From holographic RG to information flows

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Complexity and Topology in Quantum Matter

AdS/CFT correspondence

Branes, duality

RG flows: Breaking conformal symmetry

Relevant operators

Flows to IR fixed points and C theorem

Confining flows + Chiral symmetry breaking

Overview

I. AdS/CFT correspondence and Renormalization Group

II. Information flows and improved machine learning

Gauge/Gravity Duality

Bring together fundamental and empirical aspects of physics

Fundamental:

String theory: Unification of interactions, quantization of gravity

Empirical:

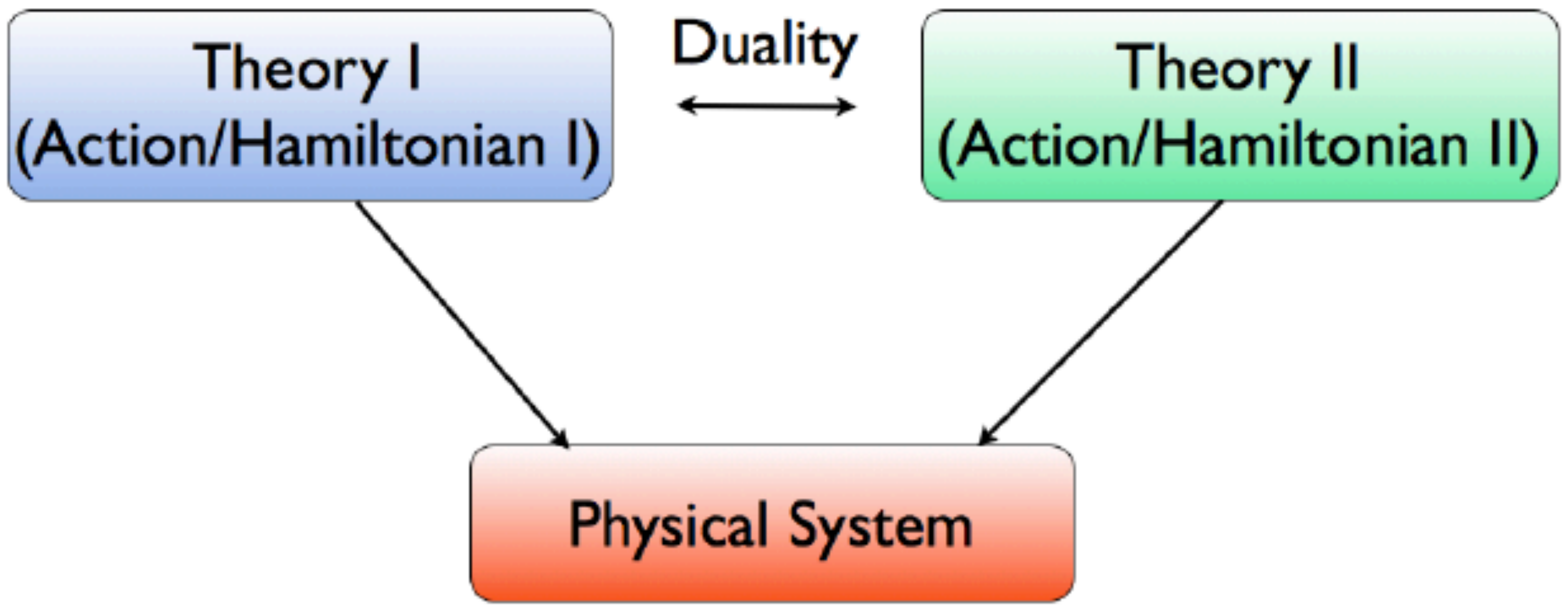
New method for describing strongly coupled systems

(comparison to experiment)

Duality: A physical theory has two equivalent formulations

Same dynamics

One-to-one map between states



Gauge/Gravity Duality:

Gauge Theory
Quantum Field Theory

$\Leftrightarrow$

Gravity theory
in higher dimensions

Gauge/gravity duality

- Conjecture which follows from a low-energy limit of string theory

- Duality:

  Quantum field theory at strong coupling

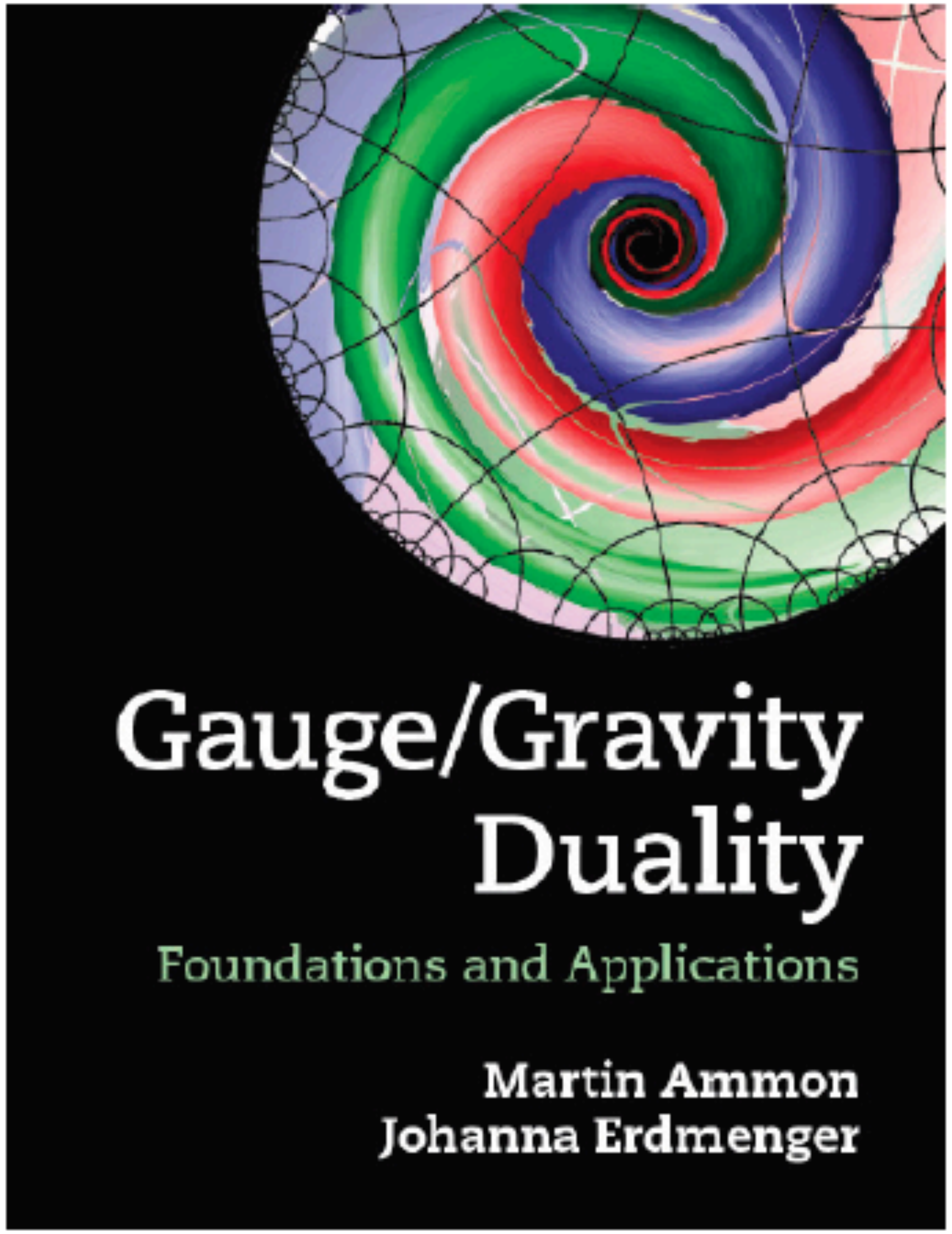
  $\Leftrightarrow$ Theory of gravitation at weak coupling

- Holography:

  Quantum field theory in d dimensions

  $\Leftrightarrow$ Gravitational theory in $d+1$ dimensions

Book on gauge/gravity duality



Examples for applications

- Low-energy QCD Chiral symmetry breaking, mesons

- Quark-gluon plasma
  Shear viscosity over entropy density, $\eta/s = 1/(4\pi)\hbar/k_B$

  Kovtun, Son, Starinets 2004

- Condensed matter physics
  - Non-Fermi liquids
  - Superconductivity
  - Quantum phase transitions
  - Interactions with magnetic impurities

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Relation to quantum information

Entanglement entropy

Consider product Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

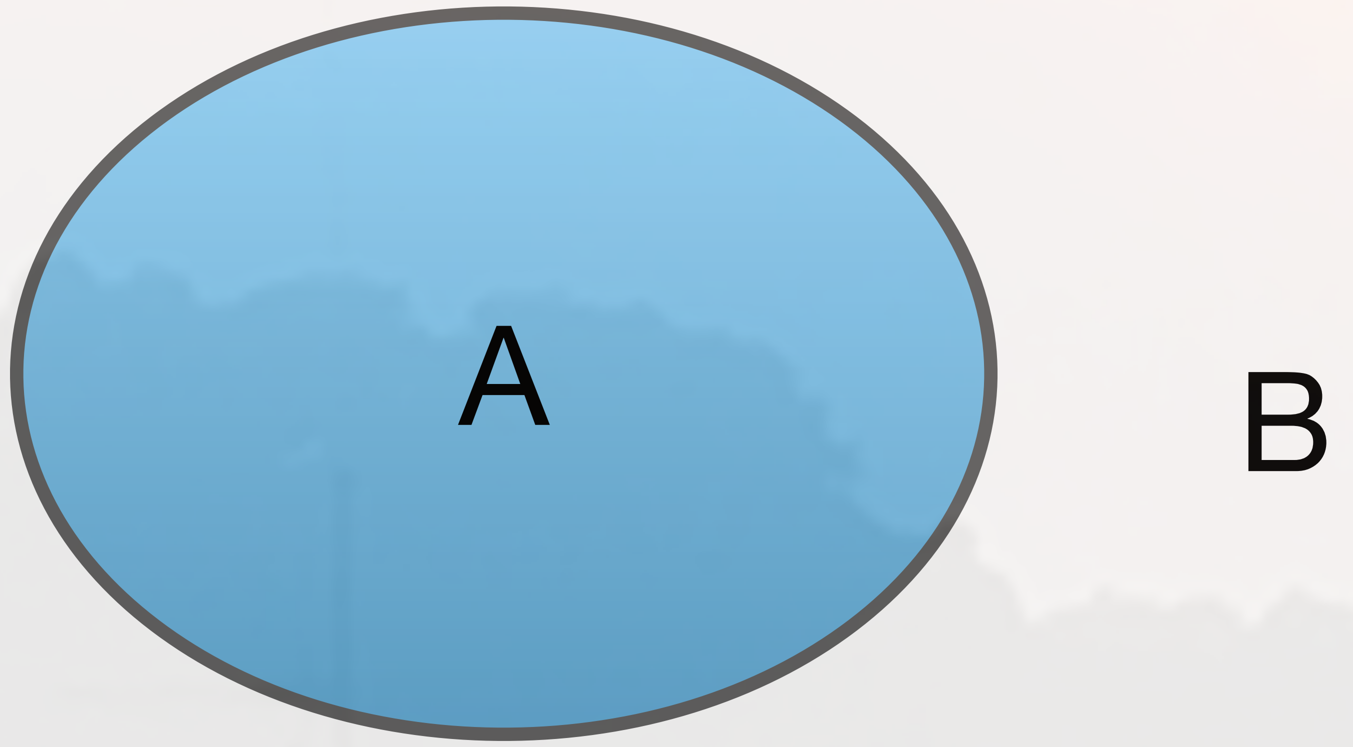
Reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho_{\mathrm{tot}}$$

Entanglement entropy

$$S_A = -\mathrm{Tr}_A \rho_A \ln \rho_A$$

Entanglement for space regions



AdS/CFT correspondence

AdS: Anti-de Sitter space, CFT: Conformal field theory

$\mathcal{N} = 4$ SU(N) Super Yang-Mills theory

Anti-de Sitter space:

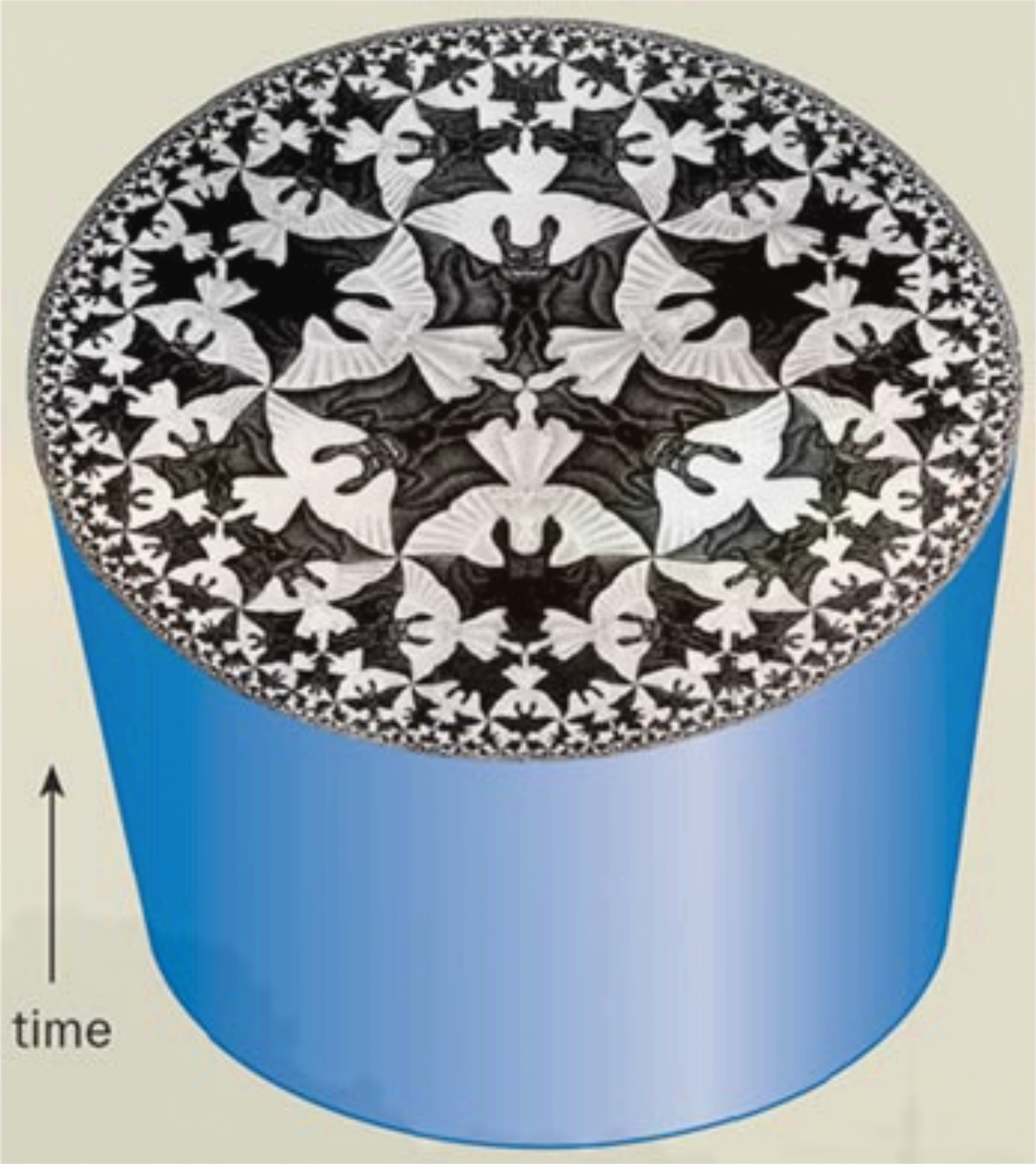
Space of constant negative curvature, has a boundary

In total: $AdS_5 \times S^5$

Isometries of AdS$_{d+1}$ and symmetry of operators in a CFT$_d$:

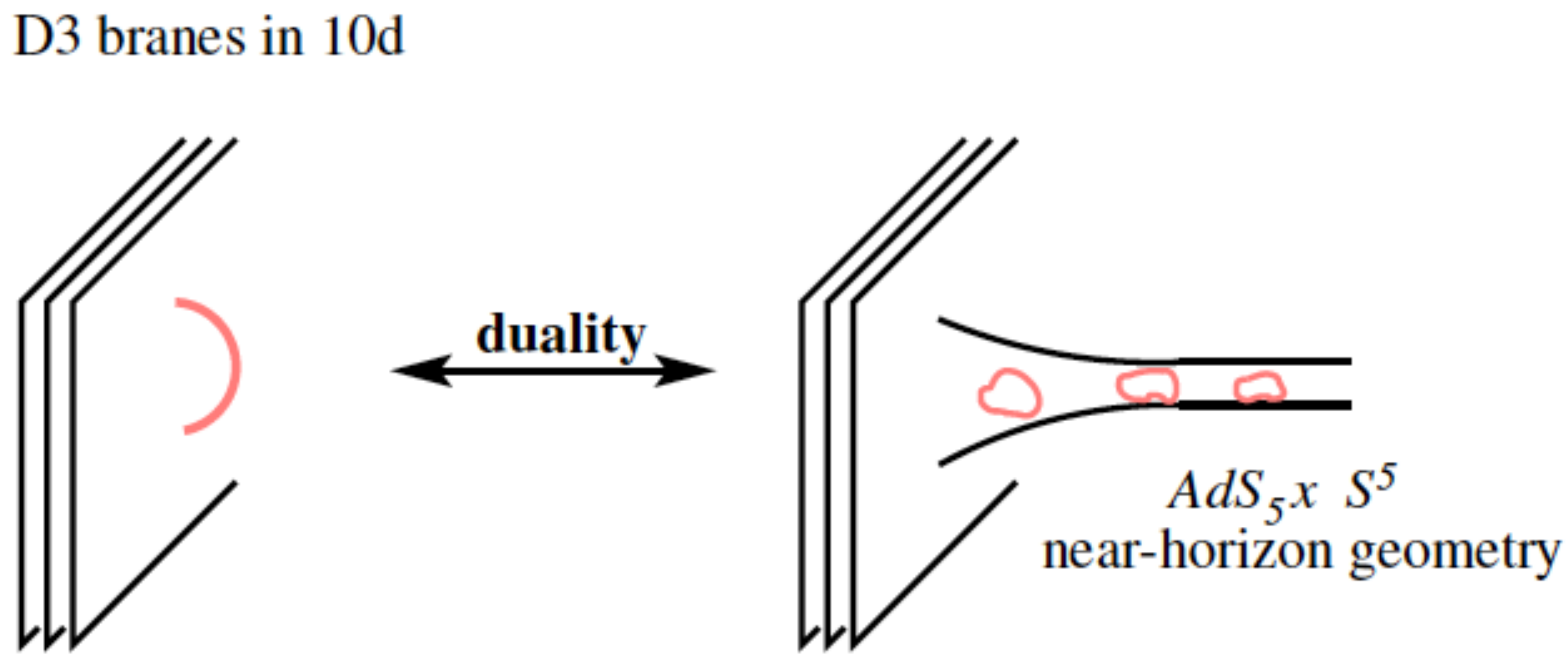
$$SO(d+1,1)$$

(Euclidean signature)



Quelle: Institute of Physics, Copyright: C. Escher

String theory origin of the AdS/CFT correspondence

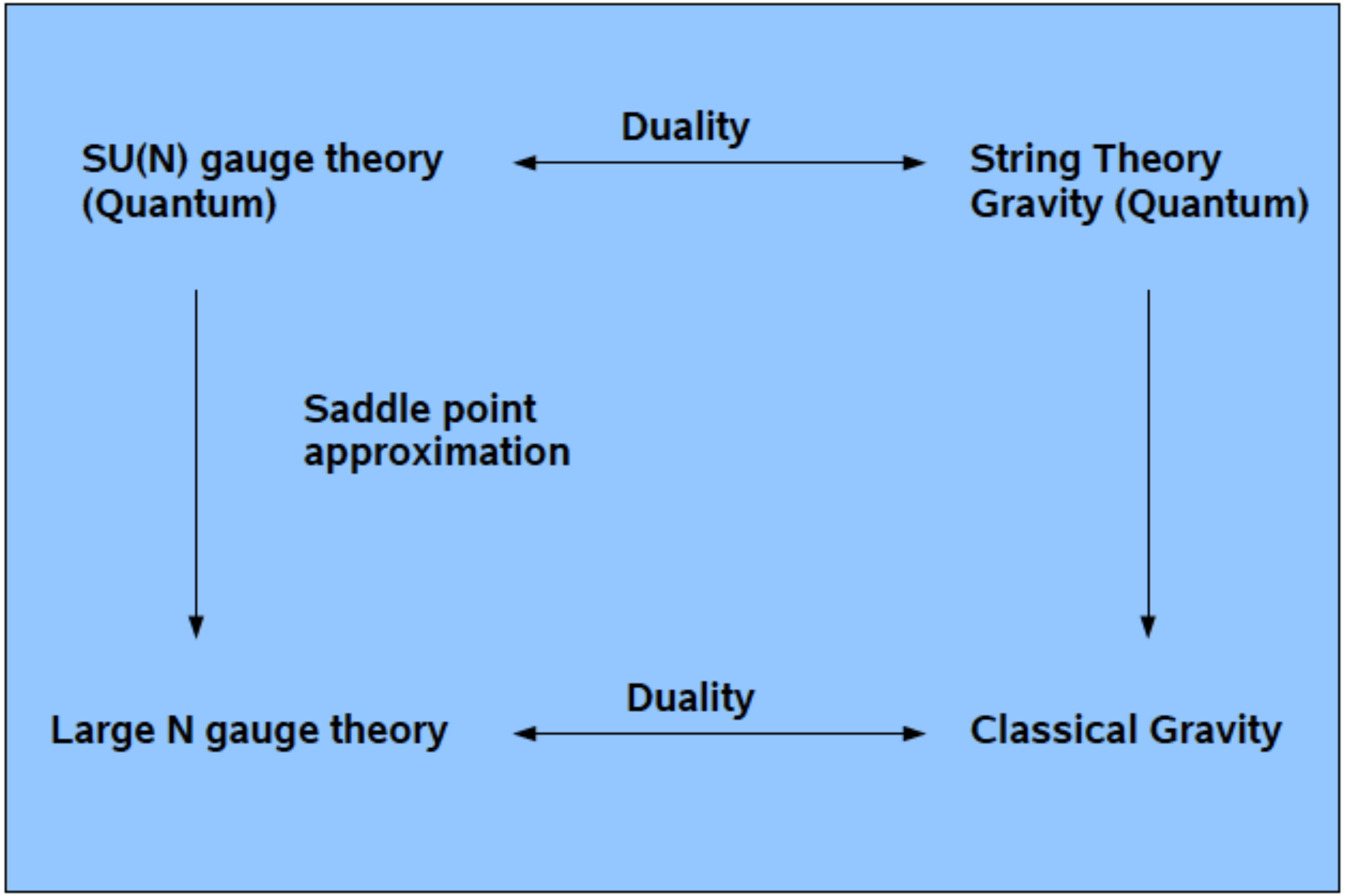


$\Downarrow$ Low energy limit

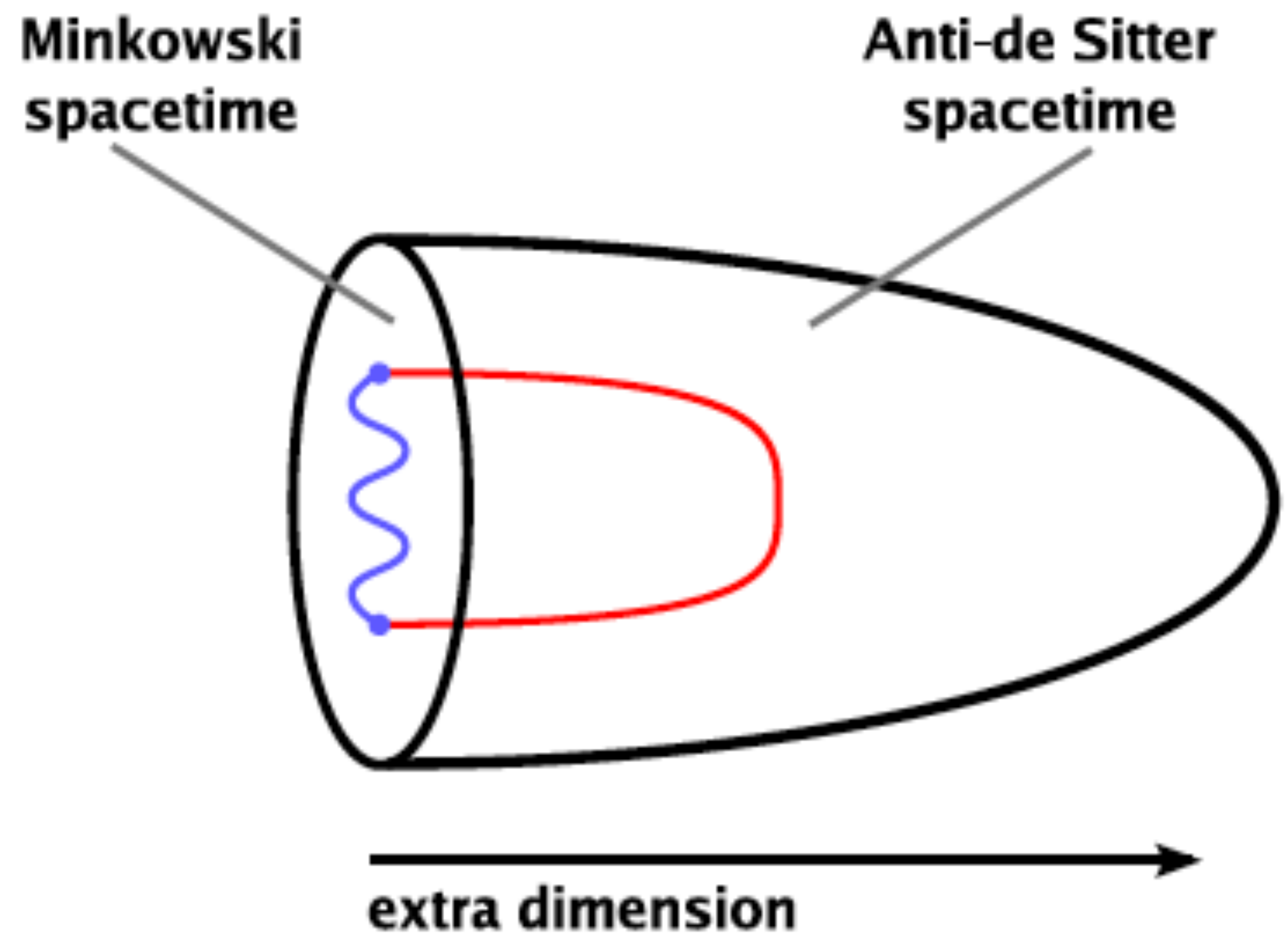
Supersymmetric $SU(N)$ gauge theory in four dimensions ($N \to \infty$)

Supergravity on the space $AdS_5 \times S^5$

Gauge/Gravity Duality: Low-energy limit



Gauge/Gravity Duality



'Dictionary' Gauge invariant field theory operators
$\Leftrightarrow$ Classical fields in gravity theory

Symmetry properties coincide, generating functionals are identified

Test: (e.g.) Calculation of correlation functions

AdS/CFT correspondence

- Field-operator correspondence:

$$\langle e^{\int d^d x \, \phi_0(\vec{x}) \mathcal{O}(\vec{x})} \rangle_{CFT} = Z_{sugra} \Big|_{\phi(0,\vec{x}) = \phi_0(\vec{x})}$$

Generating functional for correlation functions of particular composite operators in the quantum field theory

coincides with

Classical tree diagram generating functional in supergravity

Holographic Renormalization Group Flows

Original AdS/CFT correspondence involves $\mathcal{N} = 4$ SU(N) Super Yang-Mills theory

Field content: One gauge field, four fermions, six real scalars

Beta function vanishes identically to all orders in perturbation theory

Theory at UV fixed point

Perturb by relevant operator preserving $\mathcal{N} = 1$ supersymmetry

$$W_{\rm LS} \equiv h \, {\rm Tr} \, (\Phi_3 \, [\Phi_1 \, , \, \Phi_2]) \, + \, \frac{m}{2} \, {\rm Tr} \, \Phi_3^2$$ Leigh-Strassler flow

IR fixed point: $\beta(g) \sim 2N \, (\gamma_1 + \gamma_2 + \gamma_3)$ $\quad \beta_h = \gamma_1 + \gamma_2 + \gamma_3 ,$ $\quad \beta_m = 1 - 2 \, \gamma_3$

$$\gamma_1 = \gamma_2 = - \frac{\gamma_3}{2} = - \frac{1}{4}$$

Holographic Renormalization Group Flows

Leigh-Strassler flow

$$W_{\rm LS} \equiv h\,{\rm Tr}\,(\Phi_3\,[\Phi_1\,,\,\Phi_2]) + \frac{m}{2}\,{\rm Tr}\,\Phi_3^2$$

C theorem:

$$\frac{a_{\rm IR}}{a_{\rm UV}} = \frac{c_{\rm IR}}{c_{\rm UV}} = \frac{27}{32}$$

For coefficients of conformal anomaly in conformal Ward identity

$$\langle T_\mu{}^\mu \rangle = \frac{c}{8\pi^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2\right), \qquad c = \frac{N^2}{4}$$

Holographic Renormalization Group Flows

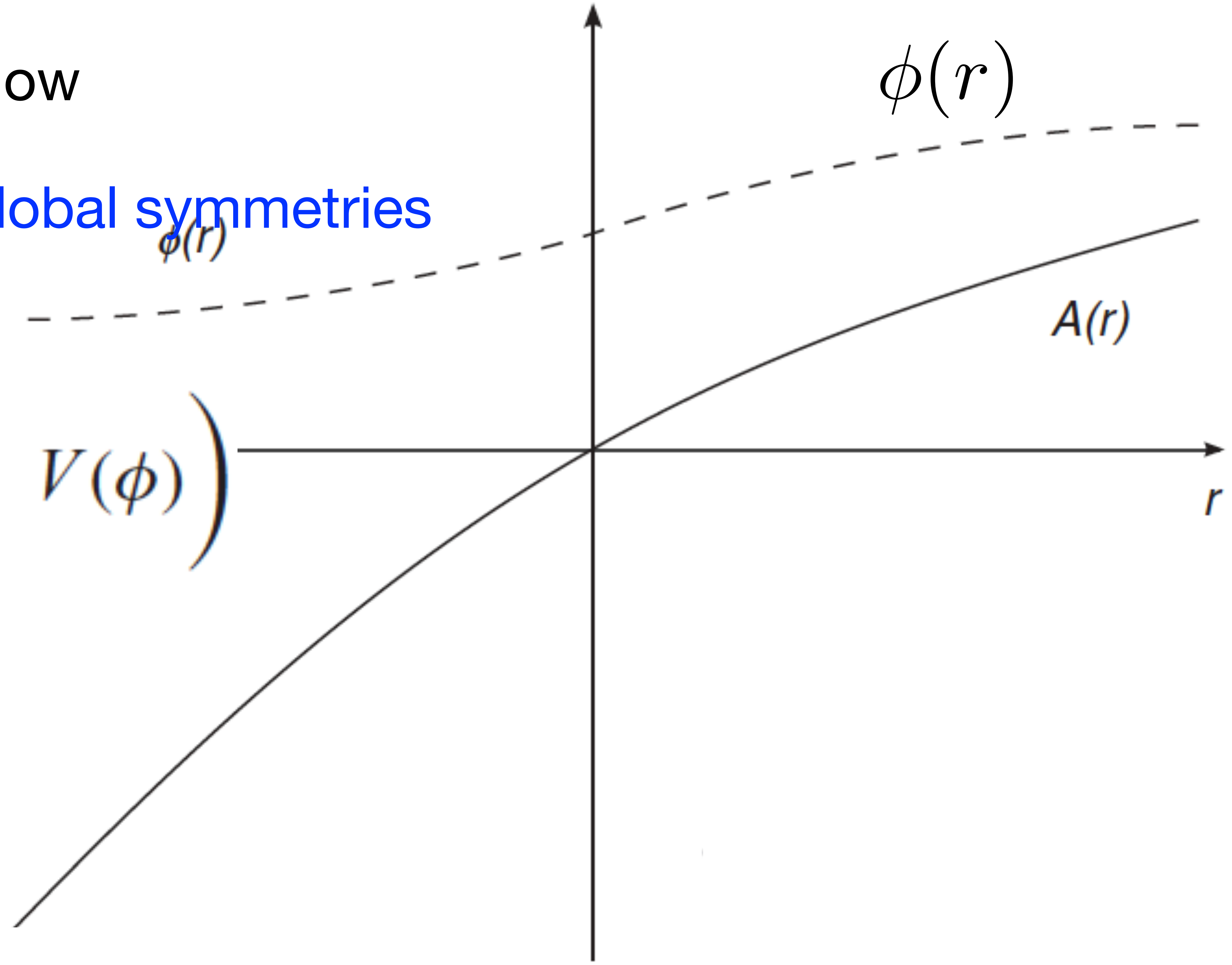
Gravity side of correspondence: Dual of Leigh-Strassler flow

Look for a solution of supergravity preserving the same global symmetries

Domain wall flow:

$$S = \int \mathrm{d}^{d+1}x \sqrt{-g} \left(\frac{R}{16\pi G} - \frac{1}{2} \partial_m\phi\, \partial^m\phi - V(\phi) \right)$$

$$\mathrm{d}s^2 = e^{2A(r)} \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + dr^2$$



Conformal anomaly at IR fixed point exactly reproduced

$$\langle T_\mu{}^\mu \rangle = \frac{L^3}{64\pi G_5} \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right)$$

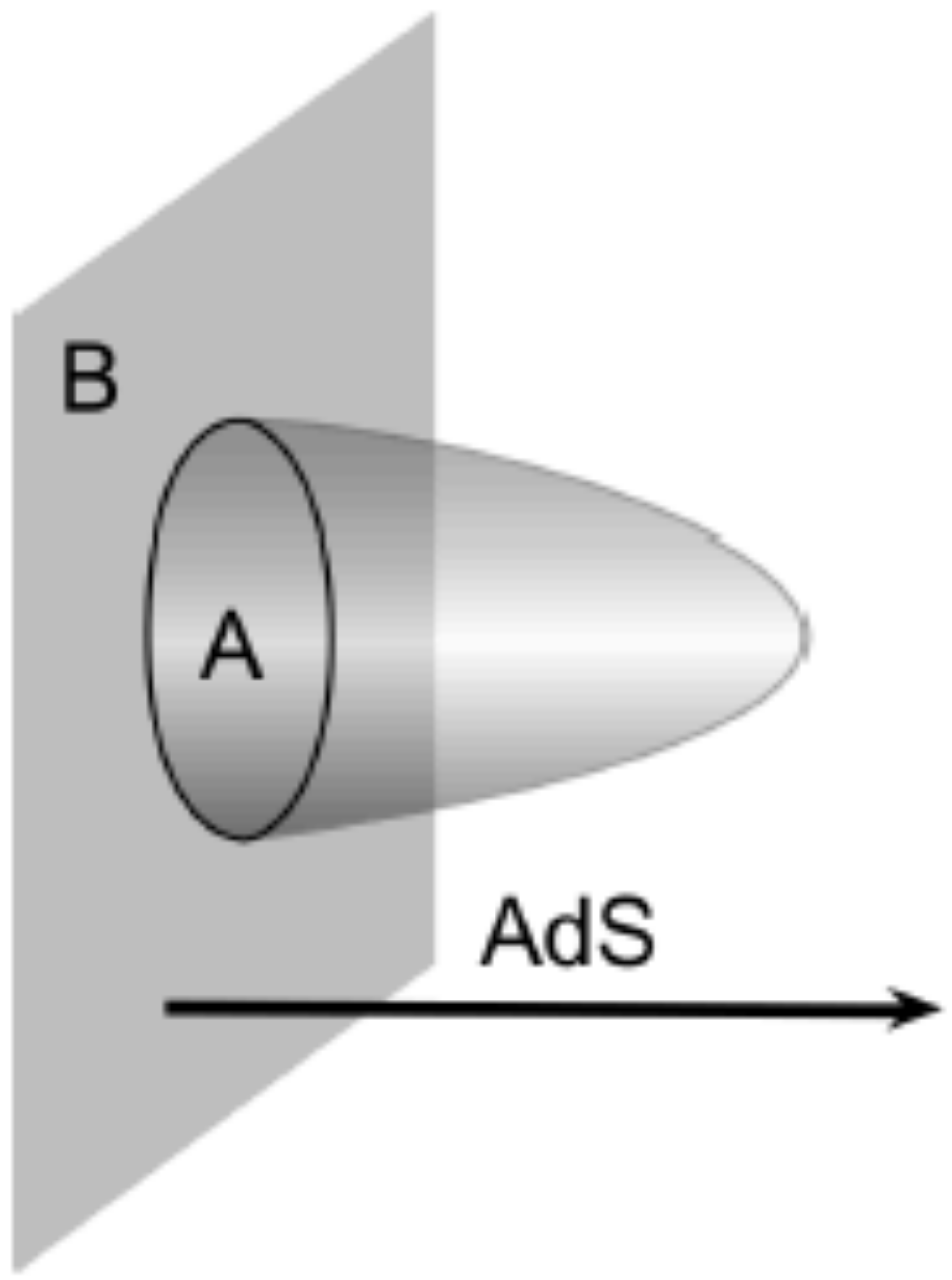
First order RG equation from gradient flow

$$V(\phi) = \frac{1}{2} \left(\frac{\mathrm{d}W}{\mathrm{d}r} \right)^2 - \frac{d}{d-1} W^2 \qquad \sqrt{8\pi G} \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\mathrm{d}W}{\mathrm{d}\phi}, \quad A' = -\frac{\sqrt{8\pi G}}{(d-1)} W$$

Entanglement entropy in Gauge/Gravity Duality

(Ryu, Takayanagi Phys.Rev.Lett. 96 (2006) 181602)

Leading term in entanglement entropy given by
area of minimal surface in holographic dimension



Entanglement entropy: Quantum field theory

Conformal field theory in 1+1 dimensions (Cardy, Calabrese, J.Stat.Mech. 0406 (2004) P06002):

$$S = \frac{c}{3}\ln(\ell\Lambda)$$

Reproduced by Ryu-Takayanagi result

$\Lambda \propto 1/\epsilon$, ϵ boundary cut-off in radial direction

$c = 3L/(2G_3)$ central charge of CFT

Finite temperature (at small ℓ):

$$S(\ell) = \frac{c}{3}\ln\left(\frac{1}{\pi\epsilon T}\sinh(2\pi\ell T)\right)$$

Part II: Quantifying information flows by relative entropy

Deep neural networks and RG

J.E., Grosvenor, Jefferson 2107.06898 (SciPost)

Relative entropy - Kullback-Leibler divergence

$$D(p||q) = \mathrm{tr}_{\mathbf{x}}\, p(\mathbf{x}) \ln \frac{p(\mathbf{x})}{q(\mathbf{x})}$$

quantifies how much the target distribution *q* differs from initial distribution *p*

Relative entropy for 1d Ising model

Decimation RG

$$Z = \sum_{\sigma_i \in \mathcal{H}} e^{-H(\sigma_i)} = \sum_{\sigma'_j \in \mathcal{H}'} \sum_{\sigma_i \in \mathcal{H}\setminus\mathcal{H}'} e^{-H(\sigma_i)} = \sum_{\sigma'_j \in \mathcal{H}'} e^{-H'(\sigma'_i)} = Z'$$

$$p'(\sigma'_j) = \sum_{\sigma_i \in \mathcal{H}\setminus\mathcal{H}'} p(\sigma_i) = \frac{1}{Z} \sum_{\sigma_i \in \mathcal{H}\setminus\mathcal{H}'} e^{-H(\sigma_i)} =: \frac{1}{Z} e^{-H'(\sigma'_j)}$$

Relative Entropy

$$S(p||p') := \sum_{\sigma_i \in \mathcal{H}} p(\sigma_i) \ln\left(\frac{p(\sigma_i)}{p'(\sigma'_j(\sigma_i))}\right) + \ln|\mathcal{H}\setminus\mathcal{H}'|$$

$$\begin{aligned} S(p||p') &= \frac{1}{Z} \sum_{\sigma_i \in \mathcal{H}} e^{-H(\sigma_i)} [H'(\sigma'_j(\sigma_i)) - H(\sigma_i)] + \ln|\mathcal{H}\setminus\mathcal{H}'| \\ &= \langle H' \rangle_{p'} - \langle H \rangle_p + \ln|\mathcal{H}\setminus\mathcal{H}'| \ , \end{aligned}$$

Relative entropy for 1d Ising model

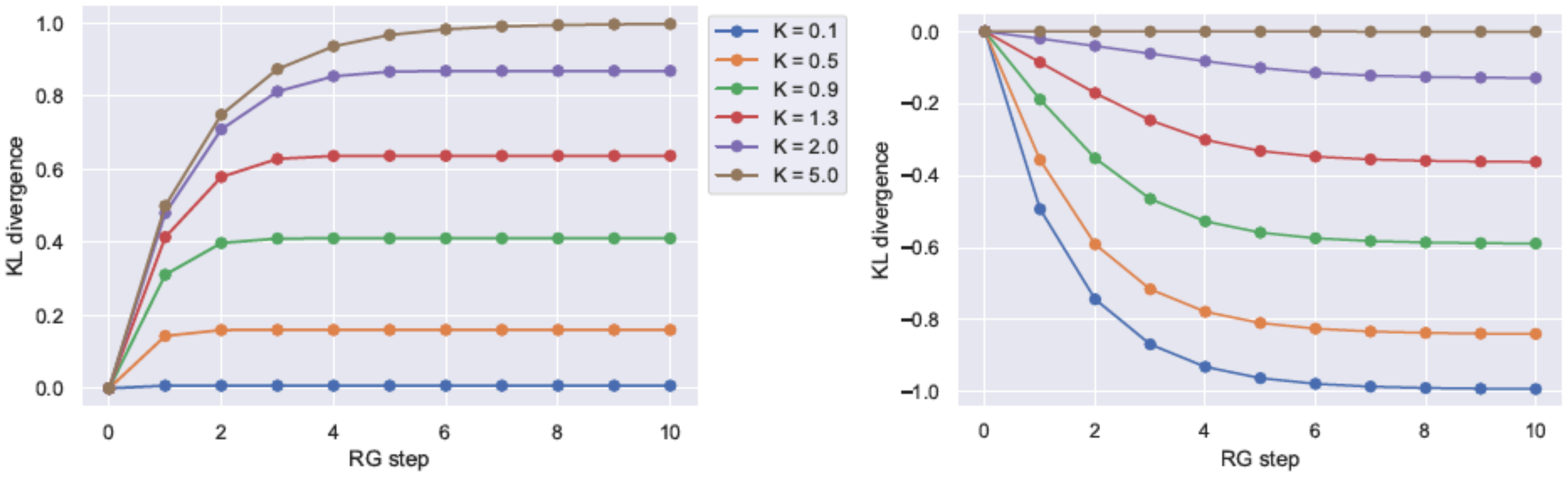


Figure 1: Relative entropy (Kullback-Leibler divergence) of the 1d classical Ising model as a function of real-space decimation step for various values of the nearest-neighbor coupling K. The left plot shows the correctly normalized entropy, while the right shows the same data without the normalization factor.

The stronger the coupling, the larger the information stored in correlations

J.E., Grosvenor, Jefferson 2107.06898

Feedforward random neural network

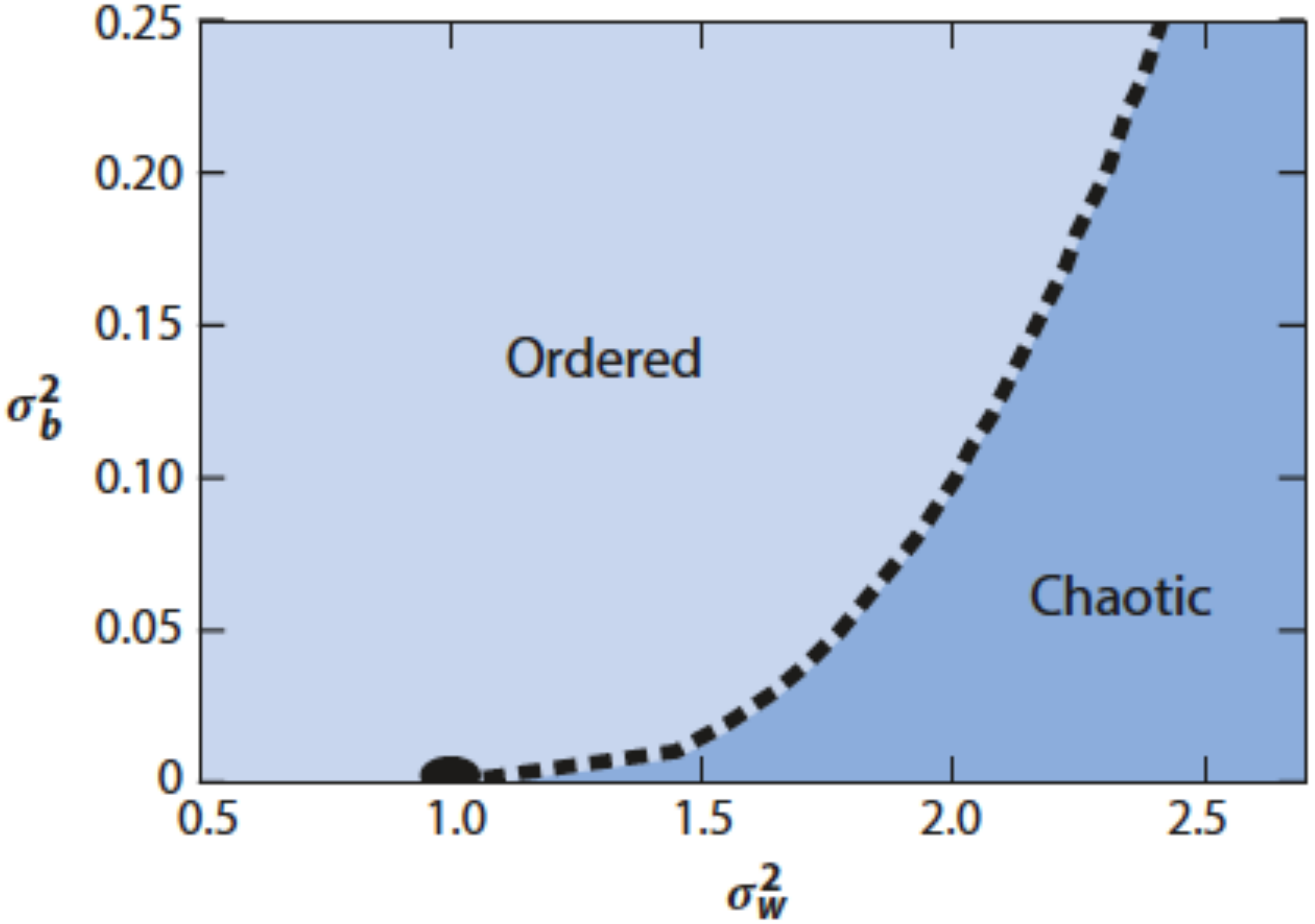
layers $\ell \in \{0, \ldots, L-1\}$, with $i \in \{0, \ldots, N^\ell - 1\}$ neurons per layer

Pre-activation
$$z_i^\ell = \sum_i W_{ij}^\ell y_j^{\ell-1} + b_i^\ell$$

$y_j^{\ell-1} = \phi(z_j^\ell)$ activation function of variables on previous layer, $\phi(z) = \tanh(z)$

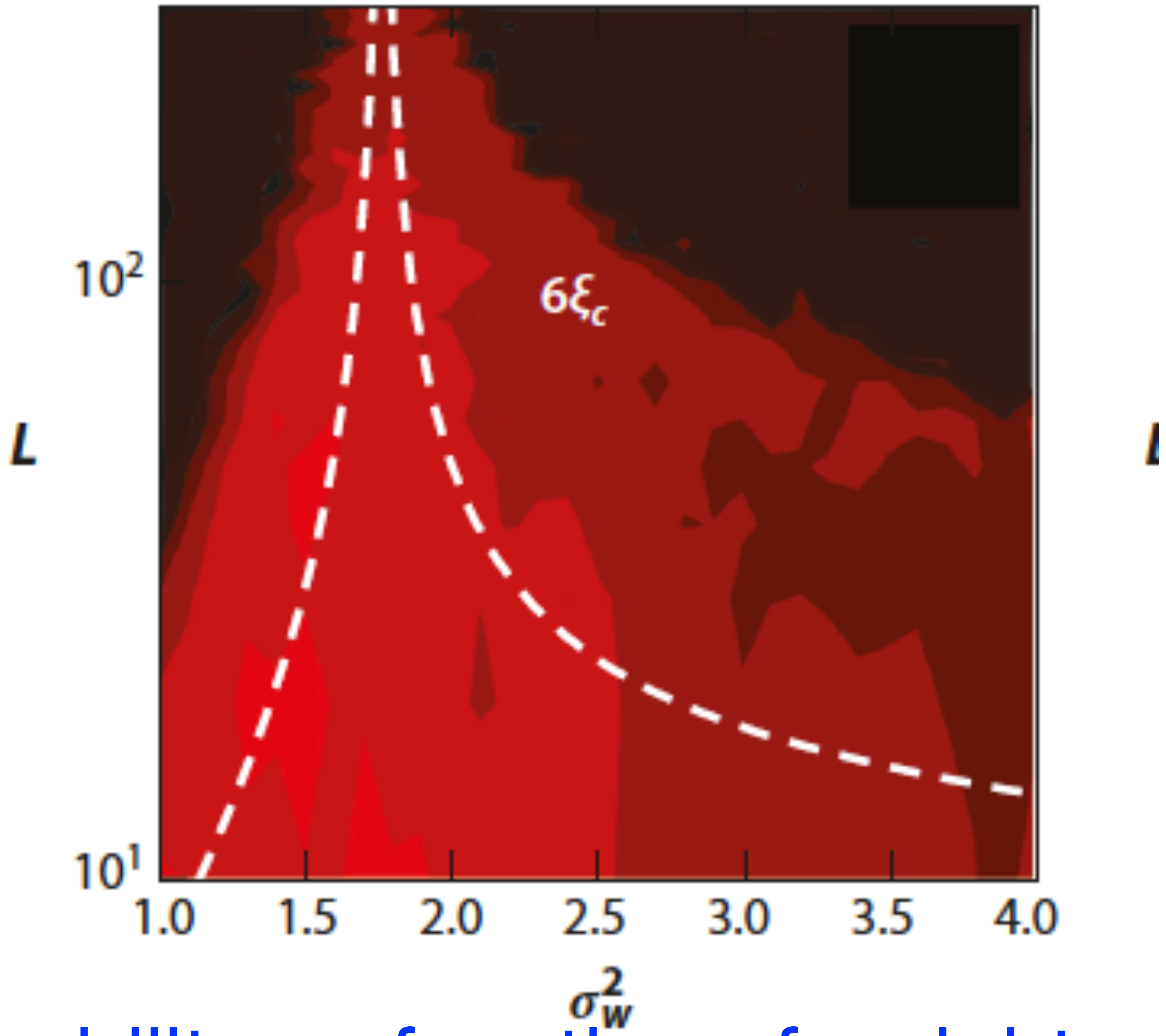
Weights and biases randomly distributed

Feedforward random neural network



Phase diagram

Figures from `Statistical mechanics of deep learning'
by Bahry, Kadmon, Pennington, Schoenholz, Sohl-Dickstein, Ganguly,
Annual Review of Condensed Matter Physics 2020



Trainability as function of weight variance and depth

Feedforward random neural network - KL divergence

Variance $\sigma_z^2 = \frac{\sigma_w^2}{N^{\ell-1}} \sum_i (y_j^{\ell-1})^2 + \sigma_b^2$ ~ $\sigma_w^2 \int \mathcal{D}z \, \phi\left(\sqrt{q^{\ell-1}}\, z\right)^2 + \sigma_b^2$

Correlations of pre-activations

$$\langle z_{i,a}^{\ell} z_{i,b}^{\ell} \rangle = \frac{\sigma_w^2}{N^{\ell-1}} \sum_j y_{j,a}^{\ell-1} y_{j,b}^{\ell-1} + \sigma_b^2$$

$$p(z) = \frac{1}{\sqrt{2\pi\sigma_p^2}} e^{-\frac{1}{2}\left(\frac{z-\mu_p}{\sigma_p}\right)^2}$$

$$\xi_c^{-1} = -\ln \sigma_w^2 \int \mathcal{D}z_1 \mathcal{D}z_2 \, \phi'(z_a^*) \phi'(z_b^*)$$

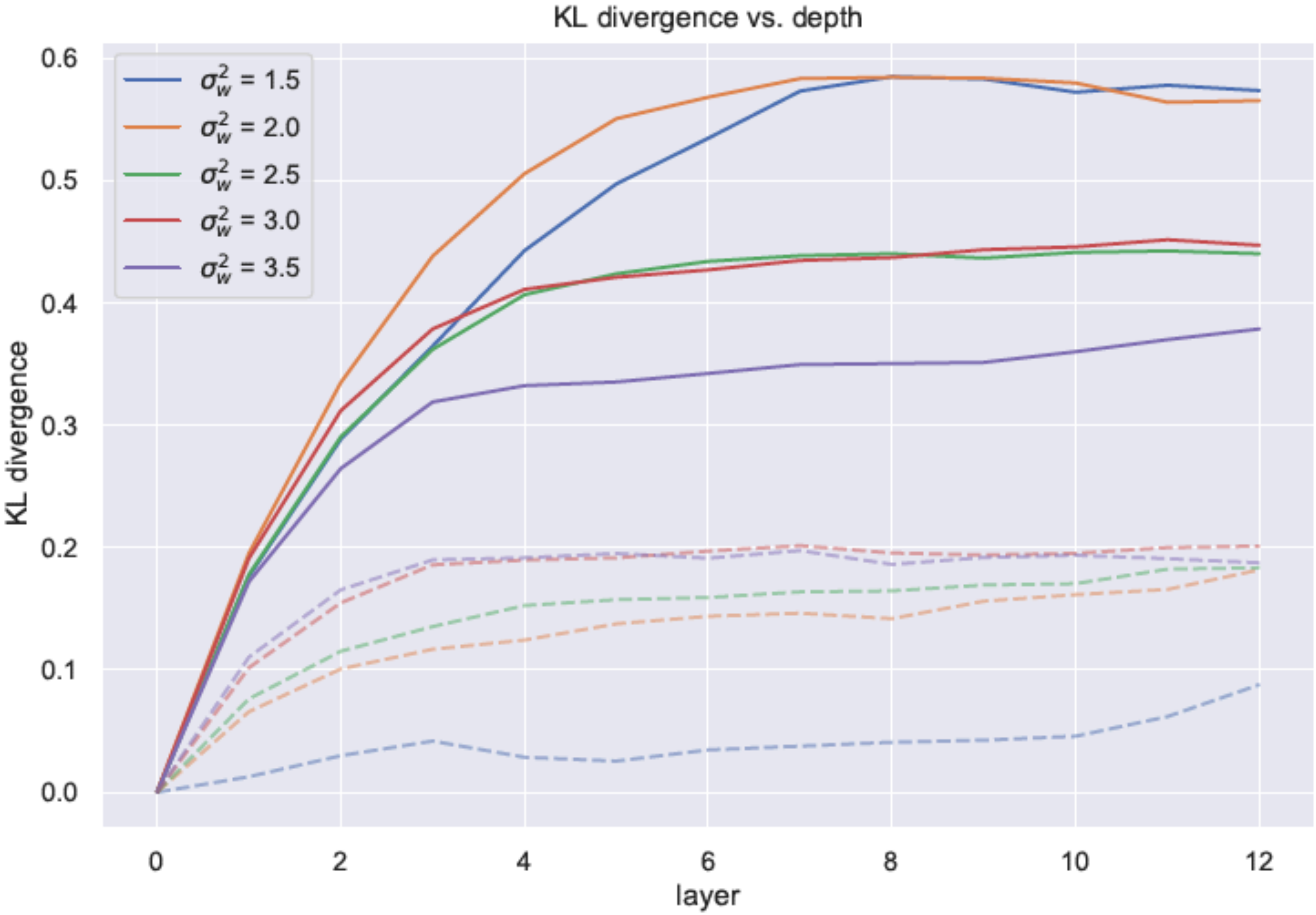
Relative entropy

$$D(p||q) = \int \mathrm{d}z \, p(z) \ln \frac{p(z)}{q(z)} = \langle \ln p(z) \rangle_p - \langle \ln q(z) \rangle_p = -S(p) - \langle \ln q(z) \rangle_p$$

After one step - thereafter more involved due to recursive nature

$$D(p||q) = -\frac{1}{2} \ln 2\pi e \sigma_p^2 + \frac{1}{2} \ln 2\pi \sigma_q^2 + \frac{1}{2\sigma_q^2} \left[\sigma_p^2 + (\mu_p - \mu_q)^2\right]$$

Relative entropy (KL divergence) for feedforward random network (MNIST/CIFAR data)



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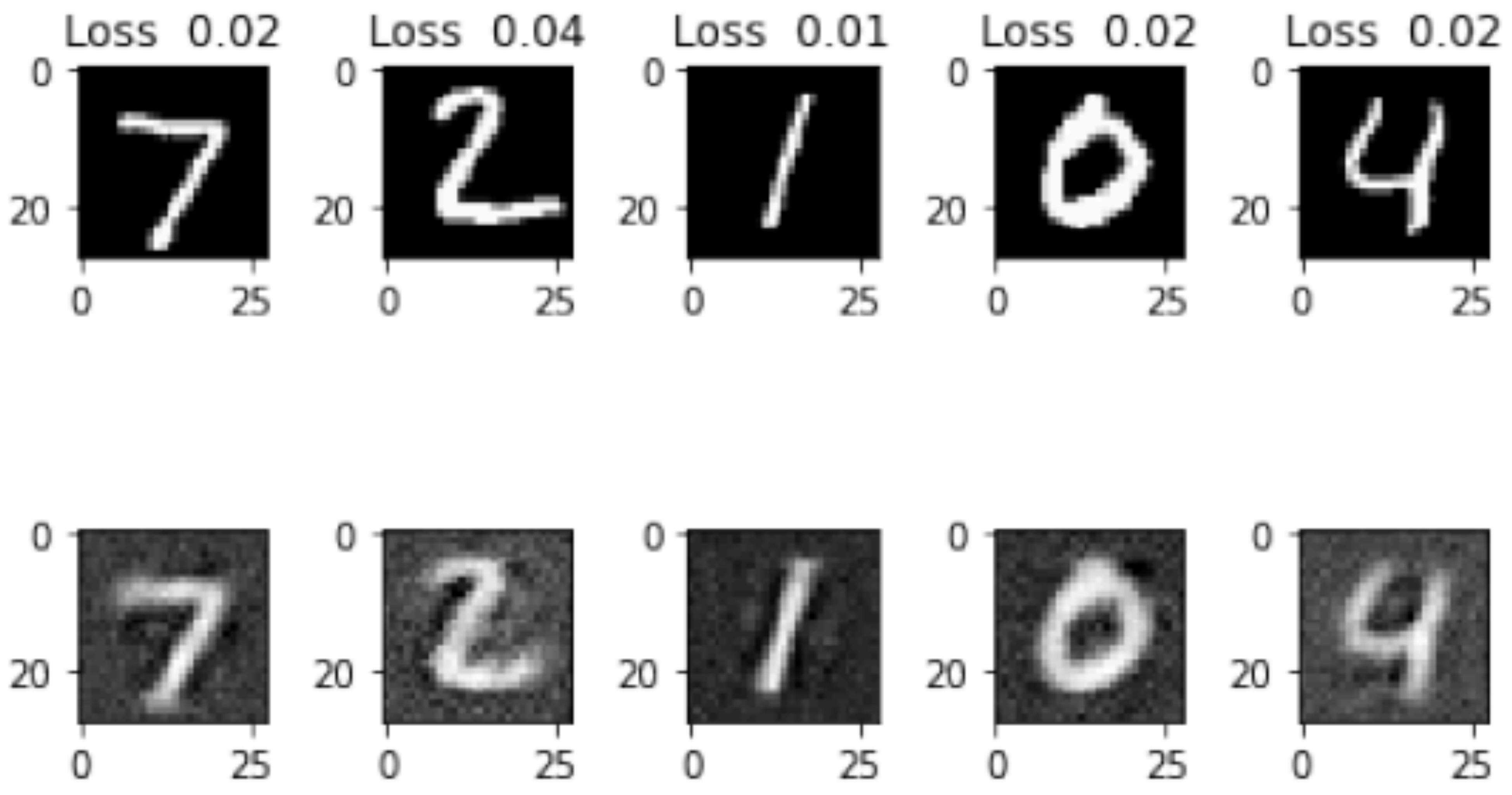

(a) Results after 50 epochs of training for an initial variance of $\sigma_w^2 = 1.25$.

(Master thesis Yanck Thurn)

Conclusion and Outlook

Gauge/Gravity Duality

both fundamental interest and applications

Holographic RG flows implemented from supergravity equations of motion

(Gradient flow) - Constructed to match global symmetries

Relative entropy (Kullback-Leibler divergence)

shows similar behaviour in Ising model and neural networks

May potentially be used to improve training in networks