Gauge (in)dependence of Essential Quantum Einstein Gravity

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Essentials of the Essential Renormalisation Group

Consider the diffeomorphism invariant action in the derivative expansion

$$\Gamma_k[g] = \int d^4x \sqrt{\det g} \left\{ \frac{\rho_k}{8\pi} - \frac{1}{16\pi G_k} R + a_k R^2 + b_k R_{\mu\nu} R^{\mu\nu} + c_k E + O(\partial^6) \right\} .$$

where $\rho_k = \frac{\Lambda_k}{G_k}$ and $E = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$.

[Baldazzi & Falls '21]

One can identify equivalence classes of the theory space with each member of a class describing the same physics.

Two equivalence classes are of interest:

- Einstein Gravity
- Higher Derivative (Quadratic) Gravity

We can use the field redefinitions to fix the flow of inessential couplings. In Essential RG we fix their value in order to simplify calculations.

Essentials of the Essential Renormalisation Group

$$\Gamma_k[g] = \int d^4x \sqrt{\det g} \left\{ \frac{\rho_k}{8\pi} - \frac{1}{16\pi G_k} R + a_k R^2 + b_k R_{\mu\nu} R^{\mu\nu} + c_k E + O(\partial^6) \right\} .$$

We choose the RG kernel for the metric to be given by

$$\Psi^g_{\mu\nu}[g] = \gamma_g g_{\mu\nu} + \gamma_R R g_{\mu\nu} + \gamma_{Ricci} R_{\mu\nu} + O(\partial^4) ,$$

while $\Psi^{c\mu} = 0 = \Psi^{\bar{c}}_{\mu}$. Here the $\gamma's$ are functions which, along with the beta functions, will be determined as functions of the couplings that appear in $\Gamma_k[g]$.

Each gamma function allows us to impose a renormalisation condition which fixes the flow of an inessential coupling. They allow us to impose three renormalisation conditions which are constraints on the form of $\Gamma_k[g]$ that we impose along the RG flow.

Essential FRG equation

$$\left(\partial_t + \Psi_k^g \cdot \frac{\delta}{\delta g}\right)\Gamma_k = \frac{1}{2}\mathrm{Tr}\,\mathcal{G}_k^{gg}\left(\partial_t + 2\cdot\frac{\delta}{\delta g}\Psi_k^g\right)\cdot\mathcal{R}_k^{gg} - \mathrm{Tr}\,\mathcal{G}_k^{\bar{c}c}\cdot\partial_t\mathcal{R}_k^{\bar{c}c}\,,$$

$t = log(k/k_0)$ [Pawlowski '07, Baldazzi & Falls '21]

$$\mathcal{G}_k^{gg} := \frac{1}{\frac{\delta^2\Gamma_k}{\delta g\delta g} + K_{gg}\cdot\Delta_{\mathrm{gf}} + \mathcal{R}_k^{gg}}\,,$$

$$\mathcal{G}_k^{\bar{c}c} := \frac{1}{K_{\bar{c}c}\cdot\Delta_{\mathrm{gh}} + \mathcal{R}_k^{\bar{c}c}}\,,$$

$$\mathcal{R}_k^{gg}[g] = K_{gg}R_k(\Delta)\,, \qquad \mathcal{R}_k^{\bar{c}c}[g] = K_{c\bar{c}}R_k(\Delta)\,, \qquad \Delta = -g^{\mu\nu}\nabla_\mu\nabla_\nu.$$

$$K_{gg}^{\mu\nu,\alpha\beta} := \frac{1}{\kappa_k^2}\sqrt{\det g}\left(\mathbb{1}^{\mu\nu,\alpha\beta} - 2g^{\mu\nu}g^{\alpha\beta}\right)\,, \qquad K_{\bar{c}c}^{\mu\nu} := \frac{\sqrt{2}}{\kappa_k}\sqrt{\det g}\; g_{\mu\nu}.$$

Minimal Essential Scheme

$$\Gamma_k[g] = \int d^4x \sqrt{\det g} \left\{ \frac{\rho_k}{8\pi} - \frac{1}{16\pi G_k} R + c_k E + O(\partial^6) \right\}$$

Minimal Essential scheme consists of identifying the inessential couplings at the Gaußian fixed point: $G_k = a_k = b_k = 0$ and ρ_k is determined from $\partial_t \rho = 0$. Their values are fixed in order to simplify the computations.

Additional advantage of Minimal Essential scheme when applied to gravity is that it maintains the simple form of the propagator and does not introduce any additional poles usually associated with a breakdown of unitarity.

Reuter fixed point has only one relevant direction.

Studies with matter
B. Knorr '22: Safe essential scalar-tensor theories.

Gauge-fixing contribution

the background covariant gauge fixing and ghost terms. In particular, we shall take

$$\hat{\Gamma}_k[g,c,\bar{c};\bar{g}] = \frac{1}{2\alpha}\int d^4x\sqrt{g}F_\mu F^\mu + \frac{1}{2}\int d^4x\sqrt{g}\,\bar{c}_\mu \mathcal{Q}^\mu{}_\nu c^\nu \, ,$$

where, to simplify calculations, we adopt background covariant harmonic gauge

$$F^\mu = \frac{1}{\sqrt{16\pi G_k}}\left(\bar{g}^{\mu\lambda}\bar{g}^{\nu\rho} - \frac{1+\beta}{4}\bar{g}^{\nu\mu}\bar{g}^{\rho\lambda}\right)\bar{\nabla}_\nu g_{\lambda\rho} \, ,$$

The ghosts operator is then given by

$$\mathcal{Q}^\mu{}_\nu c^\nu \equiv \mathcal{L}_c F^\mu = \frac{\sqrt{2}}{\kappa_k}\left(\bar{g}^{\mu\lambda}\bar{g}^{\nu\rho} - \frac{1}{2}\bar{g}^{\nu\mu}\bar{g}^{\rho\lambda}\right)\bar{\nabla}_\nu\left(g_{\rho\sigma}\nabla_\lambda c^\sigma + g_{\lambda\sigma}\nabla_\rho c^\sigma\right) .$$

Schwinger-DeWitt (Heat Kernel) technique

[Schwinger '51, DeWitt '65]

A formal way to treat functional traces and determinants of local pseudo-differential operators (including but not necessarily Laplace-type). We define

$$H(s) = e^{-s\Delta}\,.$$

for $\Delta = -\Box + E$. It gives, for example, definitions of the propagator and 1-loop effective action as

$$\frac{1}{\Delta} = \int_0^\infty ds\; e^{-s\Delta} \qquad\qquad \Gamma_{1-\mathrm{loop}} = \frac{1}{2}\mathrm{Tr}\log\Delta = \frac{1}{2}\int_0^\infty \frac{ds}{s}\mathrm{Tr}\,e^{-s\Delta}$$

$$\mathrm{Tr}\; H(s) = \frac{1}{(4\pi s)^{d/2}}\sum_{n\geq 0}\int d^d x\sqrt{g}\, s^n\,\mathrm{tr}\,\overline{a_n}\,,$$

$\overline{a_n}(x)$ are local functions of the curvature invariants and their covariant derivatives.

What happens in the nonminimal case?

$$\Delta_{\mu\nu} = -g_{\mu\nu}\Box + \left(\frac{1}{\alpha} - 1\right)\nabla_\mu\nabla_\nu .$$

Generalised Schwinger-DeWitt (Off-diagonal Heat Kernel) technique

$F\ G = 1.$ [Barvinsky, Vilkovisky '85]

$F_0\ G_0 = 1, \qquad \nabla_\mu \to n_\mu, \quad R \to 0.$

$F\ G_0|_{n_\mu \to \nabla_\mu} = 1 + M(\nabla, R, R^2),$

$$G = G_0 \frac{1}{1+M} = G_0 \left[1 - M + M^2 - \dots\right]$$

sort derivatives: commute all contracted derivatives to the right (to form $\Box$'s)

$$[X, f(\Box)] = \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n-1} \left[\, X\,,\, \Box \,\right]_n f^{(n)}(\Box)$$

A digression: 1-loop.

$$F(\lambda) = F_{min} + \lambda N$$

$$\Gamma_{1-loop} = \frac{1}{2} Tr\ log\ F$$

$$\Gamma_{1-loop}(\lambda) = \Gamma(\lambda = 0) + \frac{1}{2}\int_0^{\lambda} d\lambda\ Tr\ F^{-1}(\lambda)\cdot\frac{dF}{d\lambda},$$

$$\Gamma_{1-loop}(\lambda = 1) = \frac{1}{2} Tr\ log\ F_{min} + \frac{1}{2}\int_0^1 d\lambda\ Tr\ G(\lambda)\cdot N.$$

[Barvinsky, Vilkovisky '85]

Taking functional traces

$$\mathrm{Tr}[\nabla_{\mu_1}\ldots\nabla_{\mu_N} f(\Delta)] = \int d^dx\sqrt{g}\ \langle x|\,\nabla_{\mu_1}\ldots\nabla_{\mu_N} e^{-s\Delta}\,|x\rangle\,\tilde{f}(s)\,,$$

$$f(\Delta) = \int_0^\infty ds\ \ e^{-s\Delta}\tilde{f}(s)\,.$$

$$\mathrm{Tr}[\nabla_{\mu_1}\ldots\nabla_{\mu_N} e^{-s\Delta}] = \int d^dx\sqrt{g}\ \langle x|\,\nabla_{\mu_1}\ldots\nabla_{\mu_N} e^{-s\Delta}\,|x\rangle = \int d^dx\sqrt{g}\ H_{\mu_1\ldots\mu_N}(x,s)\,,$$

$$H_{\mu_1\ldots\mu_N}(x,s) = \frac{1}{(4\pi s)^{d/2}}\sum_{n\geq 0} s^{n-\lfloor N/2\rfloor} K^{(n)}_{\mu_1\ldots\mu_N}(x)\,,$$

$$Q_n[f] := \int_0^\infty ds\ s^{-n}\tilde{f}(s)\,,$$

$$\mathrm{Tr}[\nabla_{\mu_1}\ldots\nabla_{\mu_N} f(\Delta)] = \frac{1}{(4\pi)^{d/2}}\sum_{n\geq 0} Q_{-n+\frac{d}{2}+\lfloor N/2\rfloor}[f]\ \cdot tr\int d^dx\ \sqrt{g}K^{(n)}_{\mu_1\ldots\mu_N}(x).$$

$$\mathrm{Tr}[\nabla_{\mu_1}\dots\nabla_{\mu_n} f(\Delta)] = \int d^dx\sqrt{g}\int ds\,\tilde{f}(s) H_{\mu_1\dots\mu_N}(x,s) =$$

$$= \frac{1}{(4\pi)^{d/2}}\sum_{n\geq 0} Q_{-n+\frac{d}{2}+\lfloor N/2\rfloor}[f]\;\cdot tr\int d^dx\,\sqrt{g} K^{(n)}_{\mu_1\dots\mu_N}(x)$$

$$H(x,s) = (4\pi s)^{-d/2}\sum_{n\geq 0} s^n \overline{a_n}$$

$$H_\mu(x,s) = (4\pi s)^{-d/2}\sum_{n\geq 0} s^n \overline{\nabla_\mu a_n}$$

$$H_{\mu\nu}(x,s) = (4\pi s)^{-d/2}\sum_{n\geq 0} s^{n-1}\left(-\frac{1}{2} g_{\mu\nu}\overline{a_n} + \overline{\nabla_{(\mu}\nabla_{\nu)} a_{n-1}}\right)$$

$$H_{\mu\nu\rho}(x,s) = (4\pi s)^{-d/2}\sum_{n\geq 0} s^{n-1}\left(-\frac{3}{2} g_{(\rho\nu}\overline{\nabla_{\mu)} a_n} + \overline{\nabla_{(\rho}\nabla_\nu\nabla_{\mu)} a_{n-1}}\right)$$

$$\dots$$

Computation is long, but the method is very general

$$\overline{a_0} = 1\,,$$

$$\overline{a_1} = -E + \frac{1}{6}R\,,$$

$$\Delta = -\Box + E$$

$$\overline{a_2} = -\frac{1}{6}\Box E + \frac{1}{2}E^2 - \frac{1}{6}RE + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{30}\Box R + \frac{1}{72}R^2 - \frac{1}{180}R_{\mu\nu}^{\ 2} + \frac{1}{180}R_{\mu\nu\alpha\beta}^{\ 2}\,,$$

$$\Omega_{\mu\nu}h = [\nabla_\mu, \nabla_\nu]h$$

$$\overline{\nabla_{(\nu}\nabla_{\mu)}a_0} = \frac{1}{6}R_{\nu\mu}\,,$$

$$\overline{\nabla_{(\alpha}\nabla_{\nu}\nabla_{\mu)}a_0} = \frac{1}{4}R_{(\nu\mu;\alpha)}\,,$$

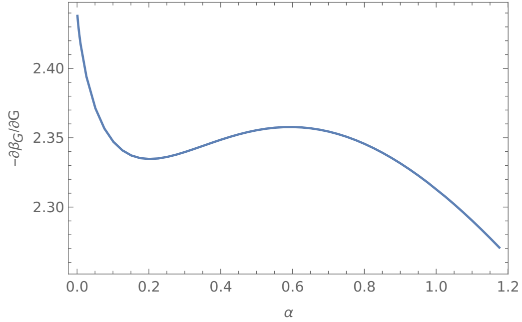
$$\overline{\nabla_{(\beta}\nabla_{\alpha}\nabla_{\nu}\nabla_{\mu)}a_0} = \frac{3}{10}R_{(\nu\mu;\alpha\beta)} + \frac{1}{12}R_{(\beta\alpha}R_{\nu\mu)} + \frac{1}{15}R_{\gamma(\beta|\delta|\alpha}R^{\gamma}{}_{\nu}{}^{\delta}{}_{\mu)}\,,$$

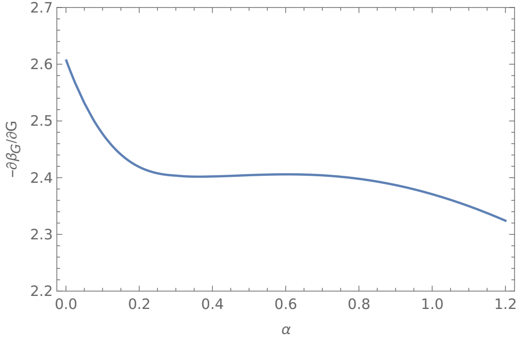
$$\overline{\nabla_\mu a_1} = -\frac{1}{2}E_{;\mu} - \frac{1}{6}\Omega_{\nu\mu;}{}^{\nu} + \frac{1}{12}R_{;\mu}\,,$$

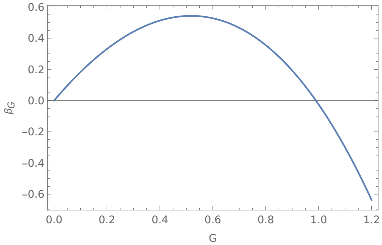
$$\overline{\nabla_{(\nu}\nabla_{\mu)}a_1} = -\frac{1}{3}E_{;(\mu\nu)} - \frac{1}{6}R_{\mu\nu}E - \frac{1}{6}\Omega_{\alpha(\mu;}{}^{\alpha}{}_{\nu)} + \frac{1}{6}\Omega_{\alpha(\nu}\Omega^{\alpha}{}_{\mu)} + \frac{1}{20}R_{;(\mu\nu)} + \frac{1}{60}\Box R_{\nu\mu} + \frac{1}{36}RR_{\nu\mu} - \frac{1}{45}R_{\nu\alpha}R^{\alpha}{}_{\mu} + \frac{1}{90}R_{\alpha\beta}R^{\alpha}{}_{\nu}{}^{\beta}{}_{\mu} + \frac{1}{90}R^{\alpha\beta\gamma}{}_{\nu}R_{\alpha\beta\gamma\mu}.$$

[Groh, Saueressig, Zanusso '11]

Results







Extrema:

$\alpha = 0.367, \quad -\partial\beta_G/\partial G = 2.40194,$

$\alpha = 0.602, \quad -\partial\beta_G/\partial G = 2.40587.$

Conclusions

- Essential Quantum Einstein Gravity is a scheme in which quantum corrections do not spoil unitarity and the simple form of the propagator is maintained.
- We identified a parameter space where gauge dependence of the critical exponent is small, which can be seen as a justification of our approximations.
- It is of interest to consider different regulators such as type II cutoff to try and cure the singularity at $\alpha \to 0$.