Holographic RG from Exact RG

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Abstract

Holographic RG is a geometric interpretation of the scale of a theory as the radial coordinate of a theory higher in dimension by one. This is realised in AdS-CFT correspondence. This allows a new physical way of looking at the correspondence. The precise way the scale change in the boundary theory should be effected is through Wilsonian RG or Exact RG. Equivalence between Holographic RG and Wilsonian RG has been shown so far by comparing the flows of *renormalised* quantities in both. We make precise the equivalence at the level of actions. We write down the ERG equation for the vector of boundary O(N) theory. Then we redefined the Exact RG evolution operator that corresponds to this equation as the bulk AdS action. This has been done upto the cubic term for the scalar[2], and kinetic term for the vector and the tensor[3]. Mapping the cubic interaction term for the vector to the Yang-Mills interaction term for the bulk gauge field is ongoing work.

Ingredients

A brief review of the concepts appearing on this poster.

Holographic Renormalisation Group

In AdS-CFT correspondence, the radial coordinate has a natural interpretation as the scale of the boundary theory. This is suggested by the AdS-metric itself written in Poincaré coordinates.

$$\mathrm{d}s^2 = \frac{1}{z^2}(\mathrm{d}z^2 + g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu). \tag{1}$$

Here $z = 0$ corresponds to the boundary. Holographic RG refers to relating the RG flow of the boundary theory with the radial equations of motion in the bulk theory. **The UV cutoff of the energy scale of the boundary theory then corresponds to fixing the boundary conditions of the bulk fields at some finite** $z = \epsilon$. This would involve adding boundary terms $S^B[\varphi]$ to the bulk action.[1]

Changing the scale cutoff of the boundary theory corresponds to changing the boundary of the bulk theory from $z = \epsilon$ to another finite value. The proper way to change the scale of a theory is via the Wilsonian or *Exact* RG. **Holographic RG then amounts to the relation that the exact RG evolution of the boundary path integral is the same as the radial Hamiltonian evolution of** $S^B[\varphi]$ **in the bulk.**

Exact Renormalisation Group

Exact RG refers to obtaining a theory at a lower UV cutoff Λ from a theory at higher or bare UV cutoff $\Lambda_0 > \Lambda$ by **integrating out all the modes in the path integral between** Λ **and and** Λ_0. Schematically, for modes

$$Z = \int \mathcal{D}\phi\, \mathrm{e}^{-S_{\Lambda_0}[\phi]} \equiv \int \mathcal{D}\phi_l\, \mathrm{e}^{-S_\Lambda[\phi_l]}, \tag{2}$$

$$\mathrm{e}^{-S_\Lambda[\phi_l]} \equiv \int \mathcal{D}\phi_h \mathrm{e}^{-S_{\Lambda_0}[\phi_l,\phi_h]}, \phi \equiv \phi_l + \phi_h, \tag{3}$$

ϕ_l being modes of ϕ below momentum scale Λ, and ϕ_h being modes at scales above Λ and below Λ_0. The propagator for ϕ, Δ is then written as

$$\Delta = \Delta_h + \Delta_l,$$
$$\ni \Delta_h(p^2) \approx 0 \; for \; p^2 < \Lambda, \; and \; \Delta_l \approx 0 \; for \; p^2 > \Lambda.$$

Δ_h is the high energy propagator—only propagates ϕ_h and Δ_l is the low energy propagator.
Polchinski's ERG equation is a convenient casting of this RG transformation as a differential equation in the parameter $t = \log \Lambda_0/\Lambda$.

$$\frac{\partial}{\partial t} e^{-S_{\Lambda,I}[\phi_l]} = \frac{1}{2}\int_x\int_y \dot{\Delta}_{hxy} \frac{\delta^2}{\delta\phi_l(x)\delta\phi_l(y)} e^{-S_{\Lambda,I}[\phi_l]}, \tag{4}$$

where the I subscript indicates the interactive part of the action and $\dot{\Delta}_h = \frac{\partial}{\partial t}\Delta_h$.

O(N) Model and Auxiliary Fields

The $O(N)$ model is given by the action

$$S_B = \int_x \Big(\frac{1}{2}\sqrt{N}\partial_\mu\phi^I\partial^\mu\phi^I + \frac{u}{4!}(\phi^I\phi^I)^2\big), \tag{5}$$

the I runs over $1 - N$, and the action is $O(N)$ invariant. (The B means bare.) Perturbing this theory by **composite operators** is to be done by introducing **auxiliary fields**. For the vector current, which is $\phi^I \overleftrightarrow{\partial^\mu} \phi^J$ in the bare theory, the auxiliary field which will proxy for it is introduced using the Hubbard-Stratanovich transformation thus.

$$S_B = \int_x \Big[\frac{1}{2}\sqrt{N}\partial_\mu\phi^I\partial^\mu\phi^I + \frac{u}{4!}(\phi^I\phi^I)^2 - \chi^{AB}_\mu(\phi_A\overleftrightarrow{\partial^\mu}\phi_B - \sigma^\mu_{AB})\Big]. \tag{6}$$

Integrating over the Lagrange multiplier field χ_μ enforces the condition $\sigma^{IJ}_\mu = \phi^I\overleftrightarrow{\partial^\mu}\phi^J$. But this is the current only in the bare theory. We want σ_μ to represent the current at all scales. So we work with the following action.

$$Z[A] = \int \mathcal{D}\chi\mathcal{D}\sigma\mathcal{D}\phi \exp\Big\{i\int_x \chi^{\mu ij}\Big(\sigma^{ij}_\mu - \frac{1}{\sqrt{N}}\frac{\delta}{\delta A^{\mu ij}}\Big)\Big\}\exp\Big\{-S_B[\phi, A_\mu]\Big\}, \tag{7}$$

$$S_B = \frac{1}{2}\sqrt{N}\int_x(\partial_\mu\phi^{iI} - A^{ij}_\mu\phi^{jI})(\partial^\mu\phi^{iI} - A^{\mu ik}\phi^{kI}). \tag{8}$$

σ_μ then stands for $\frac{\delta}{\delta A^\mu}$. One can then calculate correlation functions with multiple insertions of the current.

The Meat

The regularisation that is required to make holographic RG precise has been missing. We provide the regularisation. This involves three steps.

Step 1 Obtain ERG equation for the composite operator σ_μ from the Polchinski's equation (4) for the field ϕ^I, for a general form for regularisation, Δ_h.

This is done by using the Polchinski's equation on the action given by (7). After setting ϕ_l to 0, (as it is not the object of interest), the ERG equation for σ_μ is obtained.

$$\frac{\partial}{\partial t}\mathrm{e}^{-S_\Lambda[\sigma^{\mu ij}]} = \Big\{-\frac{1}{2}\int_p \dot{I}(p^2)\frac{\delta^2}{\delta\sigma^{\mu ij}(p)\delta\sigma^{ij}_\mu(-p)} + \frac{4i}{\sqrt{N}}\int_p\int_q\int_r \frac{\Delta_{h,r+p}\Delta_{h,r-q}\dot{\Delta}_{h,r}}{I(p^2)I(q^2)I((p+q)^2)} r_\mu r_\nu (r+p)_\rho \sigma^{\mu ik}(p)\sigma^{\nu kj}(q)\sigma^{\rho ij}(-p-q)\Big\} \mathrm{e}^{-S_\Lambda[\sigma^{\mu ij}]} + \dots \tag{9}$$

Here $I(p^2) = -\frac{2}{D}\int_r r^2\Delta_{h,r}\Delta_{h,p+r}$, and the dot indicates derivative w.r.t. t.

Step 2 In analogy with the Schrödinger equation, write the Wilson action at a scale t_f (lower momentum cutoff) as a functional integral over the **evolution operator** acting on the Wilson action at a scale t_i (higher momentum cutoff). This evolution operator gives a functional integral in $D + 1$ dimensions, with t also functioning similar to the spacetime dimensions.

$$U[\sigma_f, t_f; \sigma_i, t_i] = \int \mathcal{D}\sigma^\mu \exp\Big\{\int \mathrm{d}t\Big[\int_p \frac{1}{\dot{I}(p^2)}\dot{\sigma}^{\mu ij}\dot{\sigma}^{ij}_\mu + \frac{4i}{\sqrt{N}}\int_p\int_q\int_r \frac{\Delta_{h,r+p}\Delta_{h,r-q}\dot{\Delta}_{h,r}}{I(p^2)I(q^2)I((p+q)^2)} r_\mu r_\nu(r+p)_\rho\sigma^{\mu ik}(p)\sigma^{\nu kj}(q)\sigma^{\rho ij}(-p-q)\Big]\Big\}. \tag{10}$$

This can be considered the path integral for a $D + 1$ dimensional **holographic** theory, with the action

$$S[\sigma^\mu] = -\int \mathrm{d}t\Big[\int_p \frac{1}{\dot{I}(p^2)}\dot{\sigma}^{\mu ij}(p)\dot{\sigma}^{ij}_\mu(-p) \tag{11}$$

$$+ \frac{4i}{\sqrt{N}}\int_p\int_q\int_r \frac{\Delta_{h,r+p}\Delta_{h,r-q}\dot{\Delta}_{h,r}}{I(p^2)I(q^2)I((p+q)^2)} r_\mu r_\nu(r+p)_\rho\sigma^{\mu ik}(p)\sigma^{\nu kj}(q)\sigma^{\rho ij}(-p-q)\Big] \tag{12}$$

Step 3 Redefine the field σ_μ to obtain the **standard AdS kinetic term** in this functional integral.

$$z = e^t; f(p,z) = \sqrt{-z^{-D+1}\partial_z I(p^2)/2},$$
$$\sigma_\mu(p,z) \equiv z f(p,z) a_\mu(p,z). \tag{13}$$

The field redefinition required for this obeys a differential equation.

$$\frac{\partial}{\partial z}\Big(z^{-D+1}\frac{\partial}{\partial z}\frac{1}{f}\Big) = z^{-D+1}(p^2z^2 + m^2)\frac{1}{f}, \tag{14}$$

where $m^2 = 1 - D$ for the vector. The solutions are the modified Bessel functions $I_\nu(pz)$ and $K_\nu(pz)$, with $\nu^2 = m^2 + \frac{D^2}{4}$. The field redefinition needs to satisfy analyticity properties and give the correct falloff behaviour that comes from the boundary Green's function, and this fixes our regularisation Δ_h.

Results

- The **field redefinition** is given by

$$1/f = A(p)z^{\frac{D}{2}}K_\nu(pz) + B(p)z^{\frac{D}{2}}I_\nu(pz), \tag{15}$$

  with $A(p) \sim p^\nu$ and $B(p) \sim p^\nu$.
- We have obtained the leading order AdS action for bulk gauge field

$$S_K = \frac{1}{2}\int_p\int \frac{\mathrm{d}z}{z^{D-3}}(\partial_z a^{\mu ij}(p)\partial_z a^{ij}_\mu(-p) + p^2 a^{\mu ij}(p)a^{ij}_\mu(-p)). \tag{16}$$

- The same analysis repeated for the energy momentum tensor in the boundary theory results in bulk AdS action for metric tensor perturbations.

$$\int d^d x dz\, z^{-D+1}[\partial_z h^{T\mu}_\nu \partial_z h^{T\nu}_\mu + \partial_\rho h^{T\mu}_\nu \partial^\rho h^{T\nu}_\mu]. \tag{17}$$

It is to be noted that the AdS-CFT correspondence has not been made use of to justify any of these connections.

Bread (Future Directions)

The $O(1/\sqrt{N})$ term in the ERG equation should correspond to the cubic Yang-Mills interaction term in the AdS action. Work on this is ongoing. Similarly, for the metric, interaction terms in Einstein Hilbert action is to be derived. Locality properties of such bulk interactions are to be studied.

References

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