Benchmarking effective actions and their flow equations in zero dimensions

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Thanks

This talk is based on:

Garbrecht & PM ’16 (1509.07847)
Alexander, PM, **Nursey** & **Saffin** ’19 (1907.06503)
PM & **Saffin** ’19, ’21 and ’22 (1905.09674, 2107.12914, 2206.08865)

Thanks go to:

- my **wonderful collaborators** in this area,
- the **organisers of ERG2020**, and to **Dario Benedetti**, **Kevin Falls**, **Jan Pawlowski** and **Adam Rancon** for the discussions that followed,
- the **organisers of this super meeting**,
- and **all of you** for coming along to this talk.

nPI quantum effective actions
(à la Routh)

For each **non-connected n-point correlation function** $\Delta^{\text{nc}}_{x_1, x_2, \cdots, x_n}$ of the theory of a scalar field Φ with classical action $S[\Phi]$, we can introduce a function $J_{x_1, x_2, \cdots, x_n}$ and define a functional

$$\Gamma_{n, \{J\}}[\{\Delta^{\text{nc}}\}] = W_n[\{J\}] + J_{x_1}\Delta^{\text{nc}}_{x_1} + J_{x_1, x_2}\Delta^{\text{nc}}_{x_1, x_2} + \cdots + J_{x_1, x_2, \cdots, x_n}\Delta^{\text{nc}}_{x_1, x_2, \cdots, x_n}$$

$$W_n[\{J\}] = -\hbar \ln \int D\Phi \ \exp\left[-\frac{1}{\hbar}\left(S[\Phi] - J_{x_1}\Phi_{x_1} - J_{x_1, x_2}\Phi_{x_1}\Phi_{x_2} - \cdots - J_{x_1, x_2, \cdots, x_n}\Phi_{x_1}\Phi_{x_2}\cdots\Phi_{x_n}\right)\right]$$

Extremizing wrt **all** of the J's for a given n leads to the n**-particle irreducible effective actions**.

Extremizing wrt a **subset** of the J's for a given n leads to … **something else**.

1PI$_{\text{average}}$ [Wetterich '91]

The **1PI average effective action** amounts to taking

$$\Gamma_{2,\{J\}}[\{\Delta^{\text{nc}}\}] = W_2[\{J\}] + J_{x_1}\Delta^{\text{nc}}_{x_1} + J_{x_1,x_2}\Delta^{\text{nc}}_{x_1,x_2}$$

- extremizing wrt J_{x_1} only, which sets $J_{x_1} = J_{x_1}[\phi]$ and $\Delta^{\text{nc}}_{x_1} = \phi_{x_1}[\{J\}] = -\delta W_2[\{J\}]/\delta J_{x_1}$,
- setting $\Delta^{\text{nc}}_{x_1,x_2} = \phi_{x_1}\phi_{x_2}$, which sets $\delta\Gamma_{2,\{J\}}[\{\Delta\}]/\delta J_{x_1,x_2} \neq 0$,
- taking $J_{x_1,x_2} = -R_{k;x_1,x_2}/2$ to be proportional to the inverse FT $R_{k;x_1,x_2}$ of the regulator $R_{k;q}$.

2PI$_{\text{reg}}$ [Alexander, PM, Nursey, Saffin ’19; see also Garbrecht & PM ’16]

Instead, let’s again take

$$\Gamma_{2,\{J\}}[\{\Delta^{\text{nc}}\}] = W_2[\{J\}] + J_{x_1}\Delta^{\text{nc}}_{x_1} + J_{x_1,x_2}\Delta^{\text{nc}}_{x_1,x_2}$$

- extremize wrt J_{x_1} and J_{x_1,x_2}, which sets $\{J\} = \{J\}[\phi_{x_1}, \Delta_{x_1,x_2}]$,

$$\Delta^{\text{nc}}_{x_1} = \phi_{x_1}[\{J\}] = -\,\delta W_2[\{J\}]/\delta J_{x_1},$$

$$\Delta^{\text{nc}}_{x_1,x_2} = \hbar\Delta_{x_1,x_2}[\{J\}] + \phi_{x_1}\phi_{x_2} = -\,\delta W_2[\{J\}]/\delta J_{x_1,x_2},$$

- take $J_{x_1,x_2} = -\,R_{k;\,x_1,x_2}/2$ to be proportional to the inverse FT $R_{k;\,x_1,x_2}$ of the regulator $R_{k;\,q}$.
  [cf. Reinosa’s talk on Tuesday, i.e., Blaizot, Pawlowski and Reinosa ’11 & ’21 and Reinosa ’21]

Flow equations

$1PI_{average}$ [Wetterich '93, Morris '94, Ellwanger '94]

Natural variables:

$$\Gamma^{1PI}_{av} = \Gamma^{1PI}_{av}[\phi, R_k]$$

Fix $J_{x_1} = J_{k;\,x_1}$ such that $\partial_k \phi = 0$ and then

$$\partial_k \Gamma^{1PI}_{av} = \frac{\delta \Gamma^{1PI}_{av}}{\delta R_k} \cdot \partial_k R_k = \frac{\hbar}{2} \Delta^{1PI}_k \cdot \partial_k R_k$$

by the functional chain rule.

$2PI_{reg}$ [Alexander, PM, Nursey, Saffin '19]

Natural variables:

$$\Gamma^{2PI}_{reg} = \Gamma^{2PI}_{reg}[\phi, \Delta_k]$$

Fix $J_{x_1} = J_{k;\,x_1}$ such that $\partial_k \phi = 0$ and then

$$\partial_k \Gamma^{2PI}_{reg} = \frac{\delta \Gamma^{2PI}_{reg}}{\delta \Delta_k} \cdot \partial_k \Delta_k = -\frac{\hbar}{2} R_k \cdot \partial_k \Delta_k$$

by the functional chain rule.

Closure

1PI$_{\text{average}}$

Convexity

$$\frac{\delta^2\Gamma^{1\text{PI}}}{\delta\phi_{x_1}\delta\phi_{x_2}}\frac{\delta^2(-W)}{\delta J_{x_1}\delta J_{x_2}} = 1$$

implies

$$\left[\Delta^{1\text{PI}}_{k;\,x_1,\,x_2}\right]^{-1} = \frac{\delta^2\Gamma^{1\text{PI}}_{\text{av}}}{\delta\phi_{x_1}\delta\phi_{x_2}} + R_{k;\,x_1,\,x_2}$$

2PI$_{\text{reg}}$ [PM, Saffin '21]

Convexity

$$\text{Hess}_{\phi,\,\Delta^{\text{nc}}}\,\Gamma^{2\text{PI}} \cdot \text{Hess}_{J,\,-R/2}\,(-W) = \mathbb{1}$$

implies

$$\left[\Delta_{k;\,x_1,\,x_2}\right]^{-1} = \frac{\delta^2\Gamma^{2\text{PI}}}{\delta\phi_{x_1}\delta\phi_{x_2}} + R_{k;\,x_1,\,x_2}$$

$$-\frac{\delta^2\Gamma^{2\text{PI}}}{\delta\phi_{x_1}\delta\Delta_{k;\,y_1,\,y_2}}\left[\frac{\delta^2\Gamma^{2\text{PI}}}{\delta\Delta_{k;\,y_1,\,y_2}\delta\Delta_{k;\,y_3,\,y_4}}\right]^{-1}\frac{\delta^2\Gamma^{2\text{PI}}}{\delta\Delta_{k;\,y_3,\,y_4}\delta\phi_{x_2}}$$

[see also footnote 11 of Cornwall, Jackiw, Tomboulis '74]

Closure

$1\text{PI}_{\text{average}}$

Convexity

$$\frac{\delta^2\Gamma^{1\text{PI}}}{\delta\phi_{x_1}\delta\phi_{x_2}}\frac{\delta^2(-W)}{\delta J_{x_1}\delta J_{x_2}} = 1$$

implies

$$\left[\Delta^{1\text{PI}}_{k;\,x_1,\,x_2}\right]^{-1} = \frac{\delta^2\Gamma^{1\text{PI}}_{\text{av}}}{\delta\phi_{x_1}\delta\phi_{x_2}} + R_{k;\,x_1,\,x_2}$$

2PI_{reg} [PM, Saffin ’21]

Convexity

$$\text{Hess}_{\boxed{\phi,\,\Delta^{\text{nc}}}}\Gamma^{2\text{PI}}\cdot\text{Hess}_{\boxed{J,\,-R/2}}(-W) = \mathbb{1}$$

implies

$$\left[\Delta_{k;\,x_1,\,x_2}\right]^{-1} = \frac{\delta^2\Gamma^{2\text{PI}}}{\delta\phi_{x_1}\delta\phi_{x_2}} + R_{k;\,x_1,\,x_2}$$

$$-\frac{\delta^2\Gamma^{2\text{PI}}}{\delta\phi_{x_1}\delta\Delta_{k;\,y_1,\,y_2}}\left[\frac{\delta^2\Gamma^{2\text{PI}}}{\delta\Delta_{k;\,y_1,\,y_2}\delta\Delta_{k;\,y_3,\,y_4}}\right]^{-1}\frac{\delta^2\Gamma^{2\text{PI}}}{\delta\Delta_{k;\,y_3,\,y_4}\delta\phi_{x_2}}$$

Closure

$1\mathrm{PI}_{\mathrm{average}}$

Convexity

$$\frac{\delta^2 \Gamma^{1\mathrm{PI}}}{\delta\phi_{x_1}\delta\phi_{x_2}}\frac{\delta^2(-W)}{\delta J_{x_1}\delta J_{x_2}} = 1$$

implies

$$\left[\Delta^{1\mathrm{PI}}_{k;\,x_1,\,x_2}\right]^{-1} = \frac{\delta^2 \Gamma^{1\mathrm{PI}}_{\mathrm{av}}}{\delta\phi_{x_1}\delta\phi_{x_2}} + R_{k;\,x_1,\,x_2}$$

$2\mathrm{PI}_{\mathrm{reg}}$ [PM, Saffin '21]

Convexity

$$\mathrm{Hess}_{\phi,\,\Delta^{\mathrm{nc}}}\,\Gamma^{2\mathrm{PI}} \cdot \mathrm{Hess}_{J,\,-R/2}\,(-W) = \mathbb{1}$$

implies

$$\left[\Delta_{k;\,x_1,\,x_2}\right]^{-1} = \frac{\delta^2 \Gamma^{2\mathrm{PI}}}{\delta\phi_{x_1}\delta\phi_{x_2}} + R_{k;\,x_1,\,x_2}$$

$$- \frac{\delta^2 \Gamma^{2\mathrm{PI}}}{\delta\phi_{x_1}\delta\Delta_{k;\,y_1,\,y_2}}\left[\frac{\delta^2 \Gamma^{2\mathrm{PI}}}{\delta\Delta_{k;\,y_1,\,y_2}\delta\Delta_{k;\,y_3,\,y_4}}\right]^{-1}\frac{\delta^2 \Gamma^{2\mathrm{PI}}}{\delta\Delta_{k;\,y_3,\,y_4}\delta\phi_{x_2}}$$

2PI resummation

Zero-dimensional playground [PM, Saffin ’19, ’21 & ’22]

Take

$$S(\Phi) = \frac{1}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4$$

with $\hbar > 0$ and $\lambda > 0$.

We can calculate

$$W(J, R_k) = -\hbar \ln \int D\Phi \ \exp\left[-\frac{1}{\hbar}\left(S(\Phi) - J\Phi + \frac{1}{2}R_k\Phi^2\right)\right],$$

any effective action and its n-point variables analytically to a given order in λ.

Ansätze [PM, Saffin '21]

$1\text{PI}_{\text{average}}$

Ansatz:

$$\Gamma_{\text{av}}^{1\text{PI}} = \alpha^{1\text{PI}}(R_k) + \frac{1}{2}\beta^{1\text{PI}}(R_k)\phi^2 + \frac{1}{4!}\gamma^{1\text{PI}}(R_k)\phi^4$$

Solve flow equations and compare with explicit result:

$$\Gamma_{\text{av}}^{1\text{PI}} = S(\phi) + \frac{\hbar}{2}\left[\ln\left[\Delta^{1\text{PI}}\right]^{-1}\right] + \hbar^2\left[\frac{\lambda}{8}\left[\Delta^{1\text{PI}}\right]^2 - \frac{\lambda^2}{12}\phi^2\left[\Delta^{1\text{PI}}\right]^3\right] + \hbar^3\left[-\frac{\lambda^2}{12}\left[\Delta^{1\text{PI}}\right]^4\right]$$

2PI_{reg}

Ansatz:

$$\Gamma^{2\text{PI}} = \alpha(\Delta_k) + \frac{1}{2}\beta(\Delta_k)\phi^2 + \frac{1}{4!}\gamma(\Delta_k)\phi^4$$

Solve flow equations and compare with explicit result:

$$\Gamma^{2\text{PI}} = S(\phi) + \frac{\hbar}{2}\left[\ln\Delta^{-1} + \left(1 + \frac{\lambda}{2}\phi^2\right)\Delta - 1\right] + \hbar^2\left[\frac{\lambda}{8}\Delta^2 - \frac{\lambda^2}{12}\phi^2\Delta^3\right] + \hbar^3\left[-\frac{\lambda^2}{48}\Delta^4\right]$$

Flow equations [PM, Saffin '21]

$1PI_{average}$

$$\partial_k \alpha(\Delta_k) = \frac{\partial \alpha^{1PI}(R_k)}{\partial R_k} \partial_k R_k \text{ etc.}$$

Take derivatives of $\left[\Delta^{1PI}\right]^{-1}$ with respect to ϕ at $\phi = 0$ and solve for $\partial^n\{\alpha^{1PI}, \beta^{1PI}, \gamma^{1PI}\}/\partial R_k^n$:

$$\partial_k \alpha^{1PI} = \frac{\hbar}{2} \frac{\partial_k R_k}{\left[\beta^{1PI} + R_k\right]}$$

$$\partial_k \beta^{1PI} = -\frac{\hbar}{2} \frac{\gamma^{1PI} \partial_k R_k}{\left[\beta^{1PI} + R_k\right]^2}$$

$$\partial_k \gamma^{1PI} = 3\hbar \frac{\left[\gamma^{1PI}\right]^2 \partial_k R_k}{\left[\beta^{1PI} + R_k\right]^3}$$

$2PI_{reg}$

$$\partial_k \alpha(\Delta_k) = \frac{\partial \alpha(\Delta_k)}{\partial \Delta_k} \partial_k \Delta_k \text{ etc.}$$

Take derivatives of Δ^{-1} with respect to ϕ and Δ at $\phi = 0$ and solve for $\partial^n\{\alpha, \beta, \gamma\}/\partial\Delta_k^n$ with $R_k^0 = R_k(\phi = 0, \Delta_k)$:

$$\partial_k \alpha = \frac{\hbar}{2} R_k^0 \frac{\partial_k \beta + \partial_k R_k^0}{\left[\beta + R_k^0\right]^2}$$

$$\partial_k \beta = -\left\{ \frac{\hbar}{2}\gamma - \frac{\hbar^2}{2} \frac{\gamma^2}{\left[\beta + R_k^0\right]^2} \right\} \frac{\partial_k \beta + \partial_k R_k^0}{\left[\beta + R_k^0\right]^2}$$

$$\partial_k \gamma = \mathcal{O}(\gamma^4)$$

Flow equations [PM, Saffin '21]

1PI$_{\text{average}}$

$$\partial_k \alpha(\Delta_k) = \frac{\partial \alpha^{1\text{PI}}(R_k)}{\partial R_k} \partial_k R_k \text{ etc.}$$

Take derivatives of $\left[\Delta^{1\text{PI}}\right]^{-1}$ with respect to ϕ at $\phi = 0$ and solve for $\partial^n \{\alpha^{1\text{PI}}, \beta^{1\text{PI}}, \gamma^{1\text{PI}}\}/\partial R_k^n$:

$$\partial_k \alpha^{1\text{PI}} = \frac{\hbar}{2} \frac{\partial_k R_k}{\left[\beta^{1\text{PI}} + R_k\right]}$$

$$\partial_k \beta^{1\text{PI}} = -\frac{\hbar}{2} \frac{\gamma^{1\text{PI}} \partial_k R_k}{\left[\beta^{1\text{PI}} + R_k\right]^2}$$

$$\partial_k \gamma^{1\text{PI}} = 3\hbar \frac{\left[\gamma^{1\text{PI}}\right]^2 \partial_k R_k}{\left[\beta^{1\text{PI}} + R_k\right]^3}$$

2PI$_{\text{reg}}$

$$\partial_k \alpha(\Delta_k) = \frac{\partial \alpha(\Delta_k)}{\partial \Delta_k} \partial_k \Delta_k \text{ etc.}$$

Take derivatives of Δ^{-1} with respect to ϕ and Δ at $\phi = 0$ and solve for $\partial^n \{\alpha, \beta, \gamma\}/\partial \Delta_k^n$ with $R_k^0 = R_k(\phi = 0, \Delta_k)$:

$$\partial_k \alpha = \frac{\hbar}{2} R_k^0 \frac{\partial_k \beta + \partial_k R_k^0}{\left[\beta + R_k^0\right]^2}$$

$$\partial_k \beta = -\left\{\frac{\hbar}{2}\gamma - \frac{\hbar^2}{2} \frac{\gamma^2}{\left[\beta + R_k^0\right]^2}\right\} \frac{\partial_k \beta + \partial_k R_k^0}{\left[\beta + R_k^0\right]^2}$$

$$\partial_k \gamma = \mathcal{O}(\gamma^4)$$

Flow equations [PM, Saffin '21]

$1\mathrm{PI}_{\mathrm{average}}$

$$\partial_k \alpha(\Delta_k) = \frac{\partial \alpha^{1\mathrm{PI}}(R_k)}{\partial R_k} \partial_k R_k \text{ etc.}$$

Take derivatives of $\left[\Delta^{1\mathrm{PI}}\right]^{-1}$ with respect to ϕ at $\phi = 0$ and solve for $\partial^n \{\alpha^{1\mathrm{PI}}, \beta^{1\mathrm{PI}}, \gamma^{1\mathrm{PI}}\} / \partial R_k^n$:

$$\partial_k \alpha^{1\mathrm{PI}} = \frac{\hbar}{2} \frac{\partial_k R_k}{\left[\beta^{1\mathrm{PI}} + R_k\right]}$$

$$\partial_k \beta^{1\mathrm{PI}} = -\frac{\hbar}{2} \frac{\gamma^{1\mathrm{PI}} \partial_k R_k}{\left[\beta^{1\mathrm{PI}} + R_k\right]^2}$$

$$\partial_k \gamma^{1\mathrm{PI}} = 3\hbar \frac{\left[\gamma^{1\mathrm{PI}}\right]^2 \partial_k R_k}{\left[\beta^{1\mathrm{PI}} + R_k\right]^3}$$

$2\mathrm{PI}_{\mathrm{reg}}$

$$\partial_k \alpha(\Delta_k) = \frac{\partial \alpha(\Delta_k)}{\partial \Delta_k} \partial_k \Delta_k \text{ etc.}$$

Take derivatives of Δ^{-1} with respect to ϕ and Δ at $\phi = 0$ and solve for $\partial^n \{\alpha, \beta, \gamma\} / \partial \Delta_k^n$ with $R_k^0 = R_k(\phi = 0, \Delta_k)$:

$$\partial_k \alpha = \frac{\hbar}{2} R_k^0 \frac{\partial_k \beta + \partial_k R_k^0}{\left[\beta + R_k^0\right]^2}$$

$$\partial_k \beta = -\left\{ \frac{\hbar}{2}\gamma - \frac{\hbar^2}{2} \frac{\gamma^2}{\left[\beta + R_k^0\right]^2} \right\} \frac{\partial_k \beta + \partial_k R_k^0}{\left[\beta + R_k^0\right]^2}$$

$$\partial_k \gamma = \mathcal{O}(\gamma^4)$$

Vertex functions

$\mathbf{1PI_{average}}$

$$\partial_k \gamma^{1\text{PI}} = 3\hbar \frac{\left[\gamma^{1\text{PI}}\right]^2 \partial_k R_k}{\left[\beta^{1\text{PI}} + R_k\right]^3}$$

n-point vertices are given by

$$\Gamma^{(n>2)}_{1\text{PI}} = -\left[\Delta^{1\text{PI}}\right]^{-n} \left\{ \Delta^{1\text{PI}} \frac{\partial}{\partial \phi} \right\}^{n-2} \Delta^{1\text{PI}}$$

giving

$$\Gamma^{(4)}_{1\text{PI}}\Big|_{\phi=0} = \gamma^{1\text{PI}}$$

$\mathbf{2PI_{reg}}$

$$\partial_k \gamma = \mathcal{O}(\gamma^4)$$

n-point vertices are given by [PM, Saffin '22]

$$\Gamma^{(n>2)} = -\Delta^{-n} \left\{ \Delta \left[\frac{\partial}{\partial \phi} - \frac{\partial^2 \Gamma^{2\text{PI}}}{\partial \phi \partial \Delta} \left(\frac{\partial^2 \Gamma^{2\text{PI}}}{\partial \Delta^2} \right)^{-1} \frac{\partial}{\partial \Delta} \right] \right\}^{n-2} \Delta$$

giving

$$\Gamma^{(4)}\Big|_{\phi=0} = \gamma - \frac{3}{2} \hbar \gamma^2 \frac{1}{\left[\beta + R_k^0\right]^2}$$

Concluding remarks

- Exploit the freedom in the **2PI effective action** to fix the **two-point source** to be the **regulator**, leading to **self-consistent flow equations** that are **complementary** to the well-known ones.

- Extremization wrt the regulator involved in this approach (a) leads to the usual **2PI resummation** and (b) potentially lessens the regulator dependence.

- Many questions: e.g., can usual approximation schemes and methods, i.e., **LPA**, **derivative expansion**, **PMS**, **regulator optimisation** be applied in the same way?

Thank you for your attention.

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