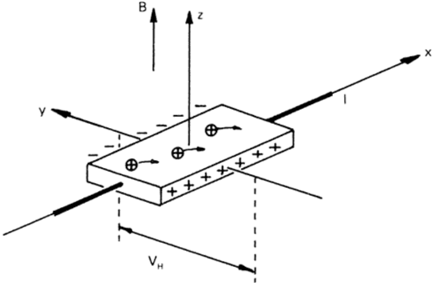
Luttinger Liquids at the Edge of Quantum Hall Systems

Marcello Porta

Joint work with V. Mastropietro (Comm. Math. Phys. 2022)

Integer quantum Hall effect

- Topological phases of matter admit dual bulk and edge descriptions.
- Paradigmatic ex.: Integer quantum Hall effect [von Klitzing *et al.* '80]
  $2d$ insulators exposed to strong magnetic field and in-plane electric field.



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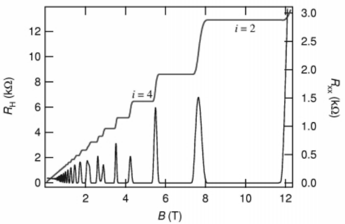
  $2d$ insulators exposed to strong magnetic field and in-plane electric field.
  Linear response: $J = \sigma E + o(E)$ with $\sigma =$ conductivity matrix:

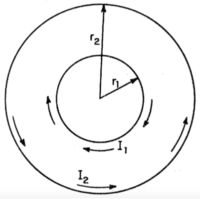
$$\sigma = \begin{pmatrix} 0 & \frac{n}{2\pi} \\ -\frac{n}{2\pi} & 0 \end{pmatrix}\,, \qquad n \in \mathbb{Z}.$$



- Insulators: Fermi energy in a spectral or mobility gap (strong disorder).
- Theory: Laughlin; Thouless, Kohmoto, Nightingale, den Nijs; Bellissard, van Elst, Schulz-Baldes; Avron, Seiler, Simon; Aizenman, Graf...

Bulk-edge duality

- Halperin '82: if $\sigma_{12} \neq 0$, the spectral gap is closed by edge modes.
  Fröhlich '91: edge modes are necessary to guarantee gauge invariance.



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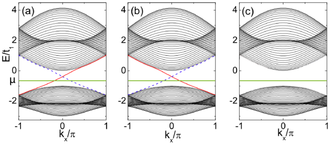
- Let H = lattice Hamiltonian on half-plane $\mathbb{Z} \times \mathbb{N}$: $\quad (H = \bigoplus_{k \in S^1} \hat{H}(k))$



- Red curve: eigenvalue branch, with generalized eigenstates:

$$\varphi_x(k) = e^{ikx_1} \xi_{x_2}(k) \,, \qquad \text{with } |\xi_{x_2}(k)| \leq Ce^{-c|x_2|} \,.$$

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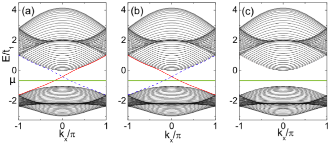

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- Bulk-edge duality: relation between σ_{12} and the edge states of H.

$$\boxed{\sigma_{12} = \sum_{\omega} \frac{\mathrm{sgn}(v_\omega)}{2\pi}}$$

Hatsugai '93; Schulz-Baldes et al. '00; Graf et al. '05... Many-body?

Many-body quantum systems

Many-body systems

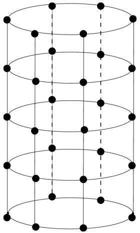
- Interacting lattice many-body Fermi system on cylinder Λ_L, $|\Lambda_L| = L^2$.
- Hamiltonian: $\mathcal{H} = \mathcal{H}_0 + \lambda\mathcal{V}$ with (ρ = spin, sublattice...)

$$\mathcal{H}_0 = \sum_{x,y} \sum_{\rho,\rho'} a^+_{x,\rho} H_{\rho\rho'}(x,y) a^-_{y,\rho'} \ , \quad \mathcal{V} = \sum_{x,y} \sum_{\rho,\rho'} v_{\rho\rho'}(x,y) a^+_{x,\rho} a^+_{y,\rho'} a^-_{y,\rho'} a^-_{x,\rho}$$

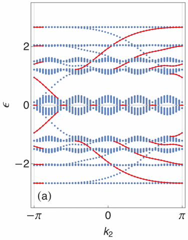
$H(x;y)$, $v(x;y)$ finite-ranged. Transl. inv.: $[H, T_1] = [v, T_1] = 0$.

- Hyp.: H has a bulk gap, and supports edge modes at the Fermi level.

Lattice:



Spectrum of H:



Edge linear response

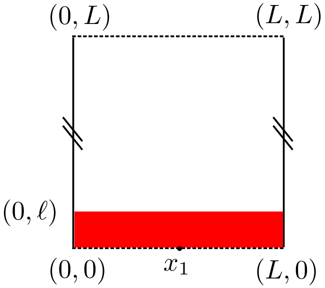
- Consider a slowly varying perturbation: $(0 < \eta, \theta \ll 1,\ t \le 0)$

$$H(t) := H + e^{\eta t}\mu(\theta x)$$

where $\mu(x_1, x_2) \neq 0$ on the $x_2 = 0$ boundary.

- Edge current in a strip of width ℓ: $(1 \ll \ell \ll L)$

$$\mathcal{J}^{\ell}_{x_1} = \sum_{x_2 \le \ell} j_{1,(x_1,x_2)} \qquad (j_{1,x} = \text{horiz. current density})$$



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- Let $\rho(t) =$ time-evolved state, $\rho(-\infty) = \rho_{\beta,L}$. Linear resp.: $(\beta, L \to \infty)$

$$\operatorname{Tr} \mathcal{J}^{\ell}_0 \rho(0) - \operatorname{Tr} \mathcal{J}^{\ell}_0 \rho(-\infty) = \int_{-\pi}^{\pi} \frac{dp}{(2\pi)}\, \hat{\mu}(p,0) \widehat{G}^{\ell}(\eta, \theta p) + \text{h.o.t.}$$

$$\boxed{\widehat{G}^{\ell}(\eta, p) = -i \lim_{a\to\infty} \lim_{\beta, L\to\infty} \int_{-\infty}^{0} dt\, e^{\eta t} \sum_{y : y_2 \le a} e^{ipy_1} \big\langle [n_y(t), \mathcal{J}^{\ell}_0] \big\rangle_{\beta,L}}$$

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Difficulties: control of real-time integral as $\eta \to 0^+$, gapless modes.

Multi-channel Luttinger liquid

- Effective $1+1$ dimensional QFT for edge modes: [Wen '90; Fröhlich '91]

$$\begin{aligned} \mathcal{Z} &= \int D\psi\, e^{-S(\psi)} \\ S(\psi) &= \sum_{\omega} \int_{\mathbb{R}^2} d\underline{x}\, Z_\omega \psi^+_{\underline{x},\omega}(\partial_0 + i v_\omega \partial_1)\psi^-_{\underline{x},\omega} \\ &\quad + \sum_{\omega,\omega'} \lambda_{\omega,\omega'} Z_\omega Z_{\omega'} \int_{\mathbb{R}^2\times\mathbb{R}^2} d\underline{x} d\underline{y}\, n_{\underline{x},\omega} n_{\underline{y},\omega'} v(\underline{x}-\underline{y})\;. \end{aligned}$$

$\psi^\pm_{\underline{x},\omega} =$ Grassmann field, $\underline{x} = (x_0, x_1)$, $\omega =$ chirality (edge modes).

- Z_ω, v_ω chosen to correctly match the scaling of edge correlations.
- Underlying hyp.: if k_F^ω is the Fermi momentum of the ω edge state,

$$(*) \qquad k_F^{\omega_1} - k_F^{\omega_2} = k_F^{\omega_3} - k_F^{\omega_4} \quad \text{only for edge modes equal in pairs.}$$

Otherwise generically false, in absence of special sym. $(k_F^{\omega_1} \equiv k_F^{\omega_1}(\mu))$.

Anomalous Ward identities

- The model is formally covariant under local chiral gauge transformations:

$$\psi^{\pm}_{\underline{x},\omega} \underset{\text{Jacobian 1}}{\longrightarrow} e^{\pm i\alpha_\omega(\underline{x})}\psi^{\pm}_{\underline{x},\omega} \quad \underset{\text{Formally!}}{\Longrightarrow} \quad \mathcal{Z}(A_\omega) = \mathcal{Z}(A_\omega + D_\omega \alpha_\omega)$$

with $D_\omega = \partial_0 + i v_\omega \partial_1$. Ward identity: $\langle \hat{n}_{\underline{p},\omega}\, ; \hat{n}_{-\underline{p},\omega'} \rangle = 0.$ (?)

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- The symmetry is broken by unavoidable regularizations, which produce anomalies in the WIs as cutoffs are removed. Correct result:

$$\langle \hat{n}_{\underline{p},\omega}\,;\hat{n}_{-\underline{p},\omega'}\rangle = T_{\omega,\omega'}(\underline{p})\frac{1}{Z^2_{\omega'}}\frac{1}{4\pi|v_{\omega'}|}\frac{ip_0 + v_{\omega'}p_1}{-ip_0 + v_{\omega'}p_1}$$

$$\left(\frac{1}{T(\underline{p})}\right)_{\omega,\omega'} = \delta_{\omega,\omega'} + \frac{ip_0 + v_\omega p_1}{-ip_0 + v_\omega p_1}\frac{1}{4\pi|v_\omega|}\frac{1}{Z_\omega}\lambda_{\omega,\omega'}Z_{\omega'}\ .$$

- Similar relations can be found for other correlations, e.g. for the vertex function $\langle \hat{n}_{\underline{p},\omega}\,;\hat{\psi}^-_{\underline{k},\omega'}\,;\hat{\psi}^+_{\underline{k}+\underline{p},\omega'}\rangle$.

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- Idea: combine these exact relations with a RG analysis of lattice model.

Main result: interacting edge transport

- We consider $\mathcal{H} = \mathcal{H}_0 + \lambda\mathcal{V}$, transl. inv. in the direction of the edge, with $\mathcal{H}_0$ displaying arbitrarily many edge modes, under the assumption $(*)$.

Theorem (V. Mastropietro, M. P. - Comm. Math. Phys. 2022)

For $|\lambda|$ small, the $\beta, L \to \infty$ edge conductance is, for $\underline{p} = (\eta, p)$ and $|\underline{p}| \ll 1$:

$$\widehat{G}^{\ell}(\underline{p}) = \sum_{\omega} g_{\omega}(\underline{p}) \frac{v_{\omega} p}{-i\eta + v_{\omega} p} \frac{sgn(v_{\omega})}{2\pi} + o(1)$$

where

$$g_{\omega}(\underline{p}) = \Big(\big(1 + \frac{1}{4\pi|v|}\Lambda\big) \frac{1}{1 + \frac{1}{4\pi|v|}\omega(\underline{p})\Lambda} \Big)_{\omega\omega}$$

with: $v_{\omega} \equiv v_{\omega}(\lambda)$, $v = diag(v_{\omega})$, $\Lambda_{\omega\omega'} = O(\lambda)$, $\omega(\underline{p}) = diag\Big(\frac{-i\eta + v_{\omega} p_1}{i\eta + v_{\omega} p_1}\Big)$.
In particular,

$$\lim_{\ell\to\infty} \lim_{p\to 0} \lim_{\eta\to 0^+} \widehat{G}^{\ell}(\underline{p}) = \sum_{\omega} \frac{sgn(v_{\omega})}{2\pi}\ .$$

Remarks

- Previous work: Antinucci, Mastropietro, P. '18; Mastropietro, P. '18.
- Bulk quantiz.: Hastings, Michalakis '15; Giuliani, Mastropietro, P. '16.
- Main technical tools for edge transport:
  - Rigorous RG analysis of the edge correlations, scaling limit.

    [Gawedzki, Kupiainen, Feldman, Magnen, Rivasseau, Sénéor, Benfatto, Gallavotti, Balaban, Knörrer, Salmhofer, Trubowitz, Kopper, Brydges, Slade...]
  - Ward identities, to prove universality & vanishing of beta function.
    Generalizing [Benfatto, Mastropietro '04], inspired by [Metzner, Di Castro '93]
- Similar ideas have been used to:
  - Prove the universality of the longitudinal conductivity of graphene [Giuliani, Mastropietro, P. - PRB11, CMP12]
  - Construct the topological phase diagram of the Haldane-Hubbard model [Giuliani, Jauslin, Mastropietro, P. - PRB16, JSP19]
  - Prove the universality of the chiral anomaly for Weyl semimetals [Giuliani, Mastropietro, P. - CMP21]

Sketch of the proof

Grassmann QFT

- Grassmann representation of the QFT:

$$\mathcal{Z}_{\beta,L} = \mathbb{E}_g\big(e^{V(\psi)}\big)$$

where:

  - $\psi \equiv \psi^{\pm}_{\mathbf{x}}$ is a complex Grassmann field, for $\mathbf{x} = (x_0, x) \in [0,\beta) \times \Lambda_L$
  - $\mathbb{E}_g$ is a Gaussian integration, with propagator:

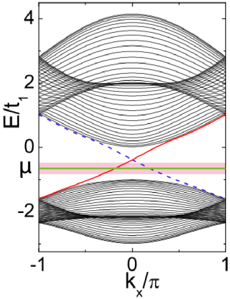
$$\mathbb{E}_g(\psi^+_{\mathbf{x}}\psi^-_{\mathbf{y}}) = \frac{1}{\beta} \sum_{k_0 \in \frac{2\pi}{\beta}(\mathbb{Z}+\frac{1}{2})} e^{ik_0(x_0-y_0)} \frac{1}{-ik_0 + H - \mu}(x,y) =: g(\mathbf{x},\mathbf{y}) \, .$$

  - $V(\psi)$ is a quartic interaction:

$$V(\psi) = \lambda \int_{[0,\beta)^2} dx_0 dy_0 \sum_{x,y\in\Lambda_L} \psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}\psi^-_{\mathbf{y}}\delta(x_0 - y_0)v(x-y) \, .$$

Reduction to an effective $1d$ model

- Integration of bulk degrees of freedom. Write $g = g_1 + g_2$, and correspondingly $\psi = \psi_1 + \psi_2$. g_2 : energies away from μ.



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- ψ_2 is integrated out via convergent exp.: [Brydges-Battle-Federbush]

$$\mathbb{E}_g\big(e^{V(\psi)}\big) = \mathbb{E}_{g_1}\mathbb{E}_{g_2}\big(e^{V(\psi_1+\psi_2)}\big) = \mathbb{E}_{g_1}\big(e^{V_{\text{eff}}(\psi_1)}\big)\;.$$

The field ψ_1 can be parametrized in terms of a truly $1+1$ dim. field:

$$\psi_{1,\underline{k}}(x_2) = \sum_{\omega} \xi^{\omega}_{k_1}(x_2)\varphi_{\omega,\underline{k}},$$

where $\xi^{\omega}_{k_1}(x_2)$ is the eigenstate of the ω-edge mode and:

$$\mathbb{E}_{\varphi}(\varphi^{+}_{\omega,\underline{k}}\varphi^{-}_{\omega',\underline{k}}) = \delta_{\omega,\omega'}\hat{g}_{\omega}(\underline{k})$$

$$\hat{g}_{\omega}(\underline{k}) = \frac{\chi(|\varepsilon_{\omega}(k_1)-\mu| \leq \delta)}{-ik_0 + \varepsilon_{\omega}(k_1) - \mu}$$

Massless propagator: close to k_F^{ω}, $\varepsilon_{\omega}(k_1) - \mu \simeq v_{\omega}(k_1 - k_F^{\omega})$.

Multiscale integration

- We end up with a (complicated, but explicit) $1d$ effective theory:

$$\mathbb{E}_g\big(e^{V(\psi)}\big) = \int \nu(d\varphi)e^{\mathcal{V}(\varphi)}$$

where $\nu = \prod_\omega \nu_\omega$ and ν_ω has propagator $g_\omega(\underline{k})$.

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- The massless $1d$ field is decomposed in scales:

$$\varphi_\omega = \sum_{h=h_\beta}^{0} \varphi_\omega^{(h)} \qquad g_\omega^{(h)}(\underline{k}) \simeq \frac{1}{Z_{\omega,h}} \frac{\chi(\|\underline{k} - \underline{k}_F^\omega\| \sim 2^h)}{-ik_0 + v_{\omega,h}(k_1 - k_F^\omega)}$$

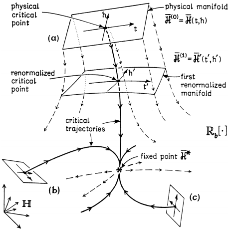
and integrated iteratively, via the Gallavotti-Nicolò tree expansion:

$$\mathbb{E}_{\varphi^{(h_\beta)}+\ldots+\varphi^{(0)}}\Big(e^{\mathcal{V}\left(\varphi^{(h_\beta)}+\ldots+\varphi^{(0)}\right)}\Big) = \mathcal{Z}_h \mathbb{E}_{\varphi^{(h_\beta)}+\ldots+\varphi^{(h)}}\Big(e^{\mathcal{V}^{(h)}\left(\sqrt{Z_h}\varphi^{(\leq h)}\right)}\Big)$$

where $(\mathcal{V}^{(h)}, Z_h, v_h)$ solve a discrete recursion equation. In particular:

$$\mathcal{V}_4^{(h)}(\xi) = \sum_{\omega,\omega'} \lambda_{\omega,\omega',h} \int dx_0 \sum_{x_1} \xi^+_{\underline{x},\omega} \xi^-_{\underline{x},\omega} \xi^+_{\underline{x},\omega'} \xi^-_{\underline{x},\omega'}$$

RG flow



- The marginal direction associated to $Z_{h,\omega}$, $v_{h,\omega}$ and to the effective couplings $\lambda_{h,\omega,\omega'}$ is controlled thanks to a key aspect of integrability:

$$\lambda_{h,\omega,\omega'} = \lambda_{h+1,\omega,\omega'} + \beta^{\lambda}_{h+1,\omega,\omega'} \qquad \beta^{\lambda}_{h+1,\omega,\omega'} = O(\lambda^2_{h+1} 2^{\theta h})$$

(asymptotic) vanishing of the beta function.

Proof based on a generalization of the method of [Benfatto-Mastropietro]

- Flow of the running coupling constants:

$$\lambda_{h,\omega,\omega'} = C_{\omega,\omega'}\lambda + O(\lambda^2) , \qquad Z_{h,\omega} \sim 2^{-h\eta_\omega \lambda^2} , \qquad v_{h,\omega} - v_\omega = O(\lambda^2) .$$

Comparison with the scaling limit description

- Representation of the edge conductance via RG (singular + regular):

$$G^{\ell}(\underline{p}) = \big(\vec{Z}_0, D^{\mathrm{LM}}(\underline{p})\vec{Z}_1\big) + R^{\ell}(\underline{p})\;, \qquad \underline{p} = (\eta, p_1) \equiv (p_0, p_1)$$

where:

  - $(\vec{A}, \vec{B}) = \sum_{\omega} A_{\omega} B_{\omega}$ (sum over edge modes at $x_2 = 0$)
  - $D^{\mathrm{LM}}_{\omega,\omega'}(\underline{p}) = \langle \hat{n}_{\underline{p},\omega}\, ; \hat{n}_{-\underline{p},\omega'} \rangle$
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  - $Z_{\mu,\omega}$ are renormalized parameters
  - $R^{\ell}(\underline{p})$ is continuous at $\underline{p} = 0$.

- From $G^{\ell}(\eta, 0) = 0$, we determine $R^{\ell}(\underline{0})$. We get:

$$\boxed{\lim_{\ell\to\infty} \lim_{p_1\to 0} \lim_{p_0\to 0^+} G^{\ell}(\underline{p}) = \big(\vec{Z}_0, \mathcal{A}\vec{Z}_1\big)}$$

with:

$$\mathcal{A} := \lim_{p_1\to 0} \lim_{p_0\to 0^+} D^{\mathrm{LM}}(\underline{p}) - \lim_{p_0\to 0^+} \lim_{p_1\to 0} D^{\mathrm{LM}}(\underline{p}) \,.$$

Combining the vertex WIs [Neglecting x_2 labels]

- From the conservation of the lattice current:

$$p_\mu \langle \mathbf{T}\, j_{\mu,\underline{p}}\, ; \hat{a}^-_{\underline{k}}\, \hat{a}^+_{\underline{k}+\underline{p}} \rangle = \langle \mathbf{T}\, \hat{a}^-_{\underline{k}}\, \hat{a}^+_{\underline{k}} \rangle - \langle \mathbf{T}\, \hat{a}^-_{\underline{k}+\underline{p}} \hat{a}^+_{\underline{k}+\underline{p}} \rangle\ .$$

- A similar (anomalous) WI holds for the effective QFT:

$$\langle \hat{n}_{\underline{p},\omega}\ ; \hat{\psi}^-_{\underline{k},\omega'}\ ; \hat{\psi}^+_{\underline{k}+\underline{p},\omega'} \rangle = T_{\omega,\omega'}(\underline{p}) \frac{1}{Z_{\omega'} D_{\omega'}(\underline{p})} \big(\langle \hat{\psi}^-_{\underline{k},\omega'} \hat{\psi}^+_{\underline{k},\omega'} \rangle - \langle \hat{\psi}^-_{\underline{k}+\underline{p},\omega'} \hat{\psi}^+_{\underline{k}+\underline{p},\omega'} \rangle \big)\ .$$

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$$\langle \hat{n}_{\underline{p},\omega}\, ; \hat{\psi}^-_{\underline{k},\omega'}\, ; \hat{\psi}^+_{\underline{k}+\underline{p},\omega'} \rangle = T_{\omega,\omega'}(\underline{p}) \frac{1}{Z_{\omega'} D_{\omega'}(\underline{p})} \big(\langle \hat{\psi}^-_{\underline{k},\omega'} \hat{\psi}^+_{\underline{k},\omega'} \rangle - \langle \hat{\psi}^-_{\underline{k}+\underline{p},\omega'} \hat{\psi}^+_{\underline{k}+\underline{p},\omega'} \rangle \big) \; .$$

- Using that, for $\underline{p}$ small and for $\underline{k}' = \underline{k} - \underline{k}^\omega_F$ small:

$$\langle \mathbf{T}\, j_{\mu,\underline{p}}\, ; \hat{a}^-_{\underline{k}} \hat{a}^+_{\underline{k}+\underline{p}} \rangle \simeq \sum_{\omega'} Z_{\mu,\omega'} \langle \hat{n}_{\underline{p},\omega'}\, ; \hat{\psi}^-_{\underline{k},\omega}\, ; \hat{\psi}^+_{\underline{k}+\underline{p},\omega} \rangle \; , \quad \langle \mathbf{T}\, \hat{a}^-_{\underline{k}} \hat{a}^+_{\underline{k}} \rangle \simeq \langle \hat{\psi}^-_{\underline{k},\omega} \hat{\psi}^+_{\underline{k},\omega} \rangle$$

Two eqs. for QFT correlations! Constraints on renormalized parameters:

$$\lim_{p_1 \to 0} \lim_{p_0 \to 0^+} T^T(\underline{p}) \vec{Z}_0 = \vec{Z} \; , \qquad \lim_{p_0 \to 0^+} \lim_{p_1 \to 0} T^T(\underline{p}) \vec{Z}_1 = v \vec{Z} \; .$$

Plugging in $G = (\vec{Z}_0, \mathcal{A}\vec{Z}_1)$, universality (remarkably) follows. ■

Conclusions and open problems

- We proved the universality of edge transport, for weakly interacting quantum Hall systems on a cylinder.
- Three main ingredients:
  - (i) Analytic continuation to imaginary times (a.k.a. Wick rotation);
  - (ii) Rigorous RG analysis and construction of the scaling limit;
  - (iii) Ward identities, to relate scaling limit and lattice model.
- As usual with RG, the method is robust, and could be used to attack various other questions.
  - Other transport coefficients (two-terminal conductance)?
  - Effect of disorder?
  - Fractional quantization?

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- Thank you!