FUNCTIONAL RENORMALIZATION FOR MATRIX MODELS

STRUCTURES CLUSTER OF EXCELLENCE

UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

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Carlos I. Pérez-Sánchez perez@thphys.uni-heidelberg.de

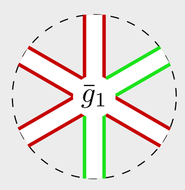
Institute for Theoretical Physics, University of Heidelberg

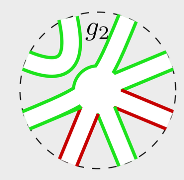
Abstract

We report a new functional renormalization framework for an ample variety of matrix models (MM)

$$\mathcal{Z} = \int_{M_N(\mathbb{C})^n_{\mathrm{s.a}}} \mathrm{d}\mu_{\mathrm{GAUSS}}(\mathbb{X}) \exp[-S^{\mathrm{INT}}(\mathbb{X})] \qquad \text{with} \qquad \mathrm{d}\mu_{\mathrm{GAUSS}}(\mathbb{X}) = C_N \prod_{j=1}^n \mathrm{e}^{-N\,\mathrm{Tr}(X_i^2/2)} (\mathrm{d}X_i)_{\mathrm{LEBESGUE}} \qquad \mathrm{Tr}(1) = N$$

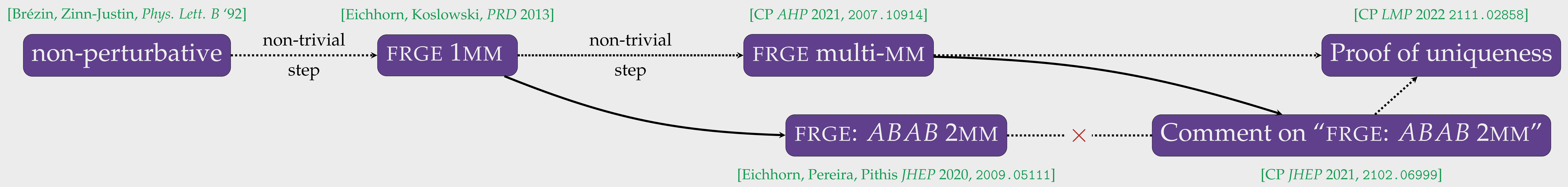
namely, we allow several random Hermitian matrices $\mathbb{X} = (X_1, \ldots, X_n)$ to interact via $S^{\mathrm{INT}}(\mathbb{X}) = \mathrm{Tr}(V_1) \times \cdots \times \mathrm{Tr}(V_k)$, for certain noncommutative polynomials $V_1, \ldots, V_k \in \mathbb{C}_{\langle n \rangle}$ in the n matrices. For $n = 2$, an example of operators is:

$g_1\,\mathrm{Tr}(ABBBAB)$ $\leftrightarrow$ 

$g_2\,\mathrm{Tr}^{\otimes 2}(AABABA \otimes AA)$ $\leftrightarrow$  (cylinder with labelled boundary).

We addressed the Wetterich-Morris equation both in top-down and bottom-up directions (as a formal series; for the outlook: put bounds).

Matrix models and renormalization



Bottom-up (finding the algebraic structure)

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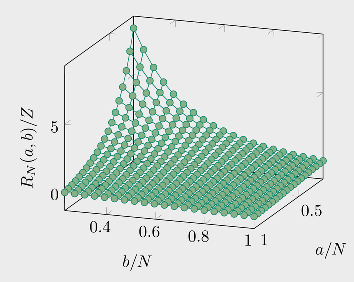
- the proof of Wetterich-Morris equation (FRGE),

$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2}\mathrm{STr}\left(\frac{\partial_t R_N}{\mathrm{Hess}\,\Gamma_N[\mathbb{X}] + R_N}\right),$$



determines the algebra that governs the geometric (Neumann) series in the Hessian of Γ_N [Benedetti, Groh, Machado, Saueressig, *JHEP* 2011] appearing in the RHS

- [Eichhorn, Koslowski, *Phys. Rev. D* '13] oriented us to find some coupling scalings and we adopted their RG-time parameter $t = \log N$, but the proof of the FRGE dictates a different algebra and we follow that, not the algebra in *op. cit.*
- β-equations found for a sextic truncation (48 operators) of 2MM's. For the unique real fixed point g^* leading to a single relevant direction (a single positive e.v. of $-(\partial\beta_i/\partial g_j)_{i,j}|_{g^*}$) yields a ($R_N$-dependent but accurate) value

$$g^*_{A^4} = 1.002 \times \left(g^*_{A^4}|_{\text{[Kazakov-Zinn-Justin, Nucl. Phys. B '99]}}\right)$$

Top-down (uniqueness of the algebraic structure)

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- write the FRGE and solve for the algebra $(\mathcal{A}_{n,N}, \star)$, where $\mathrm{STr} = \mathrm{Tr}_{M_n(\mathcal{A}_{n,N})}$
- assume an expansion of its rhs in unitary-invariant operators
- impose the one-loop structure and solve for the algebra $\mathcal{A}_{n,N}$ (see left column)
- determine from it the algebra that computes Wetterich equation; it is unique and the one reported in [CP 2007.10914], i.e. $\mathcal{A}_{n,N} = (\mathbb{C}^{(N)}_{\langle n \rangle} \otimes \mathbb{C}^{(N)}_{\langle n \rangle}) \oplus (\mathbb{C}^{(N)}_{\langle n \rangle} \boxtimes \mathbb{C}^{(N)}_{\langle n \rangle})$ (for vector spaces $\boxtimes$ is just the tensor product, but in the presence of a product on $\mathcal{A}_{n,N}$, these signs are different) whose product, for P, Q, U, W words on the matrices, reads:

$$(U \otimes W) \star (P \otimes Q) = PU \otimes WQ \qquad (1a)$$
$$(U \boxtimes W) \star (P \otimes Q) = U \boxtimes PWQ \qquad (1b)$$
$$(U \otimes W) \star (P \boxtimes Q) = WPU \boxtimes Q \qquad (1c)$$
$$(U \boxtimes W) \star (P \boxtimes Q) = \mathrm{Tr}(WP) U \boxtimes Q \qquad (1d)$$

- the supertrace is $\mathrm{Tr}_{M_n(\mathcal{A}_{n,N})} = \mathrm{Tr}_n \otimes \mathrm{Tr}_{\mathcal{A}_{n,N}}$ where

$$\mathrm{Tr}_{\mathcal{A}_{n,N}}(P \otimes Q) = \mathrm{Tr}\,P \cdot \mathrm{Tr}\,Q \qquad \text{and} \qquad \mathrm{Tr}_{\mathcal{A}_{n,N}}(P \boxtimes Q) = \mathrm{Tr}(PQ)$$

What is new in this framework?

- New is the algebra (1) above $\nearrow$ without which the β-functions cannot be correctly computed. We now interpret this algebra. Let $\mathbb{X} = (X_1, \ldots, X_n) \in M_N(\mathbb{C})^n_{\mathrm{s.a}}$ and $\mathbb{C}^{(N)}_{\langle n \rangle} = \mathbb{C}\langle \mathbb{X} \rangle$ or «words». The *noncommutative derivative* $\partial_X : \mathbb{C}_{\langle n \rangle} \to \mathbb{C}^{\otimes 2}_{\langle n \rangle}$ sums over replacements of X in a word by $\otimes$, except at the ends of the word, where one adds 1:

$$\partial_A(PAAR) = P \otimes AR + PA \otimes R \qquad \text{but} \qquad \partial_A(ALGEBRA) = 1 \otimes LGEBRA + ALGEBR \otimes 1\,.$$

Also ∂_A on traces yields the *cyclic derivative*: $\partial_A \mathrm{Tr}(PAAR) = ARP + RPA$, for instance. The *noncommutative-Hessian* is the matrix with entries $\mathrm{Hess}_{a,b}\,\mathrm{Tr}\,W = \partial_{X_a}\partial_{X_b}\mathrm{Tr}\,W$. Then $[\Gamma^{(2)}_N]^{\star k} = [\mathrm{Hess}\,\Gamma_N]^{\star k}$ in the Neumann expansion is computed using the algebra (1)

- EXAMPLE. $\mathrm{Hess}\{\mathrm{Tr}(ABAB)\}$:

$$\begin{pmatrix} \partial^A \circ \partial^A & \partial^A \circ \partial^B \\ \partial^B \circ \partial^A & \partial^B \circ \partial^B \end{pmatrix} \mathrm{Tr}(\overbrace{ABAB}) = 2\begin{pmatrix} \overbrace{B \otimes B} & \overbrace{AB \otimes 1 + 1 \otimes BA} \\ \underbrace{BA \otimes 1 + 1 \otimes AB} & \underbrace{A \otimes A} \end{pmatrix}$$

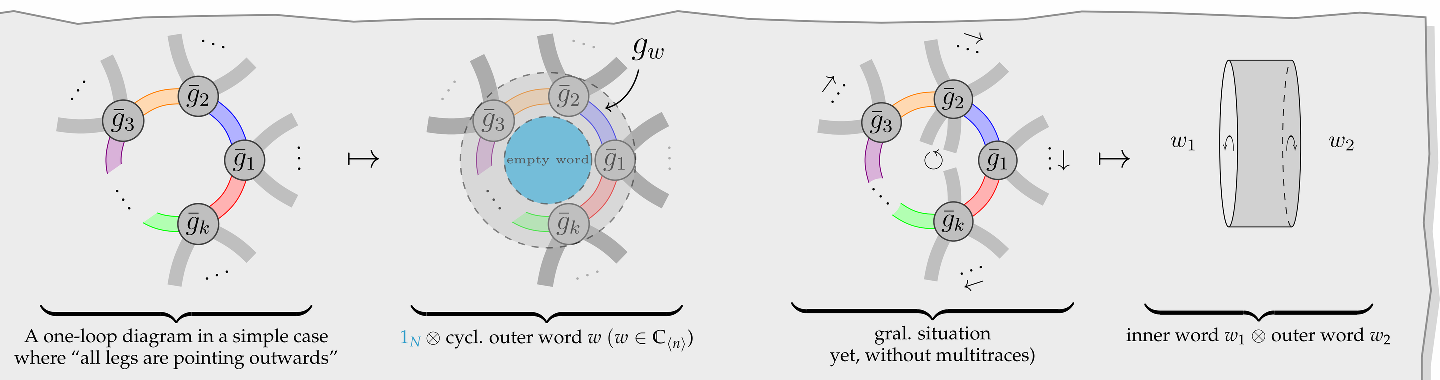

Fig. 1 How the one-loop structure of the FRGE is encoded in $M_n, (\mathcal{A}_{n,N}, \star)$.
Left map: Unrenormalized interactions $\bar{g}_i$ appearing in a k-th power of the Hessian (simplified).
Right map: The contribution to the $\beta_{w_1|w_2}$-function, w_1, w_2 formed by reading off the legs with the arrows.

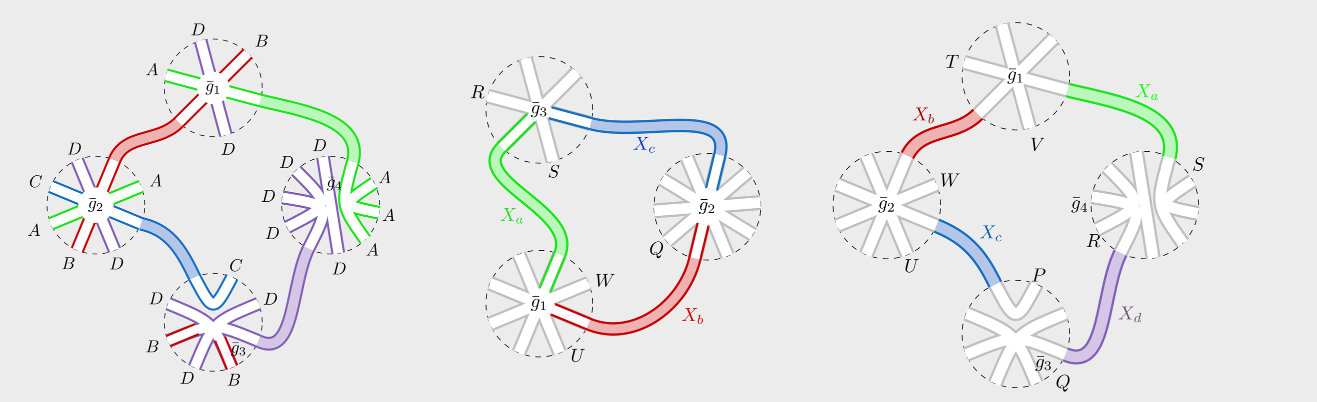

Fig. 2 Examples of graphs. From left to right: a graph of a 4-matrix model whose effective vertex is $\mathrm{Tr}(BDBD^7)\,\mathrm{Tr}(A^3DACDBACDADB)$. Next two graphs are both 1-loop (in the QFT sense) but only the one in the middle also in the topological sense. The latter is a contribution to $\mathrm{Hess}_{a,b}\,O_1 \star \mathrm{Hess}_{b,c}\,O_2 \star \mathrm{Hess}_{c,d}\,O_3 \star \mathrm{Hess}_{d,a}\,O_4$

Why a new framework?

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- [Eichhorn, Koslowski, *Phys. Rev. D* '13]'s approach is enough to treat 1MM's, but that approach projects out information needed for multimatrix models. Part of that recipe was used in [Eichhorn-Pereira-Pithis *JHEP* 2020, 2009.05111] to address the FRG for the $ABAB$-model, computing it on diagonal matrices; this leads to a β_{ABAB}-function ($\sim g^2_{ABAB}$) that *does not have the* one-loop structure *independently from the regulator*, and cannot follow from FRGE, if that β function would trully describe the full ensemble (as claimed by *op. cit.*)
- our new framework allows us to compute β-functions of the full ensemble $M_N(\mathbb{C})^2_{\mathrm{s.a}}$, and not only on subspaces of commuting matrices: (modulo $\eta = \partial_t Z$-coeffs, double-traces and cubic terms)

$$\beta(g_{ABBA}) - g_{ABBA}(2\eta + 1) \sim \underbrace{g_{AAAA} \times g_{ABBA}} + \underbrace{g_{BBBB} \times g_{ABBA}} + \underbrace{(g_{ABAB})^2} + \underbrace{(g_{ABBA})^2}$$

Hermitian 3MM

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BABY-EXAMPLE: Take operators: $O_1 = \frac{\bar{g}_1}{2}[\mathrm{Tr}(A^2/2)]^2$ and $O_2 = \bar{g}_2\,\mathrm{Tr}(ABC)$. Then

$$\mathrm{Hess}_{I,J}\,O_1 = \delta^J_I \delta^A_I \bar{g}_1 \{\underbrace{\mathrm{Tr}(A^2/2) \cdot [1_N \otimes 1_N]} + \underbrace{A \boxtimes A}\}\,.$$

Filled (resp. white) ribbons are contracted (resp. uncontracted) in the 1-loop. By (1),

$$\mathrm{Hess}\,O_2 = \bar{g}_2 \begin{bmatrix} 0 & C \otimes 1_N & B \otimes 1_N \\ 1_N \otimes C & 0 & A \otimes 1_N \\ 1_N \otimes B & 1_N \otimes A & 0 \end{bmatrix} \Rightarrow [\mathrm{Hess}\,O_2]^{\star 2} = \bar{g}_2^2 \begin{bmatrix} \overbrace{C \otimes C} + \overbrace{B \otimes B} & B \otimes A & C \otimes A \\ A \otimes B & A \otimes A + C \otimes C & C \otimes B \\ A \otimes C & B \otimes C & B \otimes B + A \otimes A \end{bmatrix}$$

Extracting coefficients

$$[\bar{g}_1\bar{g}_2]\,\mathrm{Tr}_{M_3(\mathcal{A})}\{\mathrm{Hess}\,O_1 \star [\mathrm{Hess}\,O_2]^{\star 2}\} = \mathrm{Tr}(A^2/2) \times \left[(\mathrm{Tr}\,C)^2 + (\mathrm{Tr}\,B)^2\right] + \mathrm{Tr}(ACAC + ABAB)\,,$$

which are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of {⊥, ⊥} with any of {✕, ✕}.