nu}] \Longrightarrow$ geodesic EoM, i.e., test masses ($m_1 \to 0$)
- What happens if we slowly turn m_1 on?
- Let $q \equiv \frac{m_1}{m_2} \ll 1$ be the new expansion parameter

Linear expansion in $h_{\mu\nu}$ about background metric $\mathring{g}_{\mu\nu}$: $|h_{\mu\nu}| \sim q \ll 1$

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu} \Longrightarrow g^{\mu\nu} = \mathring{g}^{\mu\nu} - h^{\mu\nu}$$

Source of the perturbation $T^{\mu\nu}[g] = T^{\mu\nu}[\mathring{g}] + \dots$

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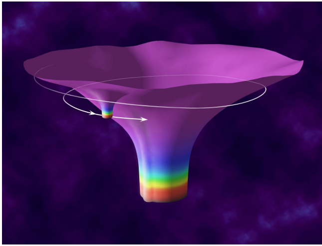
Source of the perturbation $T^{\mu\nu}[g] = T^{\mu\nu}[\mathring{g}] + \ldots$

- LO term of the field equation: $\mathring{G}_{\mu\nu}[\mathring{g}] = 0$
- NLO term of the field equation (geodesic)

$$\boxed{\mathring{G}_{\mu\nu}[h] = 8\pi T^{\mu\nu}[\mathring{g}]}$$

(post-geodesic)

- Supplement with a certain **gauge condition**.



Perturbation theory in Schwarzschild

Regge-Wheeler-Zerilli equations

Background set to Schwarzschild metric

$$\mathring{g}_{\mu\nu} = \text{diag}[-f, f^{-1}, r^2, r^2 \sin^2\theta], \quad f = 1 - \frac{2m_2}{r}$$

Decompose $h_{\mu\nu}$ into even (Y)/odd (X)-parity **tensor harmonics**[12]

$$h_{ab} = \sum_{\ell m} h_{ab}^{\ell m} Y^{\ell m},$$

$$h_{aB} = \sum_{\ell m} (j_a^{\ell m} Y_B^{\ell m} + h_a^{\ell m} X_B^{\ell m}),$$

$$h_{AB} = r^2 \sum_{\ell m} (K^{\ell m} \Omega_{AB} Y^{\ell m} + G^{\ell m} Y_{AB}^{\ell m} + h_2^{\ell m} X_{AB}^{\ell m})$$

with $a, b = t, r$ and $A, B = \theta, \phi$ and $h, j, K, G, h_2 =$ funcs.(t,r).

Regge-Wheeler gauge

$$\boxed{j_a^{\ell m} = G^{\ell m} = 0 \quad \text{(even parity)}, \qquad h_2^{\ell m} = 0 \quad \text{(odd parity)}}$$

[12] See Martel & Poisson gr-qc/0502028 for details.

Perturbation theory in Schwarzschild

Regge-Wheeler-Zerilli equations

Master equations

$$\boxed{[\Box - V_{e,o}]\Psi_{e,o} = S_{e,o}[T^{\mu\nu}]}$$

$$V_e = \frac{f}{k^2}\left[[(\ell-1)(\ell+2)]^2\left(\frac{\ell(\ell+1)}{r^2} + \frac{6m_2}{r^3}\right) + \frac{36m_2^2}{r^4}\left(\ell-1)(\ell+2) + \frac{2m_2}{r}\right)\right],$$

$$V_o = f\left[\frac{\ell(\ell+1)}{r^2} + \frac{6m_2}{r^3}\right], \qquad k = (\ell-1)(\ell+2) + 6m_2/r$$

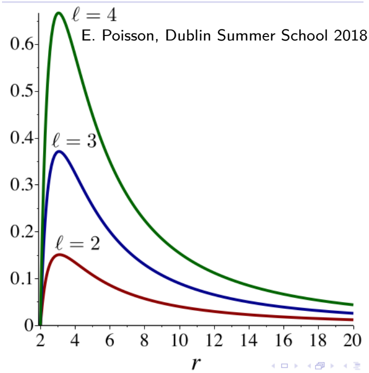
$$\Psi_e = \frac{2r}{\ell(\ell+1)}\left[K + \frac{2f}{k}(fh_{rr} - r\partial_r K)\right],$$

$$\Psi_o = \frac{2r}{(\ell-1)(\ell+2)}\left(\partial_r h_t - \partial_t h_r - \frac{2}{r}h_t\right),$$

Source: point-particle of mass m_1 along a timelike geodesic

$$T^{\mu\nu} = \frac{1}{\sqrt{-\mathring{g}}}\, m_1 \frac{d\tau}{dt}\, \mathring{u}^\mu \mathring{u}^\nu\, \delta^3(\mathbf{x} - \mathbf{z}(t))$$

$$S_{e,o} \sim \underbrace{\{\partial_a^0, \partial_a\}}_{\text{hit } \delta^3} \int T^{\mu\nu}\left\{\bar{Y}^{\ell m}_{\mu\nu}, \bar{X}^{\ell m}_{\mu\nu}\right\} d\Omega$$



Perturbation theory in Schwarzschild

Regge-Wheeler-Zerilli equations

Martel & Poisson give us everything (for circular geodesics)

$$h_+ = \frac{1}{r}\sum_{\ell m} \Psi_e^{\ell m} D^2_{\theta,\ell} Y^{\ell m} - \Psi_o^{\ell m} D_\theta Y^{\ell m},$$

$$h_\times = \frac{1}{r}\sum_{\ell m} \Psi_e^{\ell m} \frac{im}{\sin\theta} D_\theta Y^{\ell m} - \Psi_o^{\ell m} D^2_{\theta,\ell} Y^{\ell m},$$

$$\dot{E}_{\infty,H} = \frac{1}{64\pi}\sum_{\ell m} \frac{(\ell+2)!}{(\ell-2)!}\left\langle |\dot{\Psi}_e^{\ell m}|^2 + |\dot{\Psi}_o^{\ell m}|^2 \right\rangle_{t\to\infty, r_*\to\pm\infty}$$

$\langle \ldots \rangle$ is an orbital average.

Homogeneous solutions can be obtained analytically or numerically.

Fluxes are straightforward, but NOT evaluating $h_{\mu\nu}$ at $\mathbf{x} = \mathbf{z}$.

Sources $S_{e,o} \sim F\delta'(r - r_0) + G\delta(r - r_0)$ very singular!

We will talk more about regularizing δ-function sources **later**.

Quasinormal excitations of black holes

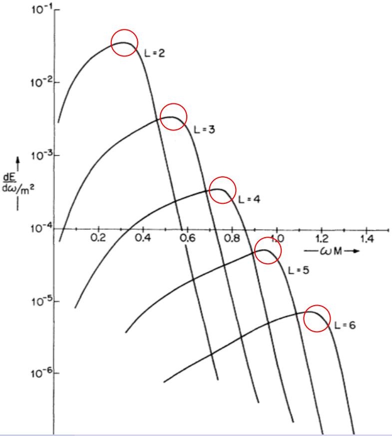
We saw previously that $\dot{E}_H \neq 0 \implies$ BH absorbs the energy

What happens?

$\implies$ **Damped** normal-mode oscillations: the BH rings!

A problem of scattering spin-2 bosons off BHs

- Vishveshwara 1970: scattering GWs off Sch. horizon: damped sinusoids
- Davis-Ruffini-Press-Price 1971: radial infall of m onto Sch. BH $g(M)$ $(m \ll M)$, solve Zerilli equation
- Press 1971: Symmetric initial pert. Numerical RWZ, $\ell >> 1, M \gtrsim \frac{\lambda}{2\pi} >> \frac{M}{\ell}$ "the black hole vibrates around **spherical symmetry** in a quasi-normal mode" $\omega \approx 27^{-1/2}\frac{\ell}{M}$



Eigenmodes of **dissipative** systems!

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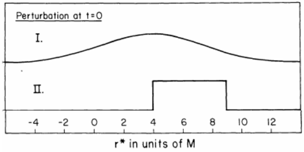
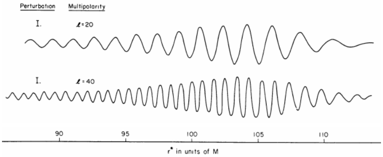
- Davis-Ruffini-Press-Price 1971: radial infall of m onto Sch. BH $g(M)$ $(m \ll M)$, solve Zerilli equation
- Press 1971: Symmetric initial pert. Numerical RWZ, $\ell >> 1, M \gtrsim \frac{\lambda}{2\pi} >> \frac{M}{\ell}$ "the black hole vibrates around **spherical symmetry** in a quasi-normal mode" $\omega \approx 27^{-1/2}\frac{\ell}{M}$





Eigenmodes of **dissipative** systems!

Quasinormal modes

One method: WKB treatment of wave scattering on the peak of the potential barrier (parabolic cylinder functions)[13]

$$(M\omega_n)^2 = V_\ell(r_p) - i\left(n+\tfrac{1}{2}\right)\left[-2\frac{d^2V_\ell}{dr_*^2}\right]^{1/2}_{r_*=r_p^*}$$

E.g., $(\ell = 2, n = 0)$: $\Re(M\omega) = 0.37$ vs. $\Re(M\omega) = 0.32$ (Davis et al. 1971)

For $M = 10M_\odot$, $f \approx 1.2\,\text{kHz}$ and damping time $\approx 0.55\,\text{ms}$

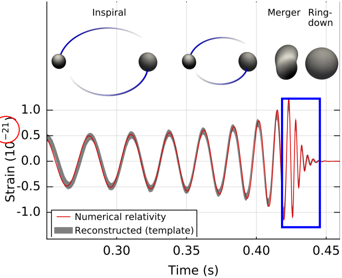
GW strain:

$$h_{\text{QNM}} \sim 5\times 10^{-22}\left(\frac{\Delta M}{10^{-3}M_\odot}\right)^{1/2}\left(\frac{1\,\text{kHz}}{f}\right)^{1/2}\frac{15\,\text{Mpc}}{D}$$

For GW150914,

$\Delta M = 3M_\odot, D = 400\,\text{Mpc}$

$h_{\text{QNM}} \lesssim 10^{-21}, \tau \approx 20\,\text{ms}$



[13] Will & Schutz Astrophys. J., **291**, L33–L36, (1985). See Berti, Cardoso, Starinets [0905.2975].

Quasinormal modes

Laplace transform the field: $\Psi(\omega, r) = \int_0^\infty \Psi(t,r) e^{i\omega t} dt$ $\qquad (s = -i\omega)$

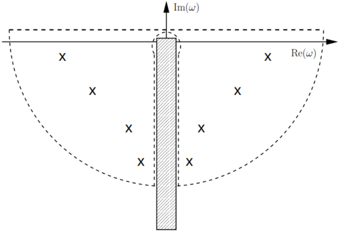
The master equation becomes

$$\frac{d^2\Psi}{dr_*^2} + (\omega^2 - V)\Psi = I(\omega, r)$$

with outer/inner **homog. sols.** $\Psi^{\pm}$ with BC

$$\lim_{r\to\infty} \Psi^- \sim A_{\text{in}}(\omega) e^{-i\omega r_*} + A_{\text{out}}(\omega) e^{i\omega r_*},$$

$$\lim_{r\to\infty} \Psi^+ \sim e^{i\omega r_*}$$



and Wronskian $W = 2i\omega A_{\text{in}}(\omega)$

Inhomogenous solution:

$$\Psi(\omega, r) = \Psi^+ \int_{-\infty}^{r_*} \frac{I(\omega,r)\Psi^-}{W} dr'_* + \Psi^- \int_{r_*}^{\infty} \frac{I(\omega,r)\Psi^+}{W} dr'_*$$

Inverse Laplace transform:

$$\Psi(t, r) = \frac{1}{2\pi} \oint_{-\infty+ic}^{\infty+ic} \Psi(\omega, r) e^{-i\omega t} d\omega$$

Poles, i.e., $A_{\text{in}}(\omega) = 0$ are the QNM frequencies.

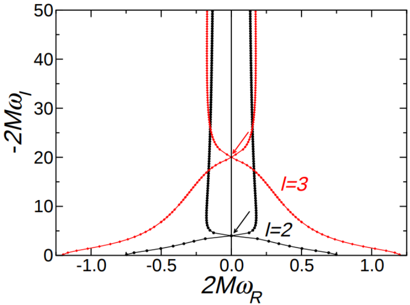
Quasinormal modes

Far away, the previous solution gives

$$\Psi_{\text{QNM}}(t,r) = -\text{Re}\left[C_n e^{-i\omega_n(t-r_*)}\right],$$

$$C_n = B_n \int_{\infty}^{\infty} \frac{I(\omega,r)\Psi^-}{A_{\text{out}}(\omega)}\,dr'_*,$$

$$B_n = \frac{A_{\text{out}}}{2\omega}\left(\frac{dA_{\text{in}}}{d\omega}\right)^{-1}\Bigg|_{\omega=\omega_n}$$



C_n QN excitation coefficients, B_n QN excitation factors

B_n depend only on the **background geometry**! $(\Psi^{\pm} \oplus V)$

Leaver 1985-86: method of continued fractions

Infinitely many ω_n for each ℓ

Monodromy for $|\omega| \gg 1$ case (Bender & Orszag)

$\text{Re}(\omega_n) \to$ constant as $n \to \infty$,

Algebraically special solutions: $\text{Re}(\omega_n) = 0$

Back to perturbation theory

Lorenz gauge

PT in Regge-Wheeler gauge: $\mathring{g} \to \mathring{g} + h \implies \dot{E}_{\infty,H}(h)$

Dissipative force: $F_\mu^{\text{diss}} = \dot{E}_H + \dot{E}_\infty$

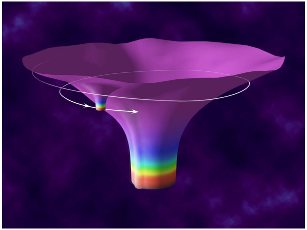
$\implies$ **adiabatic inspiral:** pushes m_1 off the geodesic orbit

Radiation-[back]reaction force $\implies$ gravitational self-force (GSF)

This force also has a conservative part

$\implies \mathcal{O}(q)$ **corrections** to:

- Redshift
- ISCO radius/frequency
- de Sitter (geodetic) precession
- Perihelion *retreat*



NEED to compute $h_{\mu\nu}$ **LOCALLY** at the particle, not at ∞.

Lorenz gauge is best suited: particle $\sim \delta^3(\mathbf{x}-\mathbf{z})$

(Recall RWZ source $\sim \delta^3(\mathbf{x}-\mathbf{z}) + \partial_i \delta^3(\mathbf{x}-\mathbf{z})$)

Isotropic singularity, rigorous regularization procedure (1990s to 2000s)

Perturbation theory in Lorenz gauge

Return to $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\mathring{g}_{\mu\nu}h$ and $\mathring{\nabla}_\mu \bar{h}^{\mu\nu} = 0$

Field equation

$$\Box\bar{h}_{\mu\nu} + 2\mathring{R}^{\alpha\ \beta}_{\ \mu\ \nu}\bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu},$$

$$T^{\mu\nu} = \frac{1}{\sqrt{-\mathring{g}}}\frac{m_1}{\mathring{u}^t}\mathring{u}^\mu\mathring{u}^\nu\delta^3(\mathbf{x}-\mathbf{z})$$

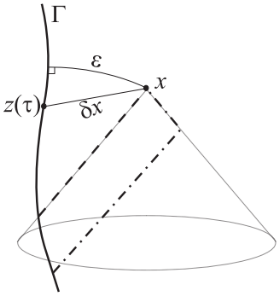
GSF: $F^\mu = m_1\mathring{\nabla}^{\mu\alpha\beta}\bar{h}^{\text{ret}}_{\alpha\beta}$ (ensures $F_\mu\mathring{u}^\mu = 0$)

Mino-Sasaki-Tanaka-Quinn-Wald (1996):

(i) $r \ll m_2$ (near zone), expand
(ii) $r \gg m_1$ (far zone), expand
(iii) $m_1 \ll r \ll m_2$ (buffer zone), match

$\bar{h}^{\text{ret}} = \bar{h}^{\text{dir}} + \bar{h}^{\text{tail}}$

$\bar{h}^{\text{dir}}_{\mu\nu} = \frac{4m_1\hat{u}_\mu\hat{u}_\nu}{\epsilon} + \mathcal{O}(\delta x^2)$



Detweiler-Whiting 2003

$\bar{h} = \bar{h}^R + \bar{h}^S$ ($\bar{h}^S_{\text{LO}} = \bar{h}^{\text{dir}}_{\text{LO}}$)

$$\boxed{F^\mu = m_1\mathring{\nabla}^{\mu\alpha\beta}\bar{h}^{\text{tail}}_{\alpha\beta} = m_1\mathring{\nabla}^{\mu\alpha\beta}\bar{h}^{R}_{\alpha\beta}}$$

Barack 0908.1664

Lorenz-gauge GSF

Mode-sum method[14]

10 field equations - 4 gauge $= 4 \oplus 2$ (even/odd parity) equations

"Spread" the δ-function singularity over an infinite ℓ-mode sum

$$\bar{h}^{S}_{\mu\nu} = \sum_{\ell=0}^{\infty} \bar{h}^{S,\ell}_{\mu\nu}, \qquad \bar{h}^{S,\ell}_{\mu\nu} \sim \mathcal{O}(\epsilon^{-1}) \text{ locally}$$

Regularization: subtract $\bar{h}^{S,\ell}_{\mu\nu}$ at each ℓ mode $\Longrightarrow \sum_{\ell=0}^{\infty}$ converges!

1. $\bar{h}^{\text{ret}}_{\mu\nu} = \frac{m_1}{r} \sum_{\ell m} \sum_{i=1}^{10} \bar{h}^{(i)\ell m}(t,r)\, Y^{(i)\ell m}_{\mu\nu}(\theta,\phi)$
2. Solve $\left[\partial^2_{uv} + V(r)\right] \bar{h}^{(i)\ell m} + \mathcal{M}^{(i)}_{(j)} \bar{h}^{(j)\ell m} = S^{(i)\ell m}\delta(r - r_0)$
3. Regularize: $F^{\mu\ell}_{\text{reg}} = \sum_{\ell m} F^{\mu,\ell m} - (A^\mu L + B^\mu + C^\mu L^{-1}) \sim \mathcal{O}(L^{-2})$
4. $F^\mu = \sum_\ell F^{\mu,\ell}_{\text{reg}}$ converges as $\mathcal{O}(L^{-2})$ (or better, $L = \ell + 1/2$)

NB: GSF is not gauge invariant! $\delta_\xi F^\mu \sim \frac{D\xi^\mu}{d\tau^2} + R(u,\xi,u)^\mu$

BUT it is **physical** (ISCO shift, perihelion retreat, EMRIs, etc.)

EMRIs are a very important source for LISA!

[14] Other approaches: moving punctures, Hadamard expansion of the retarded Green's function

Perturbation theory in **Kerr** spacetime

Teukolsky equation (1973-1974)[15]

Perturbation theory using the *Newman-Penrose* formalism

Based on **Weyl scalars** $\Psi_i \sim -C(e_1, e_2, e_3, e_4)$

e_a^μ is a null tetrad, Kinnersley tetrad: $e_a^\mu = \{\ell, n, m, \bar{m}\}^\mu$

In **Petrov Type D**, $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$
Perturb: $\Psi_i = \Psi_i + \delta\Psi_i$ then drop the δ
Linearly perturbed NP equations (nonvacuum: source $\sim m_1$):

$$R_{13[13|4]} + \text{Ricci} = 0, \quad R_{13[13|2]} + \text{Ricci} = 0$$ [16]

Ψ_4 carries the GW information
2nd order PDE for $\Psi_0 \Longrightarrow$ 2nd order PDE for Ψ_4 (null rotations)
$\Longrightarrow$ Separable, **2nd order PDE** in term of s: new Master equation!

$$\boxed{\hat{T}_s \psi_s = \mathcal{T}_s}$$

$$\psi_2 = \Psi_0, \psi_{-2} = \rho^{-4}\Psi_4$$

[15] See Teukolsky 2014 (1410.2130) Sec. 8 for a brief history. [16] See Chandrasekhar Sec. 1.8.

Teukolsky Equation

$$\boxed{\hat{T}_s\psi_s = \mathcal{T}_s}$$

$$\hat{T}_s \equiv \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2} - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial}{\partial t} + \frac{4Mar}{\Delta}\frac{\partial^2}{\partial t\partial\phi}$$
$$+ \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right) - \csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right]\frac{\partial}{\partial\phi}$$
$$+ \left[\frac{a^2}{\Delta} - \csc^2\theta\right]\frac{\partial^2}{\partial\phi^2} + (s^2\cot^2\theta - s),$$

$$\mathcal{T}_s = 4\pi(r^2 + a^2\cos^2\theta)T_s,$$

$$\Delta = r^2 - 2Mr + a^2, \quad \rho = (r - ia\cos\theta)^{-1},$$

$$T_{-2} = 2\rho^{-4}\left[\begin{array}{l}(\mathbb{\Delta} + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})\left\{(\bar{\delta} - 2\bar{\tau} + 2\alpha)T_{24} - (\mathbb{\Delta} + 2\gamma - 2\bar{\gamma} + \bar{\mu})T_{44}\right\} \\ (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)\left\{(\mathbb{\Delta} + 2\gamma + 2\bar{\mu})T_{24} - (\bar{\delta} - \bar{\tau} + 2\bar{\beta} + 2\alpha)T_{22}\right\}\end{array}\right]$$

$$\mathbb{\Delta} \equiv \left[\frac{r^2+a^2}{\Delta}\partial_t, \partial_r, 0, \frac{a}{\Delta}\partial_\phi\right]^T, \quad \delta \equiv \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left[ia\sin\theta\,\partial_\theta, 0, \partial_\theta, i\csc\theta\,\partial_\phi\right]^T$$

NB: for $T^{\mu\nu} \sim \delta^3(\mathbf{x} - \mathbf{z})$, $\implies T_s \sim \delta(r - r_0) + \delta'(r - r_0) + \delta''(r - r_0)$

$s = 0, \pm 1, \pm 2, \frac{1}{2}$

Solving the Teukolsky equation

Flux computations since Teukolsky (frequency domain)

Time domain: 1+1D (G. Khanna, A. Zenginoglu, S. Hughes)
2+1D $\oplus$ MPD (E. Harms, Bernuzzi et al. [1510.05548])

Frequency domain

$$\psi_s = {}_sR(r)\,{}_sS(\theta)e^{i(m\phi-\omega t)}$$

Teukolsky equation **separates**!

$$\left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right)+\frac{K^2-2is(r-M)K}{\Delta}+4iswr-\lambda\right]{}_sR(r)=-4\pi\mathcal{T}_{s\ell m\omega},$$

$K=(r^2+a^2)\omega-ma$

${}_sS(\theta)e^{im\phi}$ are the spin-weighted spheroidal harmonics

λ is the eigenvalue of the angular equation

The radial equation can be solved

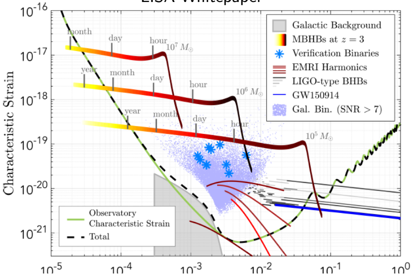
- analytically: small ω expansions of hypergeometric and Coulomb functions
- numerically as a Sasaki-Nakamura equation
- numerically directly

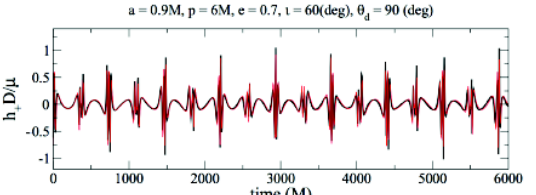
Further Motivation

Why bother with Lorenz-gauge GSF, Teukolsky equation?

$\Longrightarrow$ Extreme Mass Ratio Inspirals

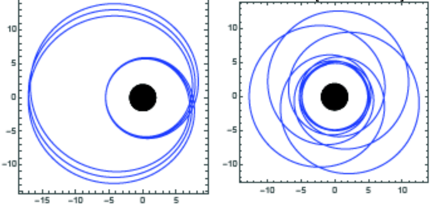
LISA Whitepaper

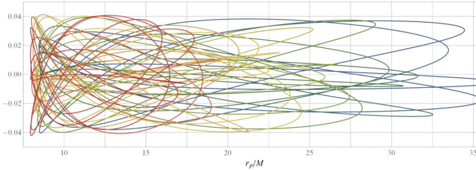




Babak et al. (2006)

Osburn et al. [1511.01498]



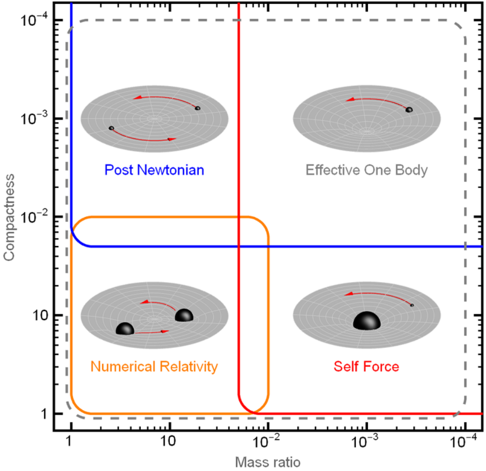


Warburton et al. [1708.03720]

Effective One Body Theory EOB

Buonanno-Damour 1998-2000s

Key idea: Map PN binary motion to geodesic motion in an *effective spacetime* using ν as a deformation parameter.



EOB dynamics

Newtonian **2-body** problem:

$$H_N = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_1^2}{2m_1} - \frac{Gm_1m_2}{r} = \frac{\mathbf{p}^2}{2\mu} - \frac{G\mu M}{r} + \frac{\mathbf{P}_{\mathsf{CM}}^2}{2M}$$

Augment to PN, CM frame (relative motion, $\mathbf{P}_{\mathsf{CM}} = 0$)

$$H_{PN} = H_N + \frac{1}{c^2}H_{1PN} + \frac{1}{c^4}H_{2PN} + \dots$$

Effective metric

$$g_{\mathsf{eff}} = \mathsf{diag}[-A(r), \frac{D(r)}{A(r)}, S^2]$$

EOB **dynamics**: Hamiltonian theory

$$H_{\mathsf{EOB}} = \mu\hat{H}_{\mathsf{EOB}} = \frac{\mu}{\nu}\sqrt{1 + 2\nu(\hat{H}_{\mathsf{eff}} - 1)},$$

$$\hat{H}_{\mathsf{eff}} = \sqrt{p_{r_*}^2 + A\left(1 + \frac{p_\phi^2}{r^2} + z_3\frac{p_{r_*}^4}{r^2}\right)} + \hat{H}_{\mathsf{spin}},$$

$$A = 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + \mathcal{O}(u^5),$$

$$D = \left[1 + 6\nu u^2 + 2\nu(26 - 3\nu)u^3\right]^{-1} + \mathcal{O}(u^4) \qquad [u \equiv \tfrac{GM}{c^2 r}]$$

EOB EoM

Work in Damour-Jaranowski-Schäfer gauge
$\Longrightarrow$ Simplified Hamilton's equations

$$\frac{dr}{dt} \sim \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_{r_*}}, \qquad \frac{dp_{r_*}}{dt} \sim -\frac{\partial \hat{H}_{\text{EOB}}}{\partial r},$$
$$\frac{d\phi}{dt} = \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_\phi} \equiv \Omega,$$
$$\frac{dp_\phi}{dt} = \mathcal{F}_\phi \qquad \text{[RR force]}$$

NB: $\mathcal{F}_r = 0$ for convenience (not zero)[17]
$\mathcal{F}_\phi \Rightarrow$ inspiral (special factorization and resummation, [Damour-Nagar 2007])

$$\mathcal{F}_\phi \sim \sum_{\ell m} |h_{\ell m}|^2,$$
$$h_{\ell m} = \hat{h}_{\ell m}^{\text{Newt}} \hat{S}_{\text{eff}} \hat{h}_{\ell m}^{\text{tail}} f_{\ell m} \hat{h}_{\ell m}^{\text{NQC}}$$

[17] $\mathcal{F}_r$ has been derived by Bini-Damour [1210.2834], but no resumming strategy exists for it (Damour-Nagar [1406.6913]).

Our EOB: TEOBResumS

Nagar-Bernuzzi et al.

Time-domain effective-one-body gravitational waveforms for coalescing compact binaries with nonprecessing spins, tides and self-spin effects

Alessandro Nagar[1,2,3], Sebastiano Bernuzzi[4,5,6], Walter Del Pozzo[7], Gunnar Riemenschneider[2,8], Sarp Akcay[4], Gregorio Carullo[7], Philipp Fleig[9], Stanislav Babak[10], Ka Wa Tsang[12], Marta Colleoni[13], Francesco Messina[14,15], Geraint Pratten[13], David Radice[16,17], Piero Rettegno[2,8], Michalis Agathos[18], Edward Fauchon-Jones[19], Mark Hannam[19], Sascha Husa[13], Tim Dietrich[12,20], Pablo Cerdá-Duran[21], José A. Font[21,22], Francesco Pannarale[19,23], Patricia Schmidt[24], and Thibault Damour[3]

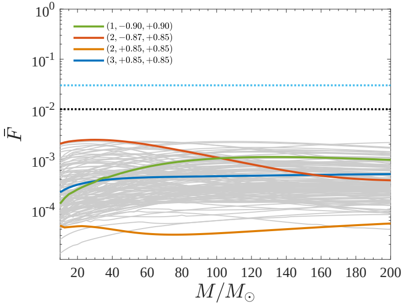
Circular, **spin** [anti]aligned inspirals with tides enhanced by NR simulations.

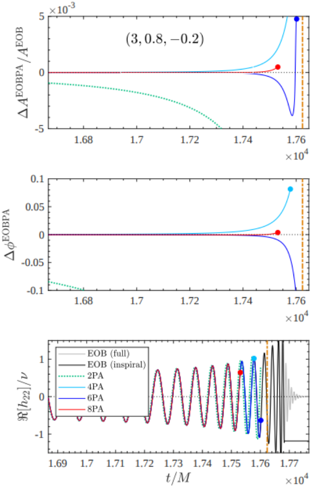
Ingredients:

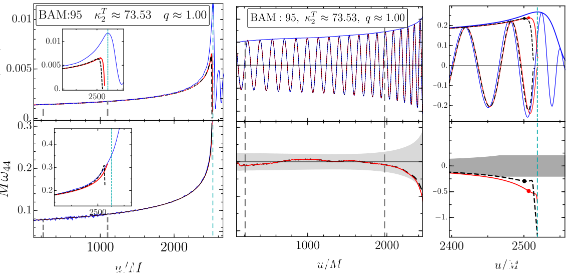
- Point-mass inspiral: (1,5)-Padé-resummed $A(u)$.
- Spin-orbit, spin-spin in the dynamics and flux (low multipoles).
- Tides LR-pole factorized GSF series at $\mathcal{O}(q) \oplus \mathcal{O}(q^2)$GSF-PN hybrid.
- Tides use **quasi-universal** fits of Yagi et al.
- Monopole-quadrupole[18] coupling upto NLO in dynamics $\oplus$ flux.
- Plunge and ringdown smoothly attached to the inspiral (phenomenological).
- "Unfaithfulness" to BBH $\lesssim 10^{-3}$, to BNS $< 10^{-2}$.
- FAST! post-adiabatic: 10 Hz inspiral in $\approx$ **0.5** sec! AN-Rettegno [1805.03891].

[18] Poisson 1997

Advertisement: TEOBResumS







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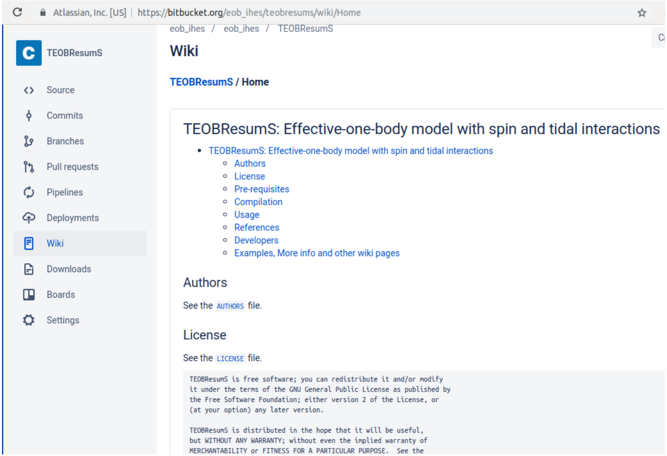


https://bitbucket.org/eob_ihes/teobresums/wiki/Home

Currently being evaluated by LVC for LAL.

Synergies

Cross-cultural comparisons of "gauge-invariant" quantities

E.g., $\mathcal{O}(q)$ correction to ISCO radius.

$$\mathring{r}_{\text{ISCO}} = 6m_2 \to \mathring{r}_{\text{ISCO}} + q\Delta\hat{r}_{\text{ISCO}}$$

Coordinates x^μ are gauge-dependent!

Frequency is gauge invariant

$$\Omega_{\text{ISCO}} \to \Omega_{\text{ISCO}}\left(1 + q\Delta\hat{\Omega}_{\text{ISCO}}\right) \equiv \Omega_{\text{ISCO}} + qC_\Omega$$

$C_\Omega = 1.2512(4)$ Barack-Sago [1002.2386] (from GSF)
$= 1.2510(2)$ Le Tiec et al. [1111.5609] (PN $\odot$ (2,3)Padé)
$= 1.25101546(5)$ Akcay et al. [1209.0964] (from the redshift)

NOTE: BS result depends on GSF, but yields a gauge invariant quantity.

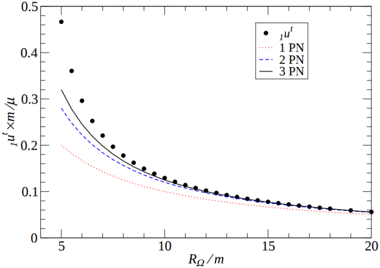
Synergies

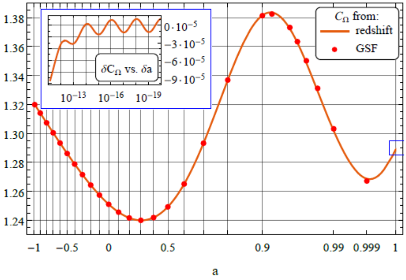
Detweiler redshift

Detweiler [0804.3529]: PN - RW agreement
redshift of a photon leaving m_1

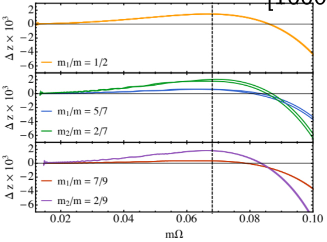
Sago-Barack-Detweiler [0810.2530]
Lorenz vs. RW gauge: $\Delta_{\text{rel}} \lesssim 10^{-5}$

Current technology: extremal Kerr





NR-PN comparison Zimmerman et al. [1606.08056]



Synergies

Perihelion retreat

$\mathcal{O}(q)$ correction to Einstein perihelion shift (negative!)

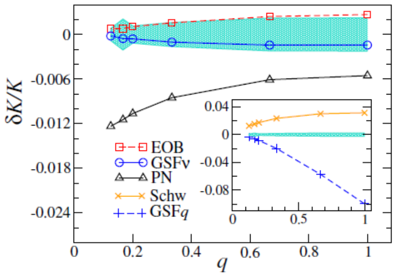
Barack-Sago [1101.3331]

p_0	e_0	$q^{-1}\Delta\delta$	$q^{-1}\Delta\delta/\delta_0$
6.1	0.02	−146(2)	−20.7(2)
6.2	0.05	−57.0(2)	−11.71(5)
6.3	0.1	−41.9(1)	−10.23(3)
6.4	0.1	−19.71(5)	−6.12(2)

Reformulated into the invariant $K = \frac{\Omega_\phi}{\Omega_r}$ (BS-Damour [1008.0935])

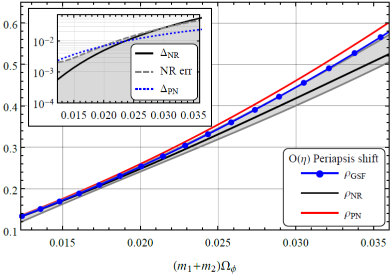
GSF-NR-PN-EOB comparison

Le Tiec et al. [1106.3278]



GSF-NR-PN synergy in Kerr

van de Meent [1610.03497]

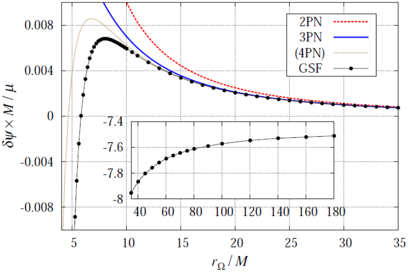


Synergies

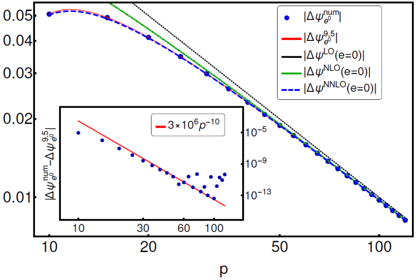
de Sitter precession

$\mathcal{O}(q)$ correction to geodetic precession

Dolan et al. [1312.0775]:
circular orbits in Schwarchild



Akcay et al. [1608.04811]:
eccentric orbits



The end

We have come a long way!

- 103.4 years of general relativity
- Two different analytical approaches to the two-body problem
- Both feed into EOB (so does numerical relativity)
- Massive challenges overcome in the last 100 years