Background-Independent field quantization with sequences of gravity-coupled approximants

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The framework: introduction & motivation

The framework of Background-Independent field quantization with sequences of gravity-coupled approximants is based upon three requirements for a quantization scheme:

- **(R1)** Background Independence
- **(R2)** Gravity-coupled approximants
- **(R3)** *N*-type cutoffs

These three requirements are not logically independent. Next, we will explain and motivate them in detail.

(R1) Background Independence

Background structures (if they are employed) must be determined by dynamical laws (self-consistence).

(R2) Gravity-coupled approximants

By an approximant we understand a "quasi-physical" auxiliary system consisting of a QM state $\Psi_{\mathbf{f}}$ with $\mathbf{f} < \infty$ degrees of freedom and a classical metric, denoted by

$$\mathrm{App}(\mathbf{f}) = \Psi_{\mathbf{f}} \otimes \mathrm{metric}\,.$$

When we invoke **(R1)** we obtain a quantum system on a self-consistent background:

$$\mathrm{App}^{\mathrm{sc}}(\mathbf{f}) = \Psi_{\mathbf{f}}^{\mathrm{sc}} \otimes \text{self-consistent metric} \xrightarrow{\mathbf{f}\to\infty} \Psi_{\mathrm{QFT}} \otimes \text{self-consistent metric}\,.$$

Motivations for **(R2)** Gravity-coupled approximants

- If approximants are physically realizable systems in their own right, we enhance chances to find sequences which converge to physically interesting limits.
- Background Independence is a natural requirement.
- Possibility of observable discreteness: experiment tells us that Nature is actually better described by the approximant $\mathrm{App}(\mathbf{f}_{\mathrm{Obs}})$, rather then in the limit $\mathbf{f} \to \infty$.
- Calculations which not invoke **(R1)** at the level of the regularized system can lead to defective background structures such as seen in the cosmological constant problem stemming from summing up vacuum energies.

The cosmological constant problem stemming from summing up vacuum energies

Astronomical observations give an experimental bound on the value of the cosmological constant:

$$\frac{\Lambda}{8\pi G} = \rho_{\text{vac}} + \frac{\Lambda_{\text{b}}}{8\pi G} \leq \rho_{\text{crit}} \approx 10^{-47}\, GeV^4 \,.$$

For a basic theoretical estimate of ρ_{vac} assume...

...ϕ is a free & massless field of Euclidean (Minkowski) space

...the field to consist of a set of harmonic oscillator with zero point energy $1/2\hbar\omega$ and dispersion relation $\omega(\vec{p}) = |\vec{p}|$

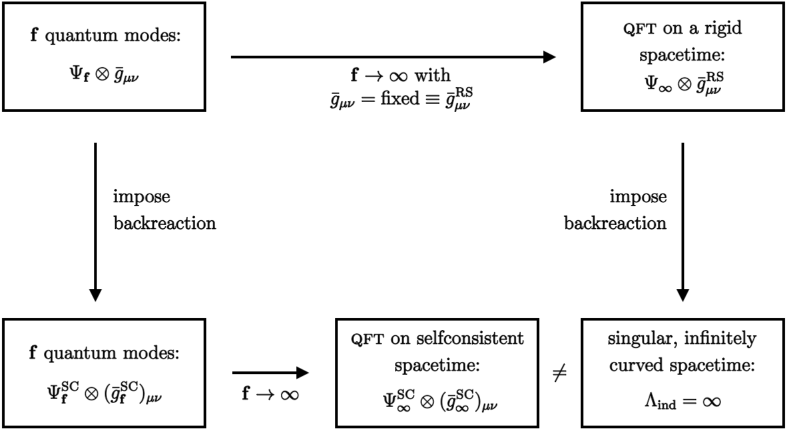
...the cutoff scale $\mathcal{P}$ at the Planck scale:

$$\Rightarrow \rho_{\text{vac}}[g_{\mu\nu} = \delta_{\mu\nu}] = \frac{1}{2}\int_{|\vec{p}|\leq\mathcal{P}} \frac{\mathrm{d}^3 p}{(2\pi)^3}|\vec{p}| \sim \mathcal{P}^4 = m_{\text{Pl}}^4 = 10^{76}\, GeV^4$$

This is a deviation theory-experiment of $\sim$ 123 digits!

Motivations for **(R2)** Gravity-coupled approximants

This further motivates the inclusion of gravity into the physical description of the approximants:



(R3) N-type cutoffs

N-type cutoffs are a specific technical realization of **(R2)**.

An N-type cutoff is a metric-independent regularization scheme for path integrals of the type

$$Z = \int_{\mathscr{F}} \mathcal{D}(\chi) \mathrm{e}^{-S[\chi]} \,,$$

constructed as follows. Let $\mathscr{B} = \{w_\alpha(\,\cdot\,) \mid \alpha \in I\}$ be a basis of $\mathscr{F}$. Let $N \in \mathbb{N}$ ($N \in [0, \infty)$ is possible, too) and define

$$\mathscr{B}_N := \{w_\alpha(\,\cdot\,) \mid \alpha \in I_N\}$$

with $\mathscr{B}_0 = \varnothing$, $\mathscr{B}_\infty = \mathscr{B}$, $N_2 > N_1 \Rightarrow \mathscr{B}_{N_2} \supset \mathscr{B}_{N_1}$. Then

$$\{\mathscr{F}_N = \mathrm{span}\mathscr{B}_N\}_{N \in \mathbb{N}}$$

is called an N-type cutoff. The regularized path integral is given by

$$Z_N = \int_{\mathscr{F}_N} -\mathcal{D}(\chi) \mathrm{e}^{-S[\chi]} \,.$$

(R3) N-type cutoffs

The metric-independent N-type cutoff is usually constructed with the eigenbasis of some metric-dependent self-adjoint operator, such as $\mathscr{K}[\bar{g}] = -\Box_{\bar{g}} = -\bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$.

Consider its EV-problem and assume its spectrum to be discrete (e.g., when M compact):

$$\mathscr{K}[\bar{g}]w_\alpha[\bar{g}](x) = \lambda_\alpha[\bar{g}]w_\alpha[\bar{g}](x)\,,\ \alpha \in I\,.$$

Then an N-type cutoff is given by

$$\mathscr{B}_N[\bar{g}] := \{w_\alpha[\bar{g}](\,\cdot\,) \mid \alpha \in I_N\}$$

For example, for $M = S^2(r)$ one has with $\alpha = \ell$ the EVs $\lambda_\ell[\bar{g}] = \ell(\ell+1)/r^2$ with $\ell = 0, 1, 2, \ldots$ and $I_N = \{\ell \mid \ell \leq N\}$.

(R3) N-type cutoffs

The N-type cutoff regularizes traces over these Hilbert spaces as follows (schematically for scalars here):

$$\mathrm{Tr}_{\mathsf{S}}[\mathscr{O}_{\mathsf{S}}] = \sum_{n=1}^{\infty} \sum_{m=1}^{D_n} \langle nm | \mathscr{O}_{\mathsf{S}} | nm \rangle \,,$$

$$\mathrm{Tr}_{\mathsf{S}} \to \mathrm{Tr}_{\mathsf{S}_N} \,, \quad \sum_{n=1}^{\infty} \to \sum_{n=1}^{N} \,.$$

Note that the degrees of freedom become a function of N then:

$$\mathbf{f} = \mathrm{Tr}_{\mathsf{S}}[\mathbb{1}_{\mathsf{S}}] \to \mathbf{f}(N) = \mathrm{Tr}_{\mathsf{S}_N}[\mathbb{1}_{\mathsf{S}_N}] < \infty \,.$$

(and fully analogously for vectors and tensors).

(R3) N-type cutoffs vs. $\mathcal{P}$-type cutoffs

The familiar momentum-space cutoff is background-dependent. If the EV $\lambda = \lambda_\alpha[\bar{g}] \Leftrightarrow \alpha = \alpha[\bar{g}](\lambda)$ is used as a label, one will have the basis

$$\mathscr{B}[\bar{g}] = \{W_\lambda[\bar{g}](\,\cdot\,) \mid \lambda \in \mathrm{spec}(\mathscr{K})\}\,,$$

with

$$W_\lambda[\bar{g}](x) := w_\alpha[\bar{g}](x)\Big|_{\alpha=\alpha[\bar{g}](\lambda)}\,,$$

and regularization works via the momentum-cutoff $\mathcal{P}$,

$$\mathscr{B}_\mathcal{P}[\bar{g}] = \{W_\lambda[\bar{g}](\,\cdot\,) \mid \lambda \leq \mathcal{P}^2\}\,.$$

However, the $\bar{g}$-dependence in $\alpha = \alpha[\bar{g}](\lambda)$ can not be self-consistently determined, and thus **(R2)** is violated!

Scalar field on $S^d(L)$

Consider the 1L effective action induced from a free scalar field with mass M and coupling ξ to gravity,

$$\Gamma[\bar{g}] = S_{\mathrm{EH}}[\bar{g}] + \Gamma_{1\mathrm{L}}[\bar{g}] \text{ with } \Gamma_{1\mathrm{L}}[\bar{g}] = \frac{1}{2}\mathrm{Tr}\log(-\Box_{\bar{g}} + M^2 + \xi\bar{g})\,.$$

On $S^d(L)$, it is sufficient to consider the equations of motion

$$\bar{\mathscr{T}}\Gamma[\bar{g}] = 0 \text{ with } \bar{\mathscr{T}} = -2\int \mathrm{d}^d x\, \bar{g}_{\mu\nu}(x)\frac{\delta}{\delta\bar{g}_{\mu\nu}(x)}\,.$$

Implementing an N-type cutoff one finds

$$\begin{aligned}\bar{\mathscr{T}}\Gamma_{1\mathrm{L}}[\bar{g}]_N &= \mathrm{Tr}_N\left[\frac{-\Box_{\bar{g}} + \xi\bar{R}}{-\Box_{\bar{g}} + M^2 + \xi\bar{R}}\right] \\ &= \sum_{n=1}^{N} D_n \frac{\mathscr{E}_n(L) + \xi\bar{R}(L)}{\mathscr{E}_n(L) + M^2 + \xi\bar{R}(L)} =: \Theta_N^{\mathrm{eff}}(L)\,.\end{aligned}$$

Scalar field on $S^4(L)$

Therewith, the equation of motion becomes

$$\bar{R}(L) = \frac{2d}{d-2}\Lambda_{\mathrm{b}} - \frac{16\pi G}{(d-2)\mathrm{vol}[S^d(L)]}\Theta_N^{\mathrm{eff}}(L)\,,$$

and its solution are the self-consistent radii $L^{\mathrm{sc}}(N;\xi, M, G, \Lambda_{\mathrm{b}})$.

For $d = 4$ there exist no solutions for $\Lambda_{\mathrm{b}} = 0$; and for $3/L_{\mathrm{b}}^2 = \Lambda_{\mathrm{b}} > 0$ and $M = 0 = \xi$ we have the self-consistent radii

$$L^{\mathrm{sc}}(N)^2 = \frac{1}{2}L_{\mathrm{b}}^2\left[1 + \sqrt{1 + \frac{G}{\pi L_{\mathrm{b}}^2}\mathbf{f}(N)}\right] \xrightarrow{N\to\infty} \infty\,,$$

where $\mathbf{f}(N) \approx \frac{1}{12}N^4$ in leading order.

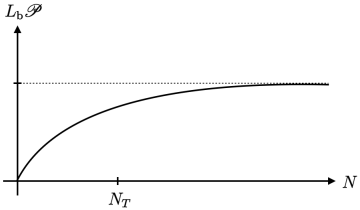
This is the opposite behavior as seen in the background-dependent calculation. Here, we have no singularity, but rather a flat universe when removing the cutoff: $S^4(L^{\mathrm{sc}}(N)) \xrightarrow{N\to\infty} \mathbb{R}^4$.

Scalar field on $S^4(L)$: another look at N vs. $\mathcal{P}$

In general, we have the equation $R = 4\Lambda = \mathcal{P}^4$. From an N-type cutoff, we can derive a "self-consistent" $\mathcal{P}$-cutoff by

$$\mathcal{P}^{\mathrm{sc}}(N)^2 := \mathscr{E}_N(L^{\mathrm{sc}}(N)) = \frac{N(N+3)}{L^{\mathrm{sc}}(N)^2} \xrightarrow{N\to\infty} (24\pi)^{1/2}\frac{m_{\mathrm{Pl}}}{N_T}\,,$$

where $N_T = (12\pi)^{1/4}\sqrt{L_{\mathrm{b}}/\ell_{\mathrm{Pl}}}$. Thus, for an N-type cutoff we have $N \to \infty \not\Leftrightarrow \mathcal{P} \to \infty$.



On the other hand, for a $\mathcal{P}$-type cutoff we have $\mathcal{P}^2 = \mathscr{E}_N(L_{\mathrm{RS}}) = \frac{N(N+3)}{L_{\mathrm{RS}}^2}$, where the rigid spacetime is needed to determine the w_α from $\lambda \le \mathcal{P}^2$. Thus, here one has $N \to \infty \Leftrightarrow \mathcal{P} \to \infty$.

Scalar field on $S^4(L)$: micro states of dS space

This result gives furthermore insights into the micro states of dS space. Therefore consider the Bekenstein-Hawking entropy of dS space (which is a purely thermodynamical quantity):

$$\begin{aligned}\mathscr{S} &= \frac{3\pi}{G\Lambda} = \frac{\pi}{G} L^2 \\ &= \frac{\pi}{G} L^{\mathrm{sc}}(N)^2 = \frac{\pi}{2G} L_{\mathrm{b}}^2 \left[1 + \sqrt{1 + \frac{G}{\pi L_{\mathrm{b}}^2} \mathbf{f}(N)} \right] .\end{aligned}$$

Therewith, we have proven an instance of a "Λ-$\mathcal{N}$-connection":

$$\mathscr{S}_{\mathrm{obs}} = \frac{3\pi}{G\Lambda_{\mathrm{obs}}} = \frac{\pi}{2G} L_{\mathrm{b}}^2 \left[1 + \sqrt{1 + \frac{G}{\pi L_{\mathrm{b}}^2} \mathbf{f}(N_{\mathrm{obs}})} \right] .$$

Universes with a strictly positive cosmological constant are described by a quantum theory with only a finite number of degrees of freedom, and this number is given by the Bekenstein-Hawking entropy of dS space.

Vacuum fluctuations of the geometry

Let us repeat the analysis for the scalar, but this time for "full-fledged" quantum gravity. Consider the standard BRST gauge-fixed "quantum action":

$$S[h, \bar{C}, C; \bar{g}] = S_{\mathrm{EH}}[\bar{g} + h] + S_{\mathrm{gf}}[h; \bar{g}] + S_{\mathrm{gh}}[h, \bar{C}, C; \bar{g}]\,,$$

where

$$S_{\mathrm{gf}}[h; \bar{g}] = \frac{1}{32\pi G}\int \mathrm{d}^d x\, \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} \big(\mathcal{F}_\mu^{\alpha\beta}[\bar{g}] h_{\alpha\beta}\big) \big(\mathcal{F}_\nu^{\gamma\delta}[\bar{g}] h_{\gamma\delta}\big)\ ,$$

involves the differential operator $\mathcal{F}_\mu^{\alpha\beta}[\bar{g}] := \delta_\mu^\beta\, \bar{g}^{\alpha\gamma} \bar{D}_\gamma - \frac{1}{2}\, \bar{g}^{\alpha\beta} \bar{D}_\mu$ and further we have the concomitant ghost action

$$S_{\mathrm{gh}}[h, \bar{C}, C; \bar{g}] = -\sqrt{2}\int \mathrm{d}^d x\, \sqrt{\bar{g}}\, \bar{C}_\mu \mathscr{M}[\bar{g} + h, \bar{g}]^\mu{}_\nu C^\nu$$

with the Faddeev-Popov operator $\mathscr{M}[\bar{g} + h, \bar{g}]^\mu{}_\nu$.

Vacuum fluctuations of the geometry

For our purpose, let us consider the dynamical fields $(h_{\mu\nu}, \bar{C}_\mu, C^\mu)$ as special sort of "matter" inhabiting the classical spacetime $(M, \bar{g}_{\mu\nu})$:

$$
\begin{aligned}
S_{\mathrm{M}}[h, \bar{C}, C; \bar{g}] &:= S[h, \bar{C}, C; \bar{g}] - S_{\mathrm{EH}}[\bar{g}] \\
&= S_{\mathrm{EH}}[\bar{g} + h] - S_{\mathrm{EH}}[\bar{g}] + S_{\mathrm{gf}}[h; \bar{g}] + S_{\mathrm{gh}}[h, \bar{C}, C; \bar{g}] \\
&= S_{\mathrm{FG}}[h; \bar{g}] + S_{\mathrm{Fgh}}[\bar{C}, C; \bar{g}] + (\text{linear}) + O(3)\,,
\end{aligned}
$$

where we have expanded around $(h = 0, \bar{C} = 0 = C)$. Here, S_{FG} denotes the "Free Graviton" action and S_{Fgh} denotes the "Free ghost" action. These are bilinear in the respective dynamical variables:

$$
\begin{aligned}
S_{\mathrm{FG}}[h; \bar{g}] &:= \frac{1}{2} \int \mathrm{d}^d x \sqrt{\bar{g}}\; h^{\mu\nu} \mathscr{K}[\bar{g}]_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \\
S_{\mathrm{Fgh}}[\bar{C}, C; \bar{g}] &:= S_{\mathrm{gh}}[0, \bar{C}, C; \bar{g}] = -\sqrt{2} \int \mathrm{d}^d x \sqrt{\bar{g}}\; \bar{C}_\mu \mathscr{M}[\bar{g}, \bar{g}]^\mu{}_\nu C^\nu\,,
\end{aligned}
$$

with $\mathscr{K}[\bar{g}]_{\mu\nu}{}^{\rho\sigma} = \frac{1}{16\pi G}[-(\bar{D}^2 + 2\Lambda_{\mathrm{b}}) K_{\mu\nu}{}^{\rho\sigma} + V_{\mu\nu}{}^{\rho\sigma}]$.

Vacuum fluctuations of the geometry

From this bare action, we can obtain in the usual way the standard 1L gravitational effective action for vanishing ghost arguments,

$$\Gamma[h,0,0;\bar{g}] = S_{\mathrm{EH}}[\bar{g}+h] + \Gamma_{1\mathrm{L}}[\bar{g}+h] + O(2\text{ loops})$$

with $\Gamma_{1\mathrm{L}}[\bar{g}] = \Gamma_{\mathrm{FG}}[\bar{g}] + \Gamma_{\mathrm{Fgh}}[\bar{g}]$. Here, self-consistency is given by the tadpole condition

$$\frac{\delta}{\delta h_{\mu\nu}(x)}\Gamma[h,0,0;\bar{g}]\bigg|_{h=0,\,\bar{g}=\bar{g}^{\mathrm{sc}}} = 0\,,$$

which for the 1L approximation for Γ amounts to the Einstein equation

$$\bar{R}^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}\bar{R} + \Lambda_{\mathrm{b}}\,\bar{g}^{\mu\nu} = (8\pi G)\,T^{\mu\nu}[\bar{g}]$$

with the stress tensor $T^{\mu\nu}[\bar{g}](x) \equiv -\frac{2}{\sqrt{\bar{g}}}\frac{\delta}{\delta\bar{g}_{\mu\nu}(x)}\Gamma_{1\mathrm{L}}[\bar{g}]\,.$

Vacuum fluctuations of the geometry

$\Gamma_{\rm FG}[\bar{g}]$ and $\Gamma_{\rm Fgh}[\bar{g}]$ are determined by the Gaussian functional integrals, respectively,

$$e^{-\Gamma_{\rm FG}[\bar{g}]} = \int \mathcal{D}_1\Big[\bar{g}^{(d-4)/(4d)}\widehat{h}_{\bullet\bullet}\Big]\ e^{-S_{\rm FG}[\widehat{h}_{\bullet\bullet};\bar{g}_{\bullet\bullet}]}\,,$$

$$e^{-\Gamma_{\rm Fgh}[\bar{g}]} = \int \mathcal{D}_1\left[\bar{g}^{(d+2)/(4d)}\,\widehat{C}^{\bullet}\right]\mathcal{D}_1\Big[\bar{g}^{(d-2)/(4d)}\,\widehat{\bar{C}}_{\bullet}\Big]\ e^{-S_{\rm Fgh}[\widehat{\bar{C}}_{\bullet},\widehat{C}^{\bullet};\bar{g}_{\bullet\bullet}]}\,.$$

For M compact, one finds

$$\Gamma_{\rm FG}[\bar{g}_{\bullet\bullet}] = \frac{1}{2}\sum_{n,m}\ln\left(\mathscr{F}_n[\bar{g}_{\bullet\bullet}]\right)$$

$$\Gamma_{\rm Fgh}[\bar{g}_{\bullet\bullet}] = -\sum_{n,m}\ln\left(\mathscr{F}_n^{\rm gh}[\bar{g}_{\bullet\bullet}]\right)\,,$$

where $\mathscr{F}_n$ and $\mathscr{F}_n^{\rm gh}$ are given by the spectral values of the operators $\mathscr{K}[\bar{g}]_{\bullet\bullet}{}^{\bullet\bullet}$ and $\mathscr{M}[\bar{g},\bar{g}]^{\bullet}{}_{\bullet}$, respectively.

Vacuum fluctuations of the geometry on $S^4(L)$

On $M = S^d(L)$ we can write $\bar{g}_{\mu\nu}(x) = L^2\gamma_{\mu\nu}(x)$, where $\gamma_{\mu\nu}$ denotes the dimensionless standard metric on the unit d-sphere.

The tadpole condition for Γ (the Einstein equation) determines the self-consistent background geometries given by $\bar{g}^{\text{sc}}_{\mu\nu} = (L^{\text{sc}})^2\gamma_{\mu\nu}$ and reads for $d = 4$, implementing an N-type cutoff,

$$4\Lambda_{\rm b} L^4 - 12L^2 = \frac{3G}{\pi}\,\Theta_N(L)\,,$$

where the integrated trace of the stress tensor yields the spectral sums

$$\Theta_N(L) \equiv \Theta_N[L^2\gamma_{\mu\nu}] = \mathbf{f}_{\rm G}(N) - \mathbf{f}_{\rm gh}(N) + \Delta\Theta_N(L;\Lambda_{\rm b})\,.$$

This representation of the trace involves the (exact) number of graviton and ghost degrees of freedom,

$$\mathbf{f}_{\rm G}(N) = \frac{1}{12}\left[10N^4 + O(N^3)\right] \quad , \quad \mathbf{f}_{\rm gh}(N) = \frac{1}{12}\left[8N^4 + O(N^3)\right]\;;$$

and a $(L;\Lambda_{\rm b})$-dependent spectral sum $\Delta\Theta_N$.

Vacuum fluctuations of the geometry on $S^4(L)$

In the leading-N approximation, for $N \to \infty$, we have

$$\mathbf{f}(N) := \mathbf{f}_{\mathrm{G}}(N) - \mathbf{f}_{\mathrm{gh}}(N) \approx \frac{1}{6} N^4$$

and $\Delta\Theta_N \sim N^2$, such that we may neglect the latter term. For $\Lambda_{\mathrm{b}} > 0$ this leads to the self-consistent radii

$$(L_N^{\mathrm{sc}})^2 = \frac{1}{2} L_{\mathrm{b}}^2 \left[1 + \sqrt{1 + \frac{1}{\pi} \left(\frac{\ell_{\mathrm{Pl}}}{L_{\mathrm{b}}} \right)^2 \mathbf{f}(N)} \right] \xrightarrow{N\to\infty} \infty\,,$$

again with the "Hubble radius" $L_{\mathrm{b}} = \sqrt{3/\Lambda_{\mathrm{b}}}$. Also adding graviton degrees of freedom (i.e., lifting the cutoff) tends to *flatten* the universe.

Summary and outlook

- We employed a new quantization scheme with the requirement of BI already at the level of the regularized precursor of a QFT.
- For quantized free matter fields coupled to gravity and quantized metric fluctuations (in terms of quantum GR as an effective theory), we found that quantizing additional modes reduces curvature, and drives the universe to flat space.
- Within its technical limitations, these results suggest that pure QEG should have a distinguished ground state, namely flat space.
- Outlook: more refined background geometries & (self) interactions & flow equation for Γ_N

THANK YOU!