Universal quantities at high orders of the derivative expansion

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PEDECIBA

UNIVERSIDAD DE LA REPÚBLICA URUGUAY

OUTLINE

- General picture of critical phenomena
- Computing with the derivative expansion
- Results and behavior of the DE
- Conclusions

GENERAL PICTURE OF CRITICAL PHENOMENA

Critical Phenomena

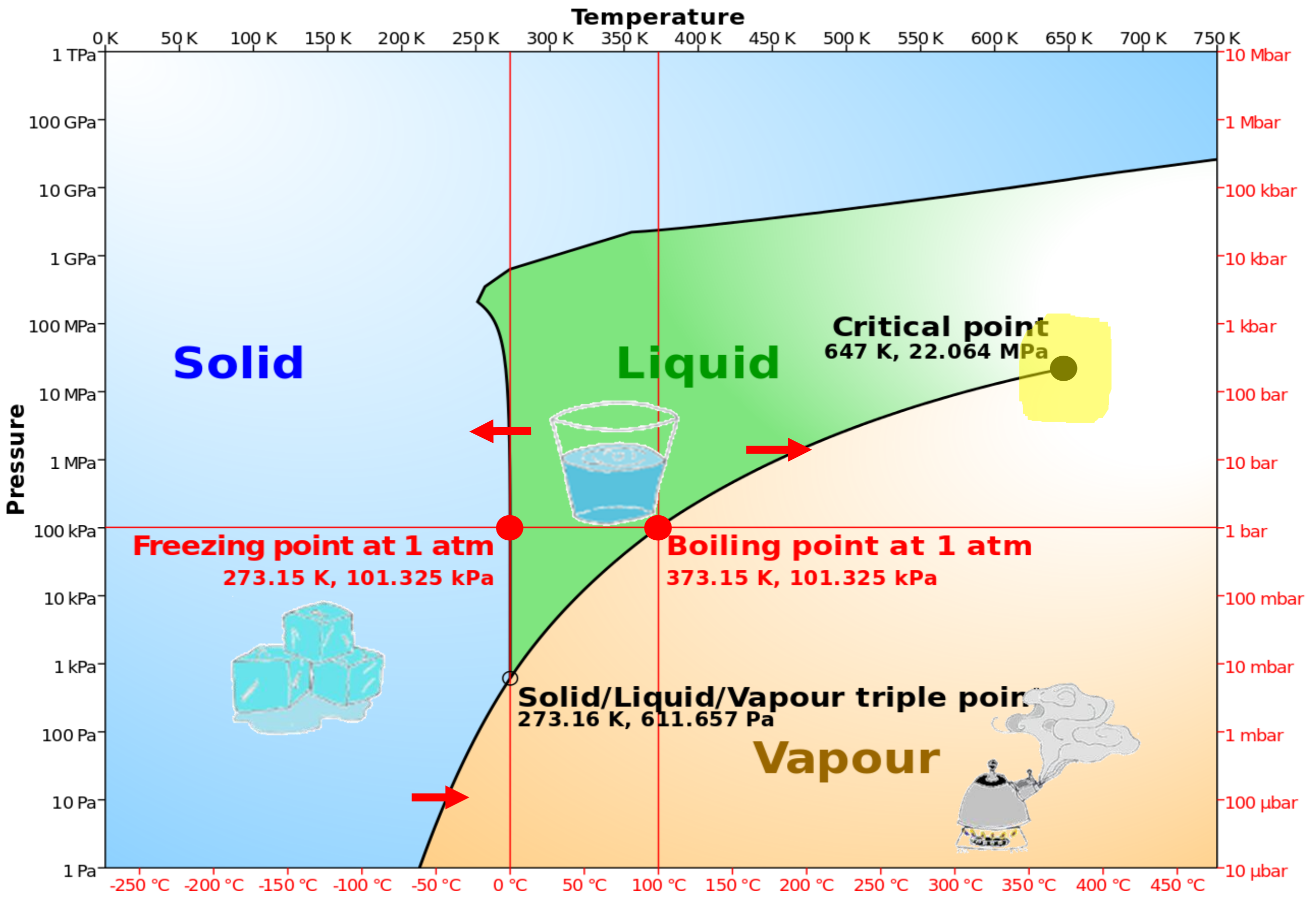
Physics around critical point

Divergence of Correlation Length

Power law behaviour of many physical quantities

$$\kappa_T \sim |T - T_c|^{-\gamma} \qquad C \sim |T - T_c|^{-\alpha}$$

Scale Invariance



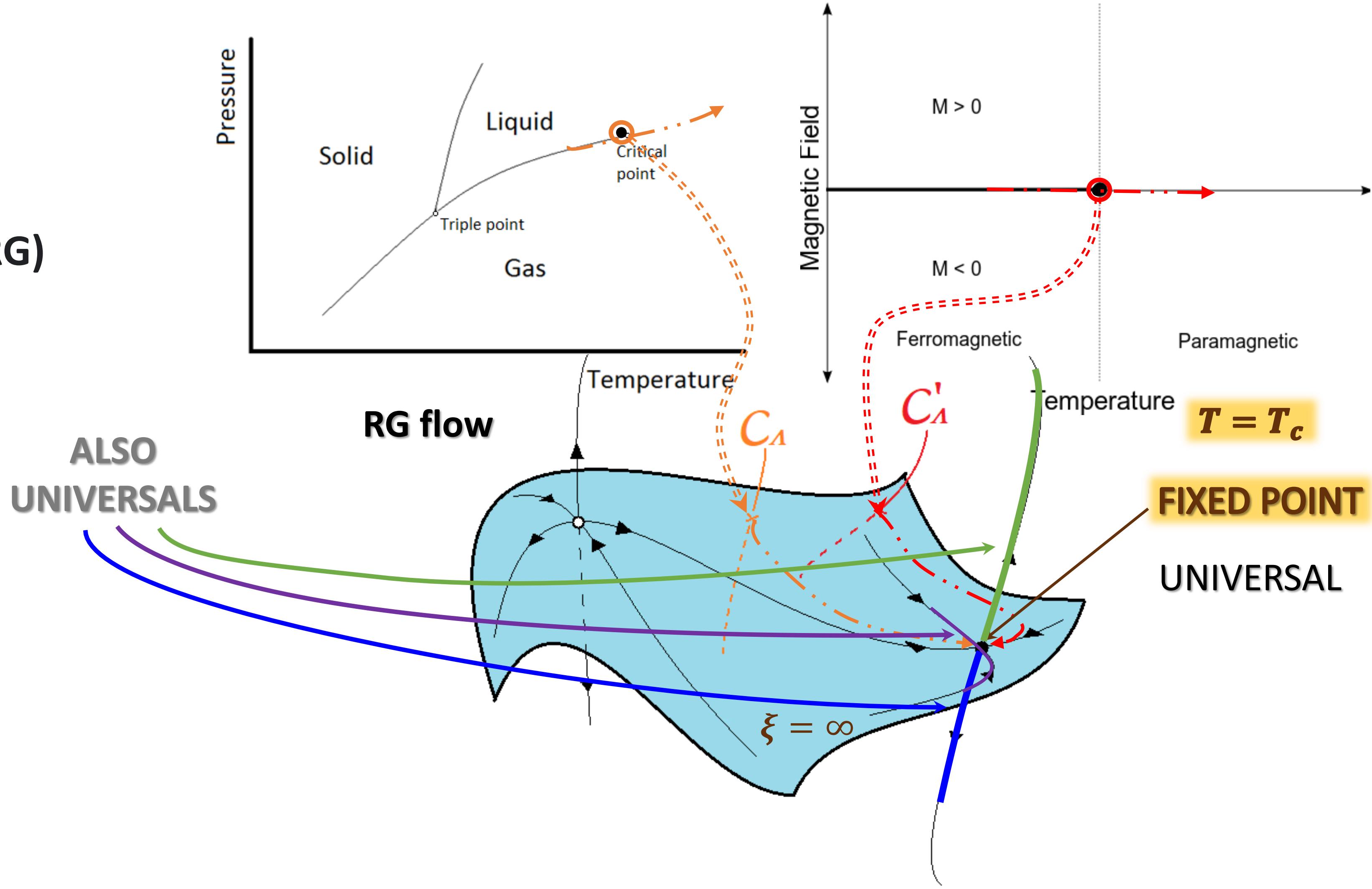
GENERAL PICTURE OF CRITICAL PHENOMENA

Critical Phenomena

Physics around critical point

Tackled with the RG (or FRG)

g_4^+	$\left(\frac{3N\chi_4}{(N+2)\chi_2^2\xi^d}\right)_{HT}$
R_χ	$\left(\frac{\chi}{Nt^{\gamma}}\right)_{HT}\left(\frac{M}{(-t)^\beta}\right)^{\delta-1}_{LT}\left(\frac{H}{M^\delta}\right)_{CT}$
R_ξ	$\left(\frac{\xi}{t^{-\nu}}\right)^{\beta/\nu}_{HT}\left(\frac{M}{(-t)^\beta}\right)_{LT}\left(\frac{H^{1/\delta}}{M}\right)^{\delta/(\delta+1)}_{CT}$
$\xi^{+(d-2)}E$	$\left(\frac{\xi}{t^{-\nu}}\right)^{d-2}_{HT}\left(\frac{\xi^2M^2}{\chi 2(-t)^{\nu(d-2)}}\right)_{LT}$
U_ξ	$\left(\frac{\xi}{t^{-\nu}}\right)_{HT}\left(\frac{\xi}{(-t)^{-\nu}}\right)^{-1}_{LT}$
U_2	$\left(\frac{\chi}{t^{-\gamma}}\right)_{HT}\left(\frac{\chi}{(-t)^{-\gamma}}\right)^{-1}_{LT}$



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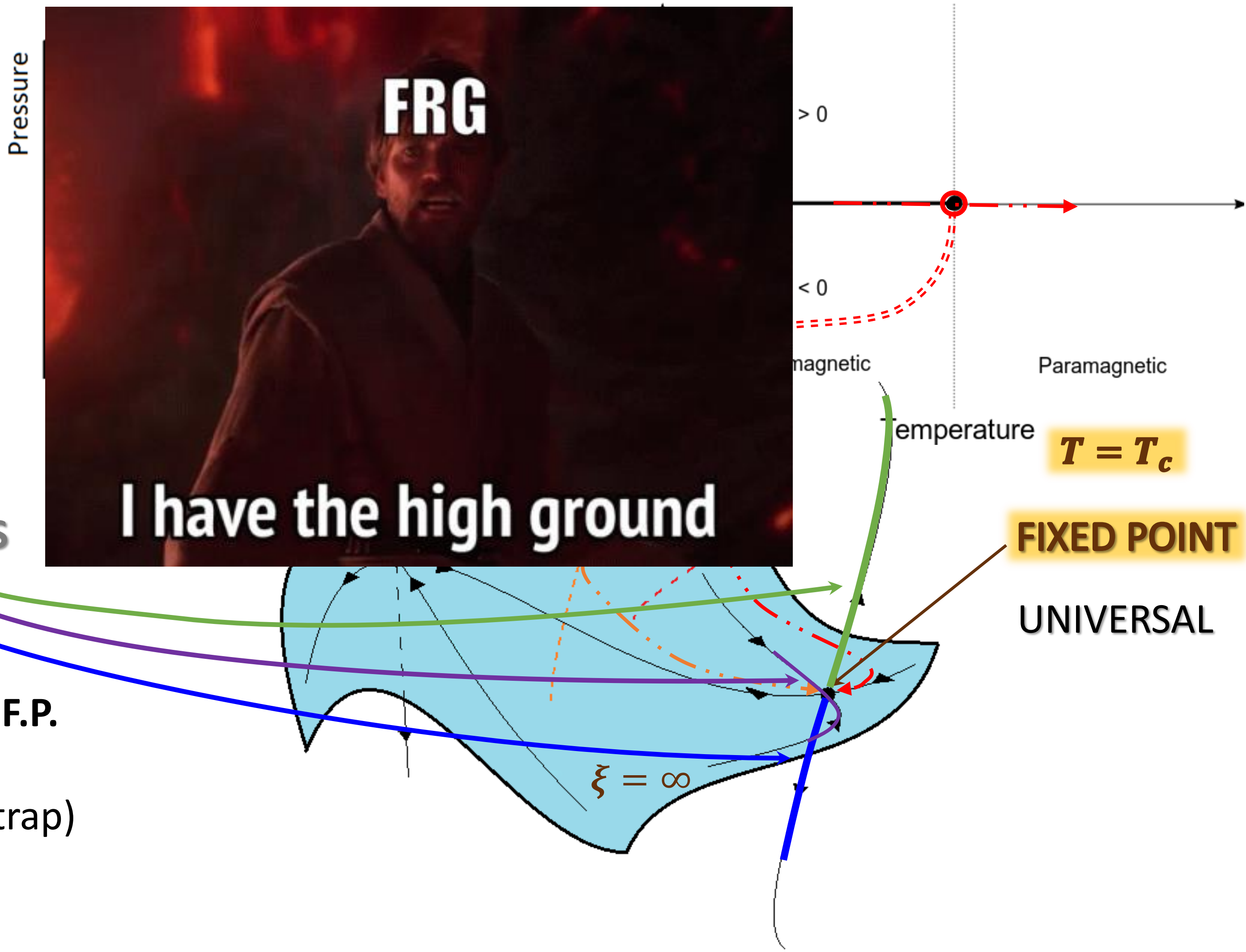
U_ξ	$\left(\frac{\xi}{t^{-\nu}}\right)_{HT}\left(\frac{\xi}{(-t)^{-\nu}}\right)^{-1}_{LT}$
U_2	$\left(\frac{\chi}{t^{-\gamma}}\right)_{HT}\left(\frac{\chi}{(-t)^{-\gamma}}\right)^{-1}_{LT}$

ALSO UNIVERSALS

¡Not accesible to F.P. methods!*

(Conformal Bootstrap)

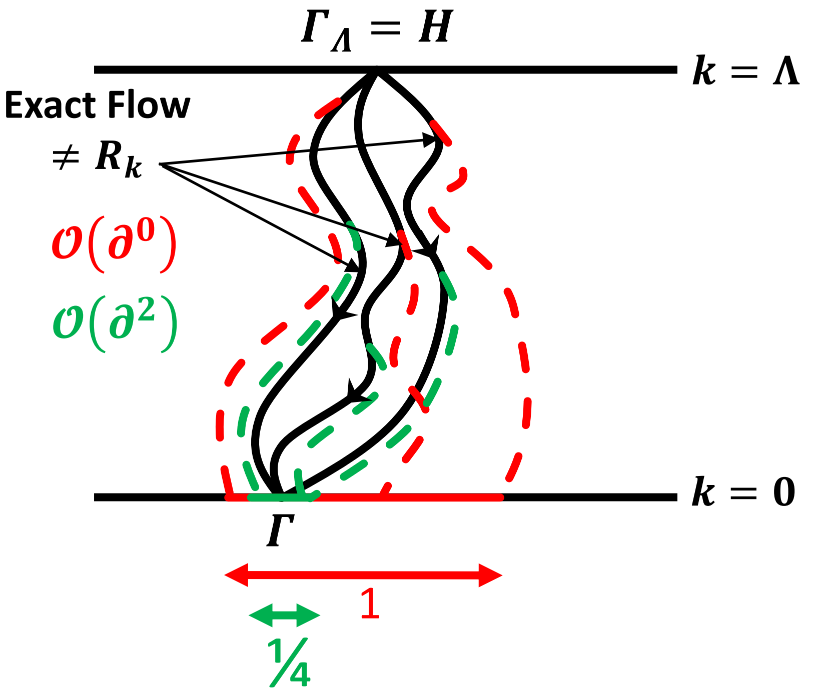


COMPUTING WITH THE DERIVATIVE EXPANSION

LONG DISTANCES $\sim q \to 0$

NEGLECTING HIGH POWERS IN q

DERIVATIVE EXPANSION *(DE)*
ORDEN $\mathcal{O}(\partial^s)$ → **Ansatz for Γ_k with S fields derivatives**



$O(N)$ model Ansatzes

$\mathcal{O}(\partial^0)$ $\mathcal{O}(\partial^2)$ $\mathcal{O}(\partial^4)$ $\rho = \frac{\varphi_a \varphi_a}{2}$

$$\Gamma_k[\vec{\varphi}] = \int_x \Big\{ U_k(\rho) + \frac{\begin{Bmatrix} 1 \\ Z_k(\rho) \end{Bmatrix}}{2} \nabla\varphi_a \cdot \nabla\varphi_a + \frac{Y_k(\rho)}{4} \nabla\rho \cdot \nabla\rho + \frac{W_{1,k}(\rho)}{2} (\partial_\mu\partial_\nu\varphi_a)^2 + \frac{W_{2,k}(\rho)}{2} (\varphi_a\partial_\mu\partial_\nu\varphi_a)^2 + W_{3,k}(\rho)\partial_\mu\rho\partial_\nu\varphi_a\partial_\mu\partial_\nu\varphi_a + \frac{W_{4,k}(\rho)}{2}\varphi_b\partial_\mu\varphi_a\partial_\nu\varphi_a\partial_\mu\partial_\nu\varphi_b + \frac{W_{5,k}(\rho)}{2}\varphi_a\partial_\mu\rho\partial_\nu\rho\partial_\mu\partial_\nu\varphi_a + \frac{W_{6,k}(\rho)}{4}\left((\partial_\mu\varphi_a)^2\right)^2 + \frac{W_{7,k}(\rho)}{4}(\partial_\mu\varphi_a\partial_\nu\varphi_a)^2 + \frac{W_{8,k}(\rho)}{2}\partial_\mu\varphi_a\partial_\nu\varphi_a\partial_\mu\rho\partial_\nu\rho + \frac{W_{9,k}(\rho)}{2}(\partial_\mu\varphi_a)^2(\partial_\nu\rho)^2 + \frac{W_{10,k}(\rho)}{4}\left((\partial_\mu\rho)^2\right)^2 \Big\}$$

FRG Equation → *RG Flow* → **Compute ν, η, UAR**

¡ERROR BARS!: WE CAN $\Rightarrow$ WE MUST

$$Q^{(s)} = \bar{Q}^{(s)} \pm \Delta Q^{(s)} \qquad \frac{\Delta Q^{(s)}}{\Delta Q^{(s-2)}} \sim \frac{1}{4} \quad [1]$$

¿HOW?

DE Quant. serious approx. scheme

[1] I. Balog, H. Chaté, B. Delamotte, M. Marohnić, and N. Wschebor Phys. Rev. Lett. **123**, 240604 (2019)

COMPUTING WITH THE DERIVATIVE EXPANSION

$$\Theta_k^n(q^2) = Z_k k^2 \alpha \left(1-\frac{q^2}{k^2}\right)^n \theta\left(1-\frac{q^2}{k^2}\right)$$
$$E_k(q^2) = Z_k k^2 \alpha e^{-q^2/k^2}$$
$$W_k(q^2) = Z_k k^2 \alpha \frac{q^2/k^2}{e^{q^2/k^2}-1}$$

Set of Regs.

↓

Strict ANSATZ
Full ANSATZ
Essential Scheme*

Impl. DE Ansatz

↓

α-DEPENDENT RESULTS (AND REGULATOR FAMILY)

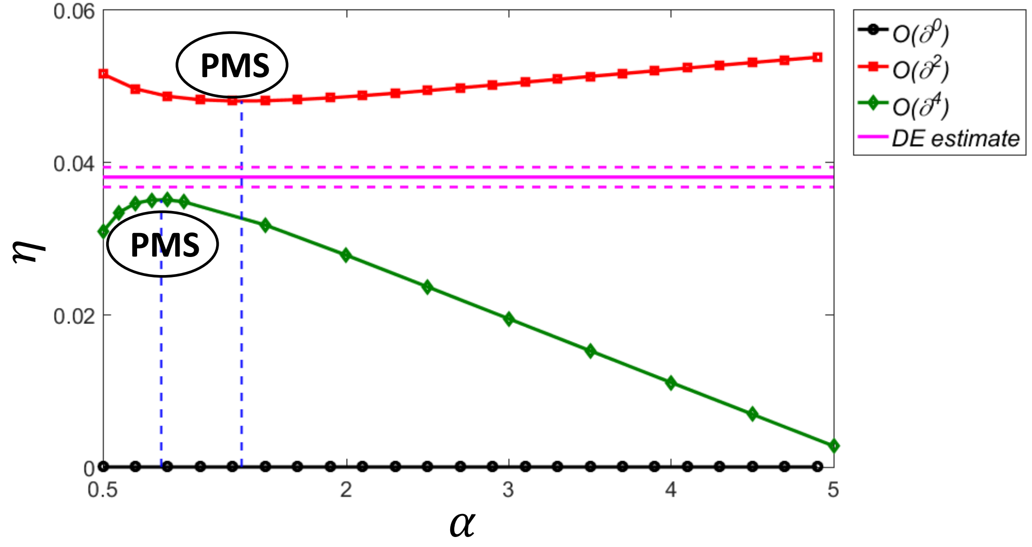
Compute RG flow / Quantities

↓

PMS ¡Needed! ¿Why? [3]

***SPIRIT*:**
REGULATOR-INDEPENDCY
⇩
PICK THEM WHEN LESS R_k DEPENDENCE

XY model ($N = 2$) [2]

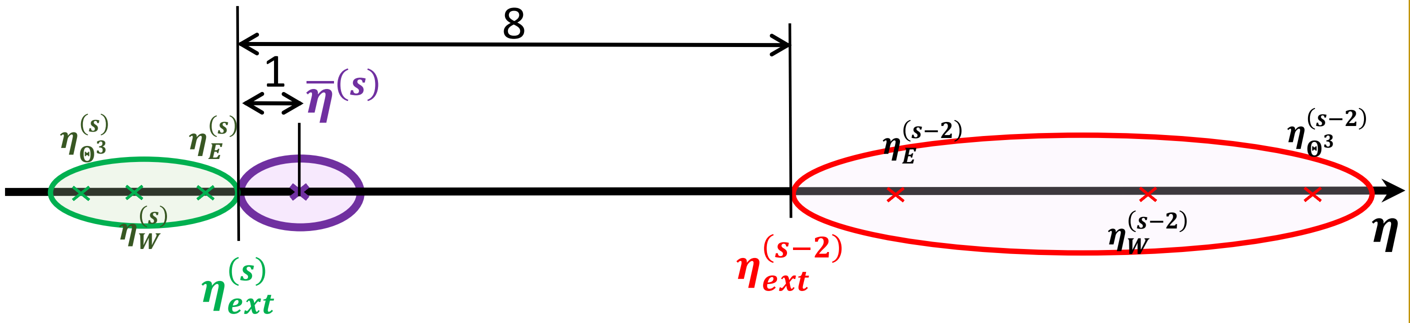


[2] Gonzalo De Polsi, Ivan Balog, Matthieu Tissier, and Nicolás Wschebor. Phys. Rev. E **101**, 042113 – (2020)

[3] Gonzalo De Polsi and Nicolás Wschebor. arXiv:2204.09170 – (2022) (Soon to appear in PRE)
Also Wschebor's Talk.

COMPUTING WITH THE DERIVATIVE EXPANSION

ALTERNATING BOUNDS



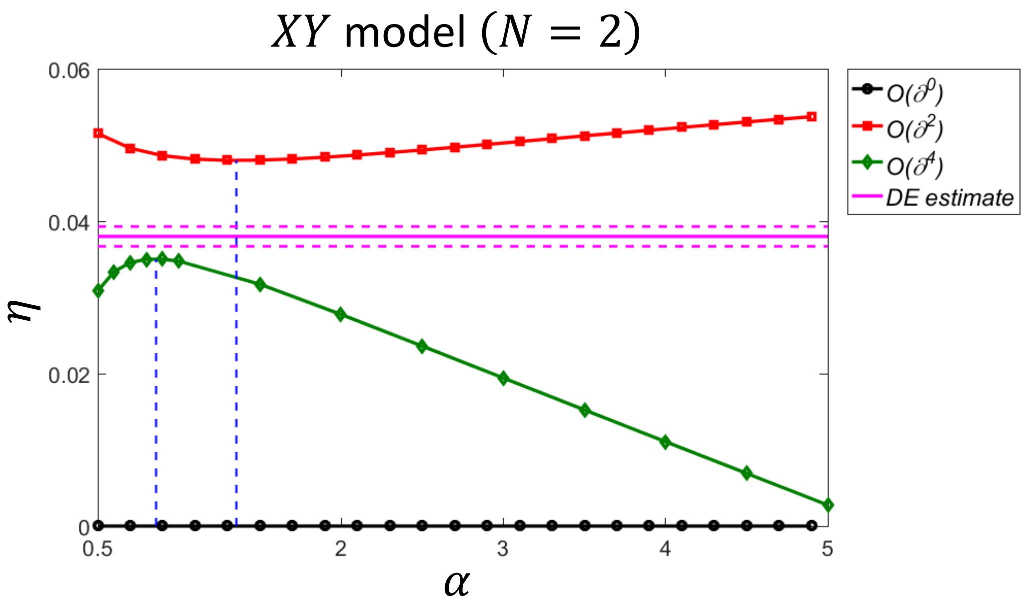
$$\eta^{(s)}_{ext} = \begin{cases} max_f \left\{\eta^{(s)}_f\right\} & \textbf{LOWER BOUND} \\ min_f \left\{\eta^{(s)}_f\right\} & \textbf{UPPER BOUND} \end{cases}$$

DE CENTRAL VALUE

$$\overline{\eta}^{(s)} = \eta^{(s)}_{ext} \pm \left|\frac{\eta^{(s)}_{ext} - \eta^{(s-2)}_{ext}}{8}\right|$$

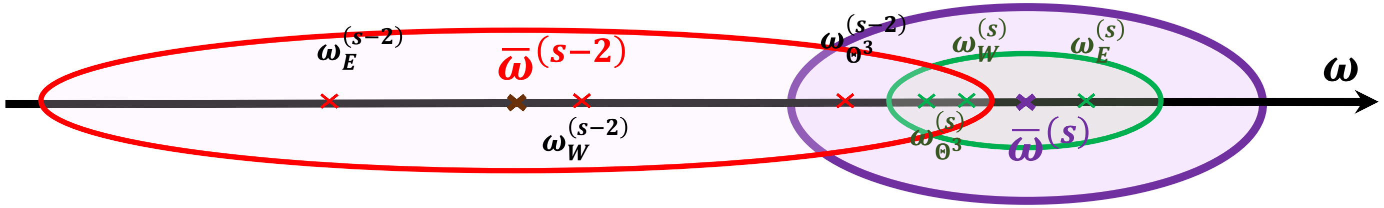
DE ERROR BAR

$$\Delta\eta^{(s)} = \left|\frac{\eta^{(s)}_{ext} - \eta^{(s-2)}_{ext}}{8}\right|$$



COMPUTING WITH THE DERIVATIVE EXPANSION

NON-ALTERNATING BOUNDS



$$\Delta_{\text{reg}}\omega^{(s)} = max_f\left\{\omega_f^{(s)}\right\} - min_f\left\{\omega_f^{(s)}\right\}$$

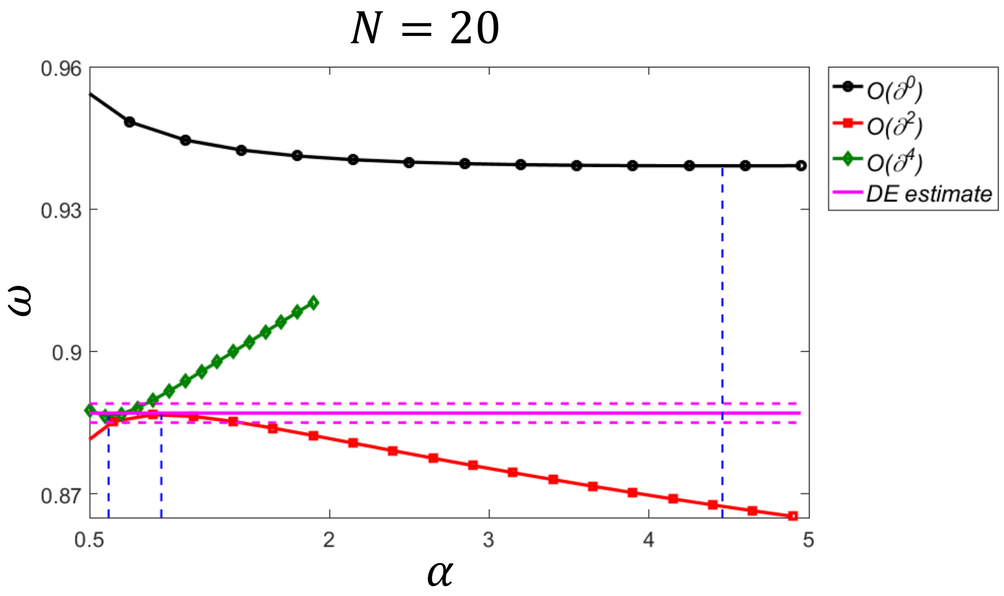
$$\widehat{\Delta}\omega^{(s)} = \frac{\overline{\omega}_f^{(s)} - \overline{\omega}_f^{(s-2)}}{4}$$

DE CENTRAL VALUE

$$\overline{\omega}^{(s)} = \frac{max_f\left\{\omega_f^{(s)}\right\} + min_f\left\{\omega_f^{(s)}\right\}}{2}$$

DE ERROR BAR

$$\Delta\omega^{(s)} = \Delta_{\text{reg}}\omega^{(s)} + \widehat{\Delta}\omega^{(s)}$$

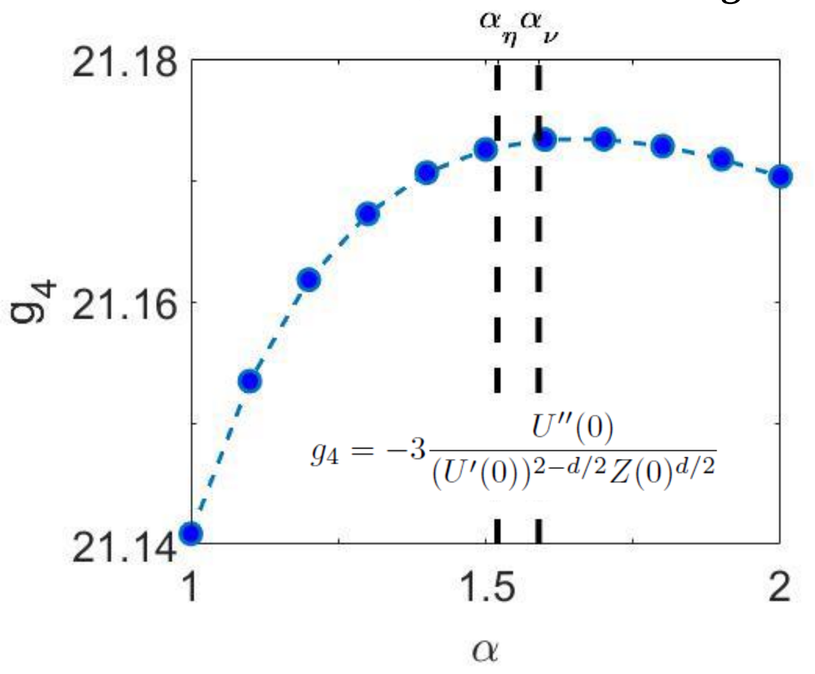


COMPUTING WITH THE DERIVATIVE EXPANSION

The truth about the breaded-beef…

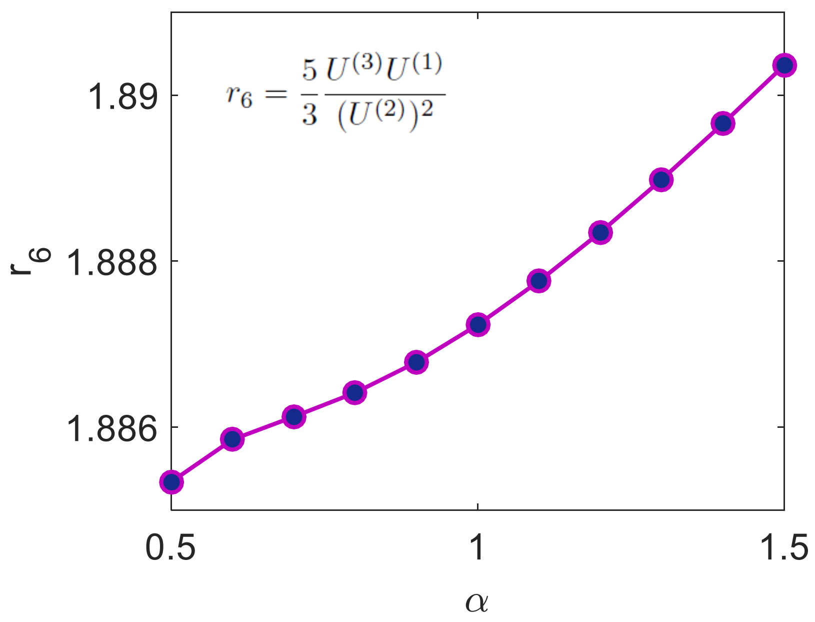
THE GOOD

$N = 2\,; Wetterich\ Reg$



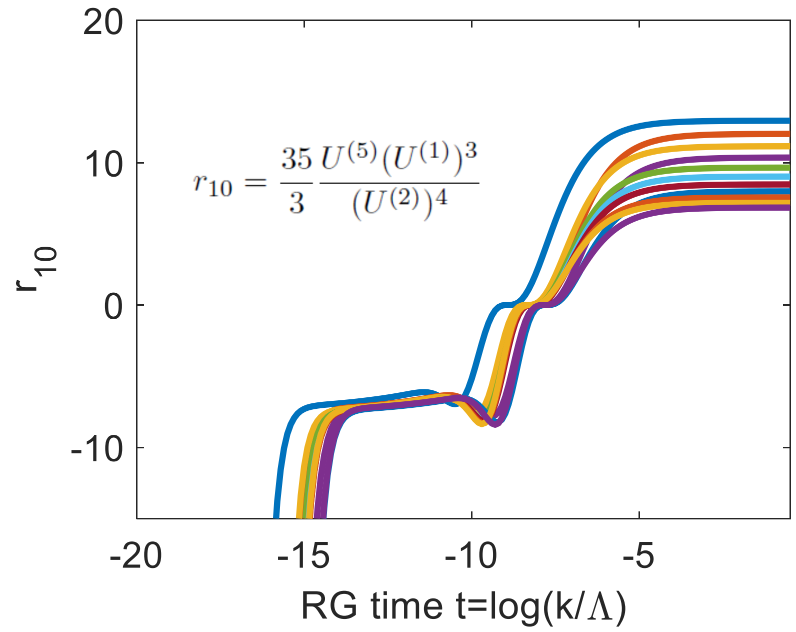
THE BAD

$N = 3\,; Exponential\ Reg$



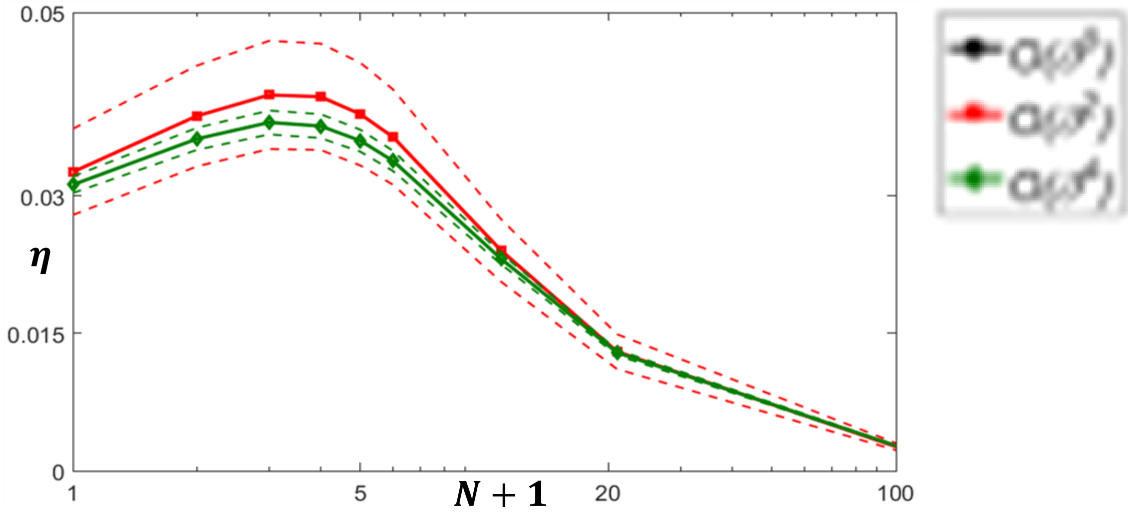
THE UGLY

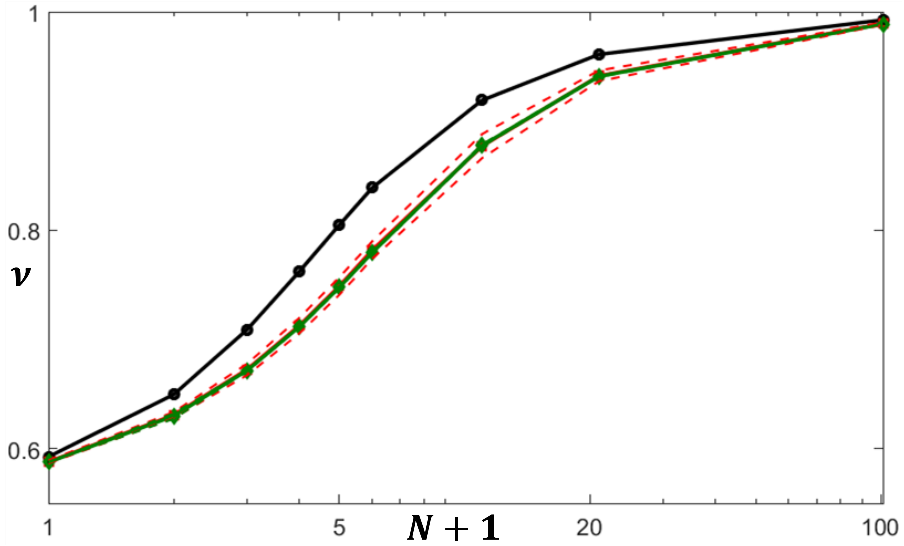
$N = 5\,; Exponential\ Reg$

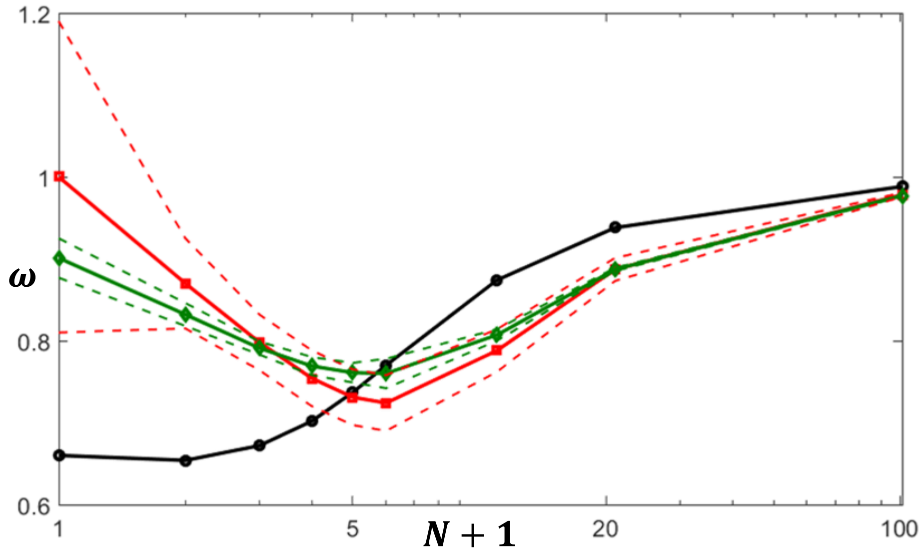


RESULTS AND BEHAVIOUR OF THE DE

On the behaviour…



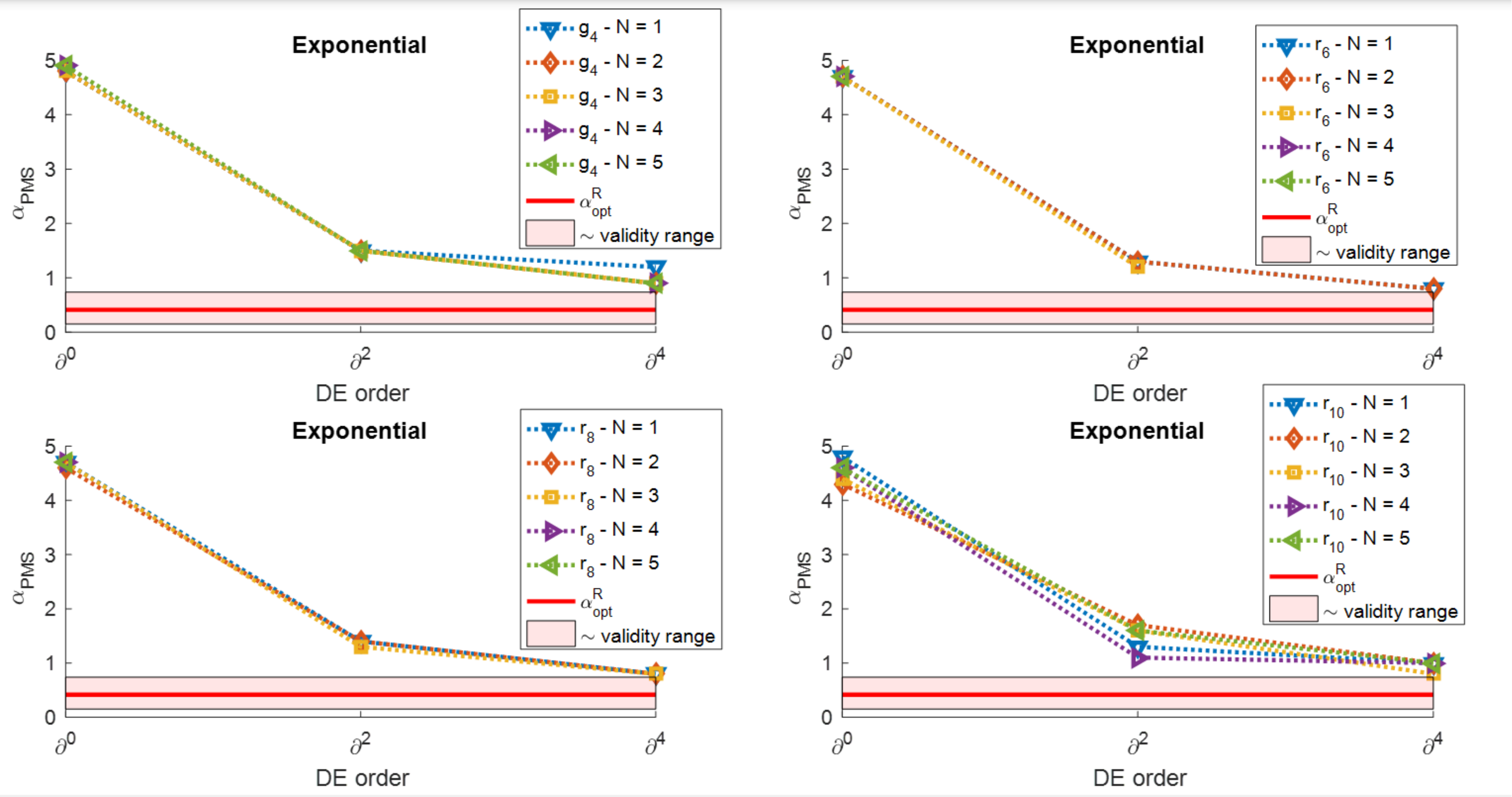




DERIVATIVE EXPANSION SELF-CONSISTENT

RESULTS AND BEHAVIOUR OF THE DE

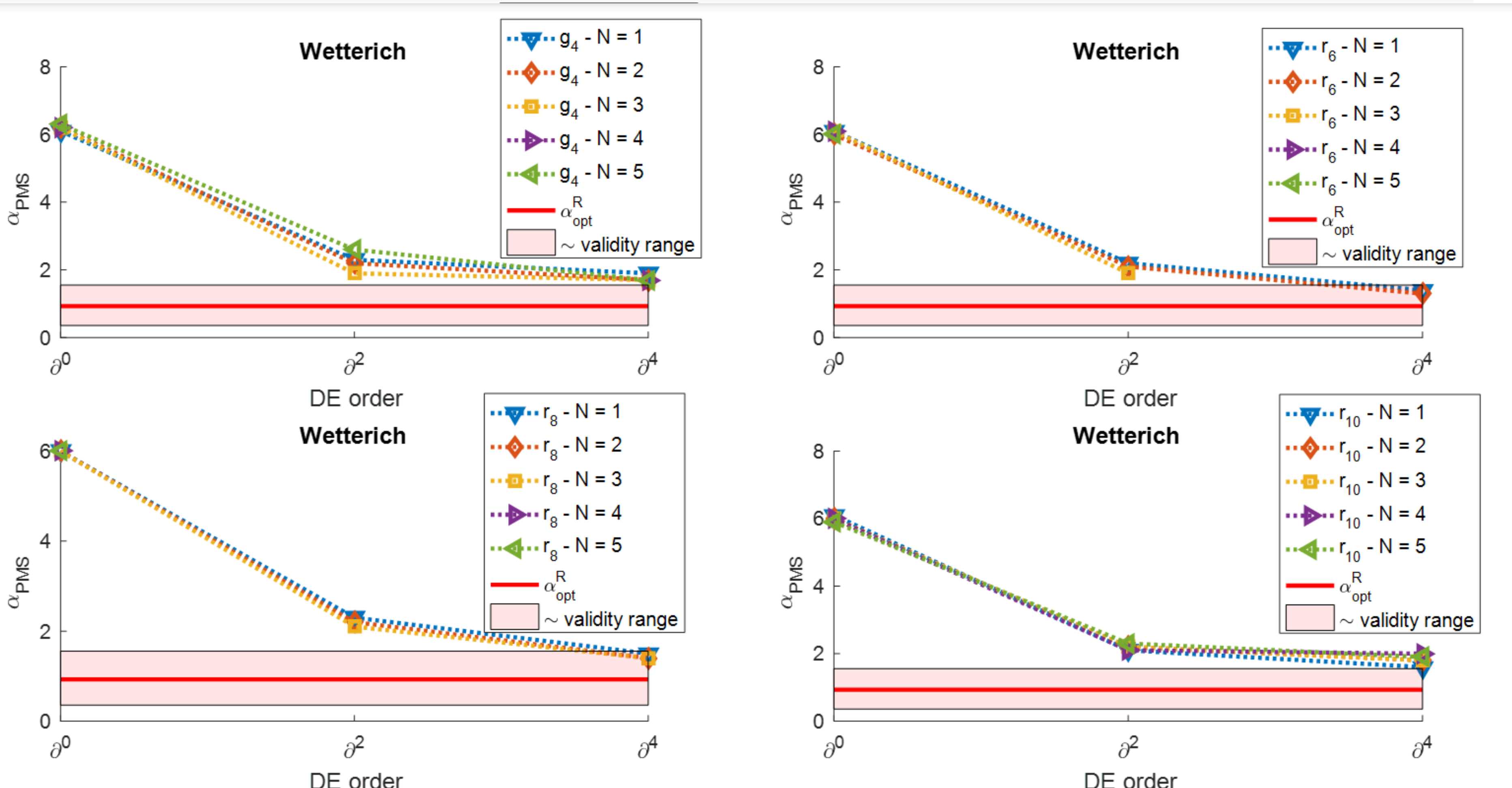
On the dependence on the regulator (α)…



PMS IS CRUCIAL (WE KNOW WHY*) BEHAVES AS EXPECTED!

RESULTS AND BEHAVIOUR OF THE DE

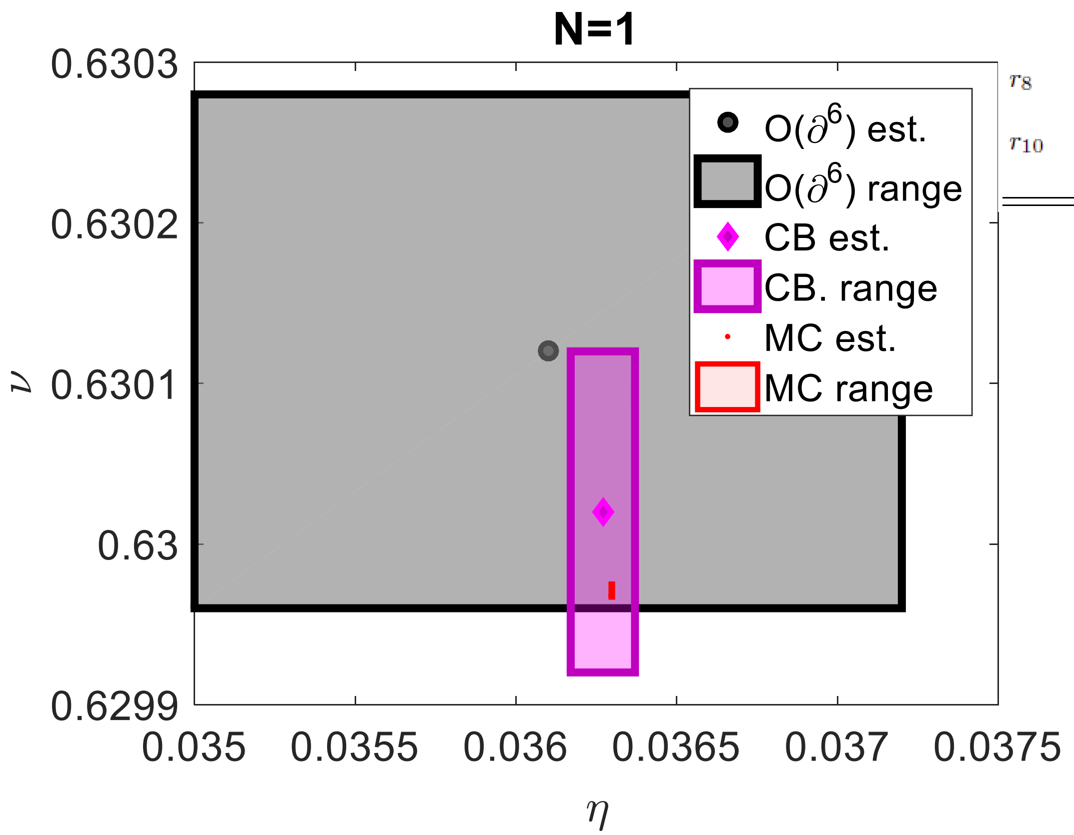
On the dependence on the regulator (α)…



PMS IS CRUCIAL
(WE KNOW WHY*)
BEHAVES AS EXPECTED!

RESULTS AND BEHAVIOUR OF THE DE

On the precisión of the results…



	HT	ϵ exp	$d=3$ exp	MC	LPA	$\mathcal{O}(\partial^2)$	$\mathcal{O}(\partial^4)$
g_4	23.56(2) [48]	23.6(2) [49]	23.64(7) [50]	23.6(2) [51] 23.4(2) [52, 53]	29.2 21(4) [54]	23.1(16)	23.60(15)
r_6	2.056(5) [48] 1.99(6) [56] 2.157(18) [58] 2.25(9) [60]	2.058(11) [52] 2.12(12) [50]	2.053(8) [50] 2.060 [57]	2.72(23) [55] 3.37(11) [51] 3.26(26) [59]	2.0 2.064(36) [54]	2.05(1)	2.064(6)
r_8	2.3(1) [48] 2.7(4) [56]	2.48(28) [52] 2.42(30) [50]	2.47(25) [50]		2.64 2.47(5) [54]	2.40(6*)	2.60(4)
r_{10}	−13(4) [61] −4(2) [56]	−20(15) [52] −12.0(1.1) [50]	−25(18) [50]		-9.5 −18(4) [54]	-14.8(14)	-14.1(3)

A. Pelissetto and E. Vicari,
Physics Reports 368, 549 , (2002).

$$N = 1$$

$$g_4 = -3\frac{U''(0)}{(U'(0))^{2-d/2}Z(0)^{d/2}}$$

$$r_6 = \frac{5}{3}\frac{U^{(3)}U^{(1)}}{(U^{(2)})^2}$$

$$r_8 = \frac{35}{9}\frac{U^{(4)}(U^{(1)})^2}{(U^{(2)})^3}$$

$$r_{10} = \frac{35}{3}\frac{U^{(5)}(U^{(1)})^3}{(U^{(2)})^4}$$

[MC] M. Hasenbusch
Phys. Rev. B **82**, 174433 (2010).
[CB] F. Kos, D. Poland, and D. Simmons-Duffin,
J. High Energy Phys. 11, 109, (2014).

RESULTS AND BEHAVIOUR OF THE DE

On the precisión of the results…

	ν	η	ω
LPA	0.7090	0	0.672
$O(\partial^2)$	0.6725(52)	0.0410(59)	0.798(34)
$O(\partial^4)$	0.6716(6)	0.0380(13)	0.791(8)
CB (2016)	0.6719(12)	0.0385(7)	0.811(19)
CB (2019)	0.6718(1)	0.03818(4)	0.794(8)
6-loop $d=3$	0.6703(15)	0.0354(25)	0.789(11)
$\epsilon-$expansion, ϵ^5	0.6680(35)	0.0380(50)	0.802(18)
$\epsilon-$expansion, ϵ^6	0.6690(10)	0.0380(6)	0.804(3)
MC+High-T. (2006)	0.6717(1)	0.0381(2)	0.785(20)
MC (2019)	0.67169(7)	0.03810(8)	0.789(4)

	g_4	r_6	r_8	r_{10}
LPA	25.7	1.91	1.79	-9.47
$O(\partial^2)$	20.8 (12)	1.96 (1)	1.64 (4*)	-14.2 (15)
$O(\partial^4)$	21.18 (10)	1.972 (5)	1.80 (6)	-13.5 (4)
High-T	21.14 (6)	1.950 (15)	1.44 (10)	-13 (7)
$d=3$ series	21.16 (5)	1.967	1.641	
$\epsilon-$expansion	21.5 (4)	1.969 (12)	2.1 (9)	

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$N = 2$

RESULTS AND BEHAVIOUR OF THE DE

On the precisión of the results…

CPU TIME τ_{CPU}

$$\frac{\tau_{CPU}(CB/MC)}{\tau_{CPU}(\mathcal{O}(\partial^4))} > \mathbf{10^5} \text{ !!!}$$

	ν	η	ω
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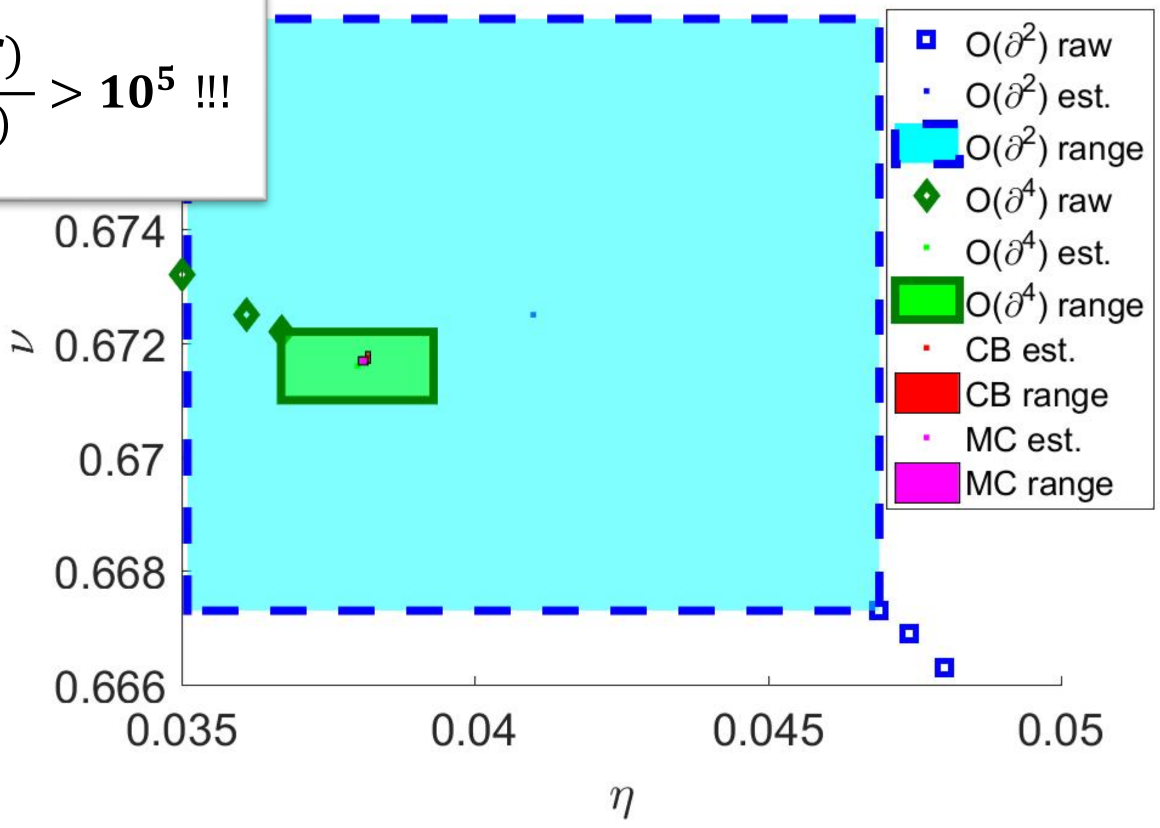

$N = 2$

RESULTS AND BEHAVIOUR OF THE DE

On the precisión of the results...

[eps, d=3] A. Butti and F. Parisen Toldin
Nuclear Physics B **704**, 527, (2005).

[MC] Martin Hasenbusch
Phys. Rev. B 105, 054428, (2022)

	ν	η	ω
LPA	0.839	0	0.770
$O(\partial^2)$	0.782(8)	0.0364(52)	0.724(34)
$O(\partial^4)$	0.7797(9)	0.0338(11)	0.760(18)
Six-loop, $d=3$	0.764(2)	0.030(1)	
ϵ expansion, ϵ^5	0.764(6)	0.034(2)	
MC	0,7808(6)	0,03397(9)	0,754(7)
Large N	0.71(7)	0.031(15)	0.51(6)

	[71]	LPA	$\mathcal{O}(\partial^2)$	$\mathcal{O}(\partial^4)$
g_4^+	15.74(2) 15.6(1)	17.9	15.8(5)	15.77(3)
r_6	1.72(2) 1.70(1)	1.65	1.73(2)	1.739(2)
r_8	-1(3) -0.3(5)	0.04	0.09(2*)	0.16(2)
r_{10}	3(8)	-3.0	-7.6(16)	-7.0(6)

$$g_4 = -3\frac{U''(0)}{(U'(0))^{2-d/2}Z(0)^{d/2}}$$

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$N = 5$

RESULTS AND BEHAVIOUR OF THE DE

On the precisión of the results

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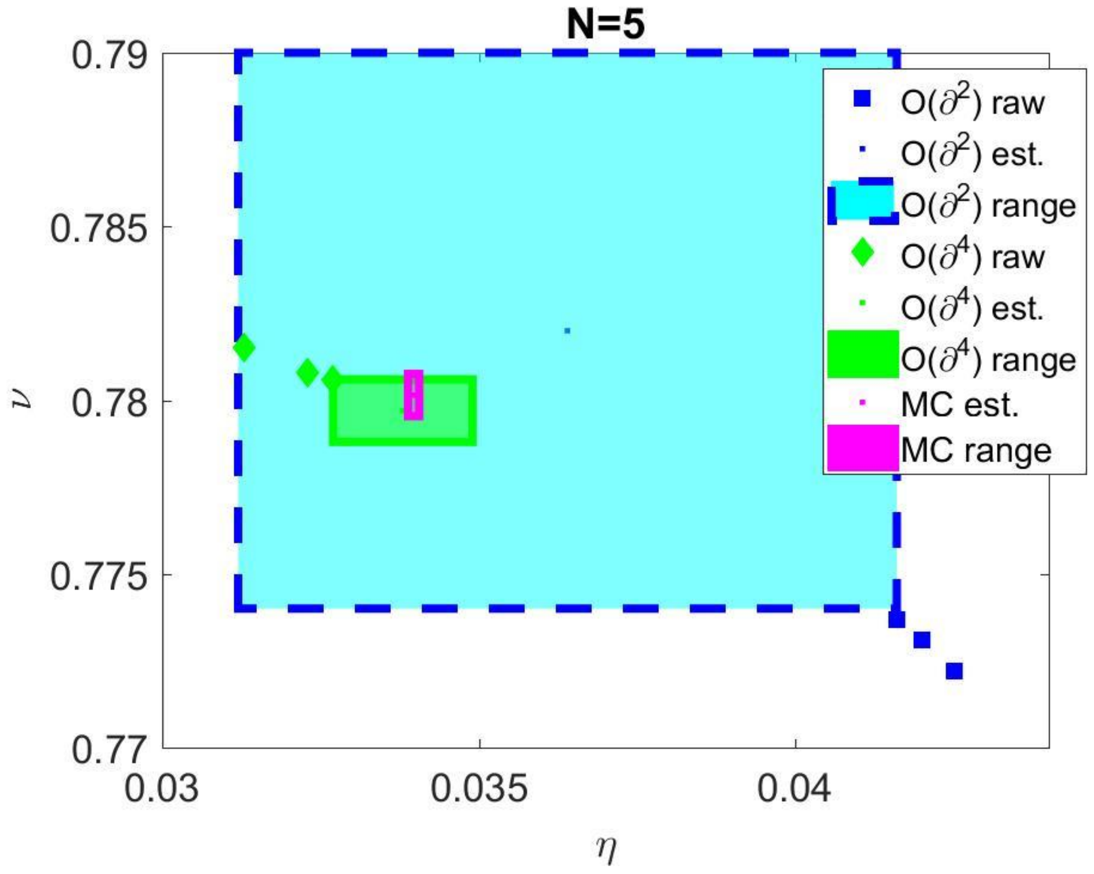
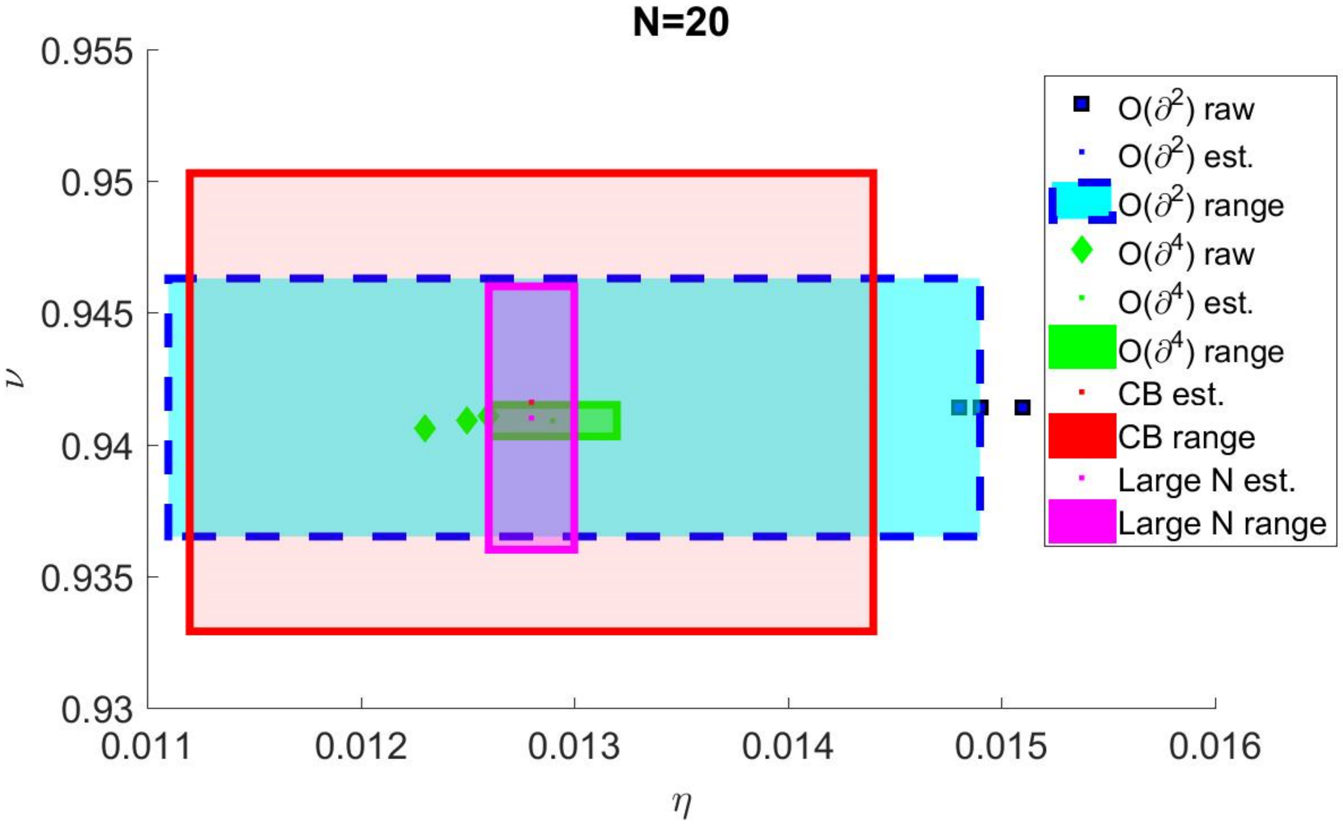


$N = 5$

RESULTS AND BEHAVIOUR OF THE DE

On the precisión of the results

	ν	η	ω
LPA	0.9610	0	0.938
$O(\partial^2)$	0.9414(49)	0.0130(19)	0.887(14)
$O(\partial^4)$	0.9409(6)	0.0129(3)	0.887(2)
CB	0.9416(87)	0.0128(16)	
Six-loop, $d = 3$	0.930	0.014	
Large N	0.941(5)	0.0128(2)	0.888(3)



$N = 20$

Conclusions

- The small parameter ($\sim 1/4$) of the DE allows for the introduction of error bars and PMS is crucial.
- Evidence shows that these error bars are consistent (and self-consisted!).
- We have used it to compute quantities with the highest quality and even quantities not accesible to fixed-point methods.
- DE produces results with a precision comparable to methods taking five orders of magnitude of CPU time! $\left(\frac{\tau_{CPU}\left(\frac{CB}{MC}\right)}{\tau_{CPU}\left(\mathcal{O}\left(\partial^4\right)\right)} > \mathbf{10^5}\right)$.

THANK YOU

?