Tetracriticality in O(N) models

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O(N) models

- They have played an important role in our understanding of second order phase transitions.

- N-component vector order parameter
  N=1…Ising, N=2…XY, N=3…Heisenberg Model

- The playground of almost all the theoretical approaches…
  Exact solution (2d Ising), Renormalization group（$d=4-\varepsilon$, $2+\varepsilon$ expansion), conformal bootstrap

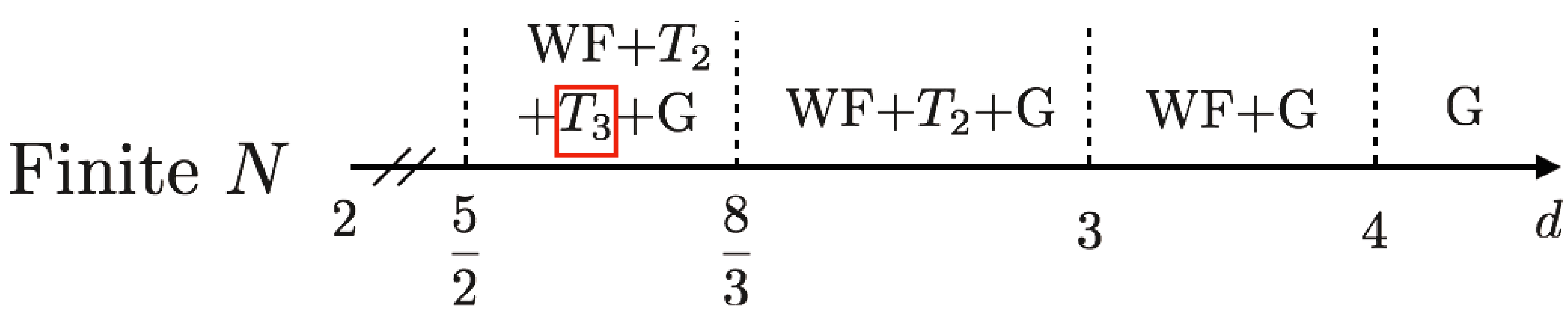
Everything is known about the criticality of O(N) models?
…This is what we want to challenge in this work.

Common wisdom on the criticality of O(N) models (finite N case)

GLW Hamiltonian $$H[\phi] = \frac{1}{2}\int_x (\nabla\phi_i)^2 + U(\phi)$$

$$U(\phi) = a_2\phi_i^2 + a_4(\phi_i^2)^2 + a_6(\phi_i^2)^3 + \ldots$$

Below the critical dimension $d_n = 2 + 2/n$, the $(\phi_i^2)^{n+1}$ term becomes relevant around the Gaussian FP (G).



WF …Wilson-Fisher FP

T_2 …Tricritical FP

T_3 …Tetracritical FP

> A nontrivial fixed point T_n with n relevant (unstable) directions branches from G at d_n .

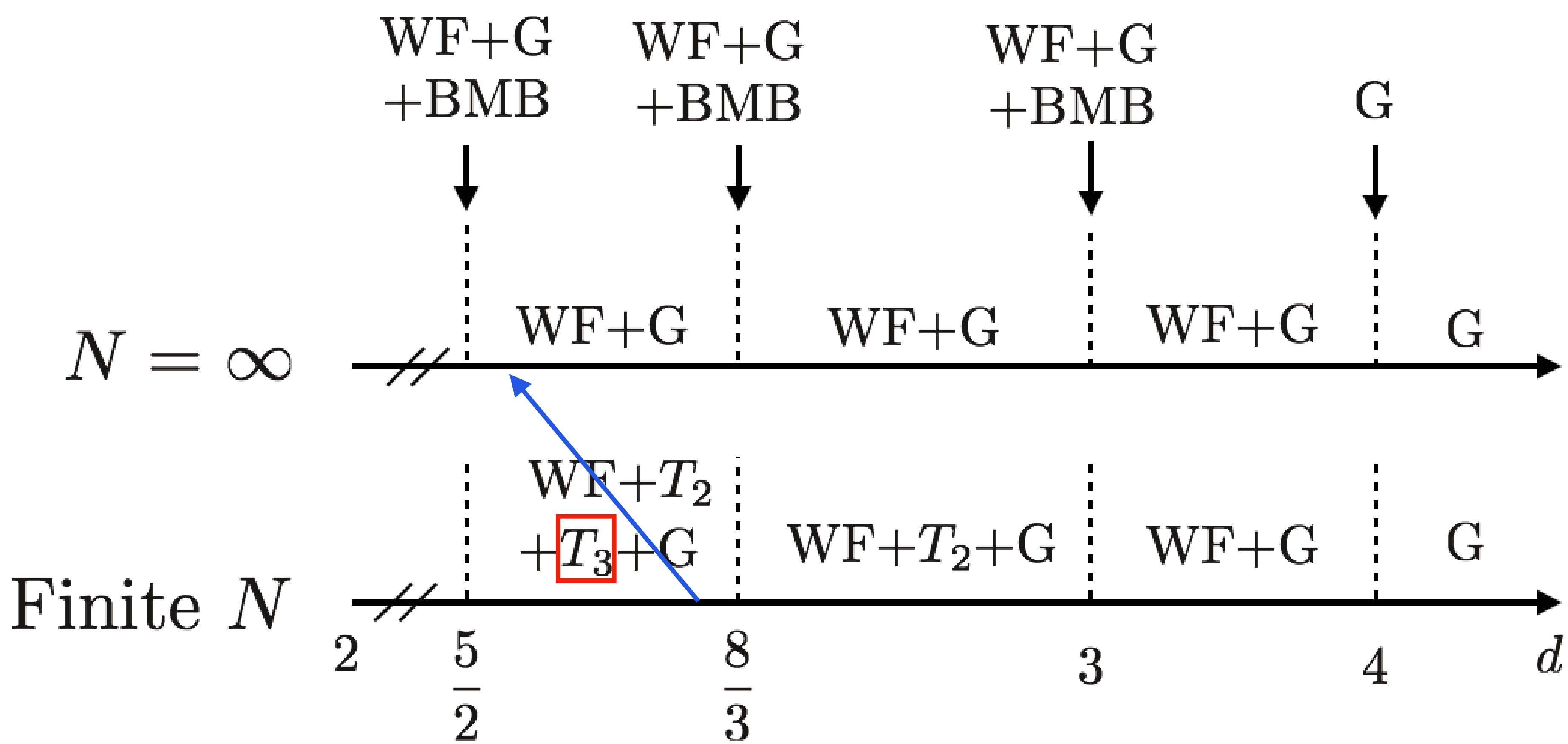
Common wisdom on the criticality of O(N) models at $N = \infty$

- At $N = \infty$, in generic dimensions 2<d<4, only Gaussian (G) and Wilson-Fisher (WF) FPs have been found, which is believed to be exact.

- LPA of NPRG is believed to be exact.

- Exceptional cases: At each $d_n = 2 + 2/n$, there exists a line of FPs starting from G and terminating at BMB (Bardeen-Moshe-Bander) FP.

Summary of common wisdom and a simple paradox



- What occurs if we follow T_3 from $d=8/3^-, N=1$ to $(d=2.6, N=\infty)$ continuously as a function of (d,N)?

We already studied a similar paradox about T_2.
(S. Yabunaka and B. Delamotte PRL 2017)

Fate of T_2 FP

Surprises in $O(N)$ Models: Nonperturbative Fixed Points, Large N Limits, and Multicriticality

Shunsuke Yabunaka and Bertrand Delamotte
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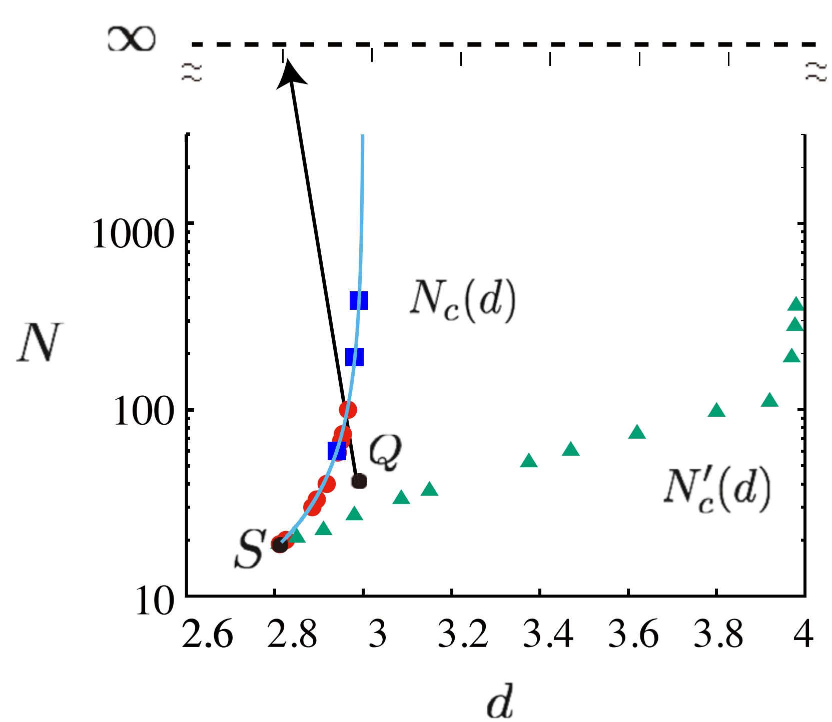

- If we continuously follow T_2 on the path indicated by the arrow in (d,N) space, T_2 collides with a nonperturbative FP C_3 and vanishes.

Possible scenarios

- T_3 disappears. (Collision with another FP?)
- T_3 exists at $N = \infty$ but is not found by usual large N methods (for which reason?)

Previous study

The fate of $O(N)$ multi-critical universal behaviour

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The multi-critical fixed points of $O(N)$ symmetric models cease to exist in the $N \to \infty$ limit, but the mechanism regulating their annihilation still presents several enigmatic aspects. Here, we explore the evolution of high-order multi-critical points in the (d, N) plane and uncover a complex mosaics for their asymptotic behaviour at large N. This picture is confirmed by various RG approaches and constitutes a fundamental step towards the full comprehension of critical behaviour in $O(N)$ field theories.

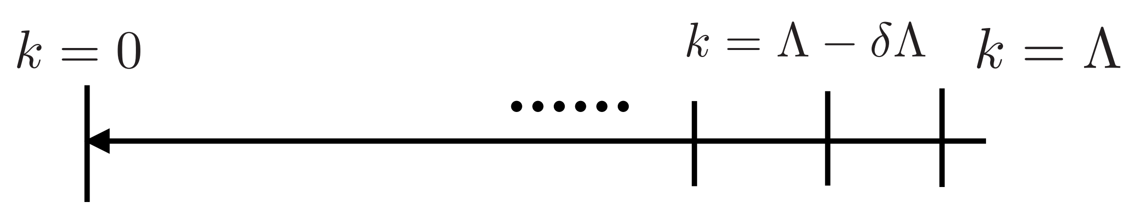
arXiv:2005.10827

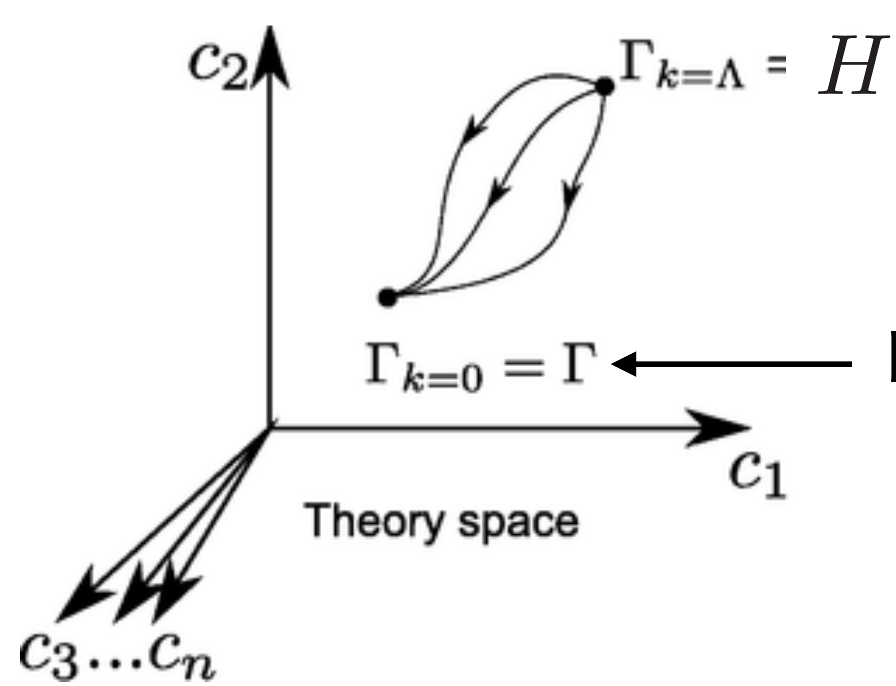
- T_3 was conjectured to exist up to N=∞ and approach the WF FP.

- However why it is not captured by conventional Large-N analysis seemed still unclear to us.

Non perturbative renormalization group (NPRG)

- Modern implementation of Wilson's RG that takes the fluctuation into account step by step in lowering the cut-off wavenumber k, in terms of wavenumber-dependent effective action Γ_k





Microscopic Hamiltonian

Effective action (Free energy), where all the fluctuations are taken into account.

Nondimensionalized NPRG eq.

- In order to capture scaling solutions as FP solutions,

$$\tilde{\phi} = \sqrt{Z_k} k^{\frac{2-d}{2}} \phi \qquad \tilde{\rho} = Z_k k^{2-d} \rho \qquad \tilde{U}_t(\tilde{\rho}) = k^{-d} U_k(\rho)$$

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Litim cutoff $y = \dfrac{q^2}{k^2}$ $R_k(q^2) = Z_k k^2 y r(y)$ $r(y) = (1/y - 1)\theta(1 - y)$

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Under LPA,

$$\partial_t \tilde{U}_t(\tilde{\phi}) = -d\, \tilde{U}_t(\tilde{\phi}) + \frac{1}{2}(d-2)\tilde{\phi}\, \tilde{U}_t'(\tilde{\phi}) + (N-1)\, \frac{\tilde{\phi}}{\tilde{\phi} + \tilde{U}_t'(\tilde{\phi})} + \frac{1}{1 + \tilde{U}_t''(\tilde{\phi})}.$$

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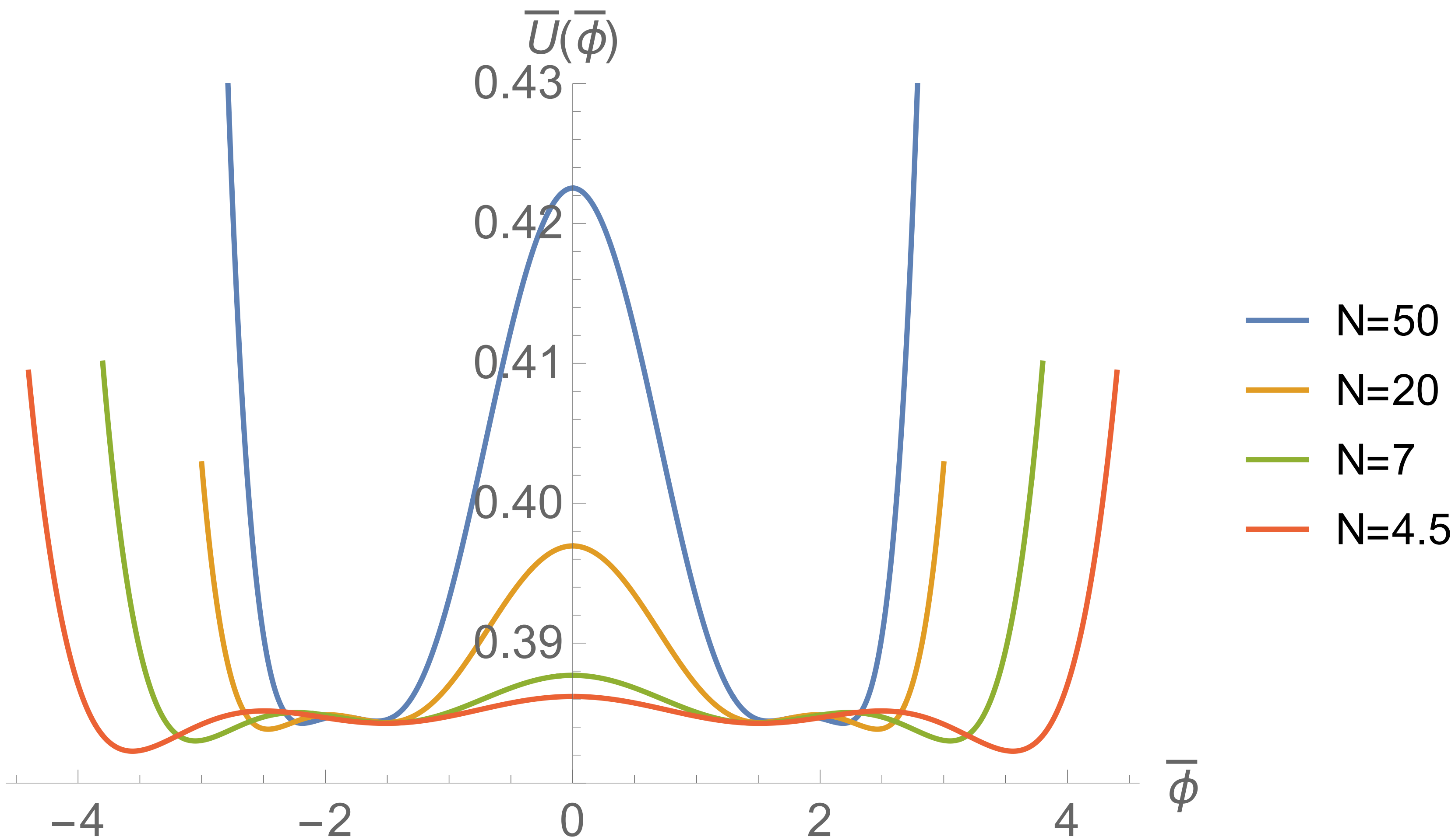
Rescaled finite N equation $\tilde{U}_t = N\bar{U}_t$ $\tilde{\phi} = \sqrt{N}\bar{\phi}$

$$\partial_t \bar{U}_t(\bar{\phi}) = -d\,\bar{U}_t(\bar{\phi}) + \frac{1}{2}(d-2)\bar{\phi}\,\bar{U}'_t(\bar{\phi}) + \left(1 - \frac{1}{N}\right)\frac{\bar{\phi}}{\bar{\phi} + \bar{U}'_t(\bar{\phi})} + \frac{1}{N}\frac{1}{1 + \bar{U}''_t(\bar{\phi})}$$

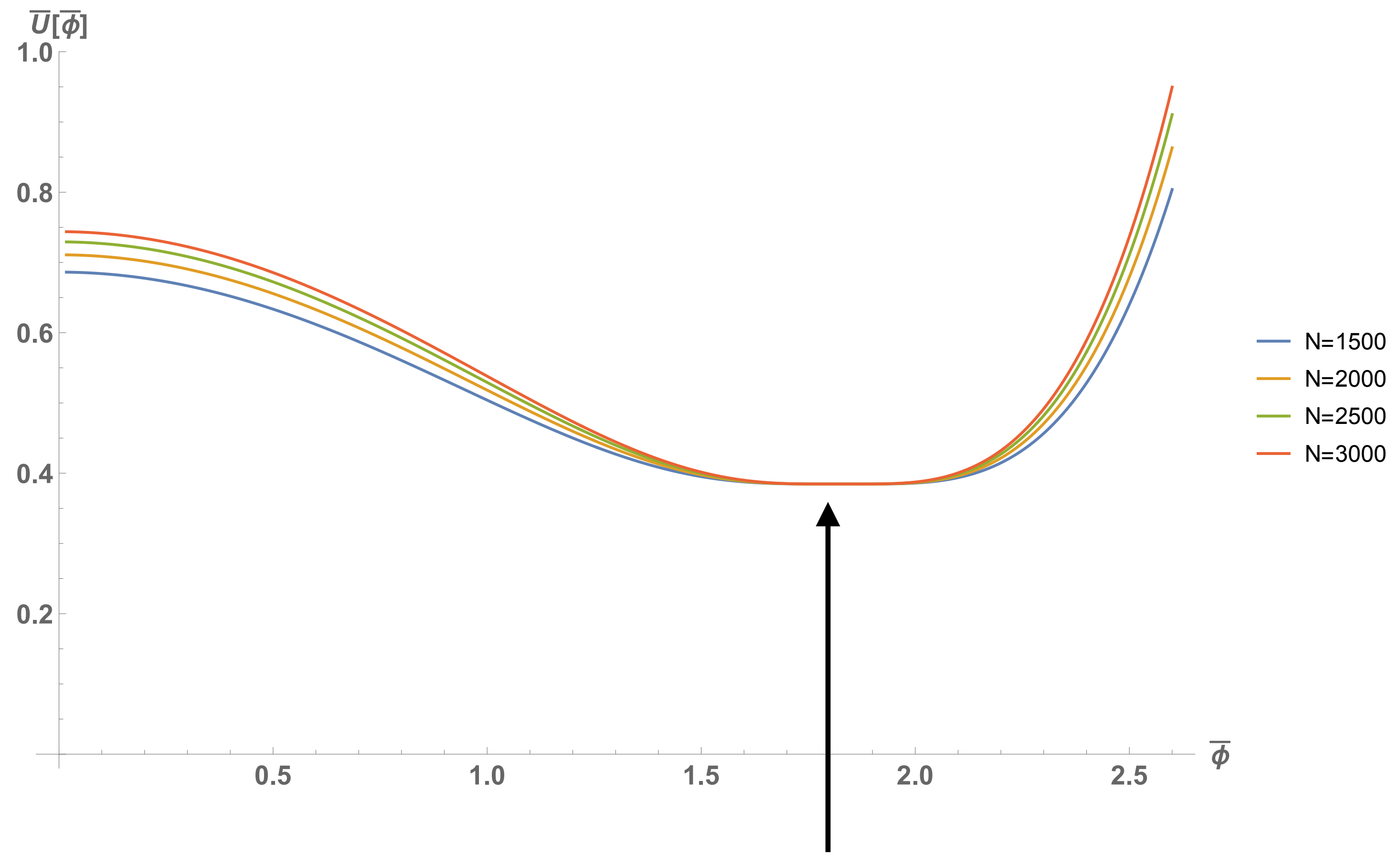
T_3 in d=2.6



T_3 has four minima.

Three extrema (one maximum and two minima) in the region $\bar{\phi} > 0$ approach when increasing N

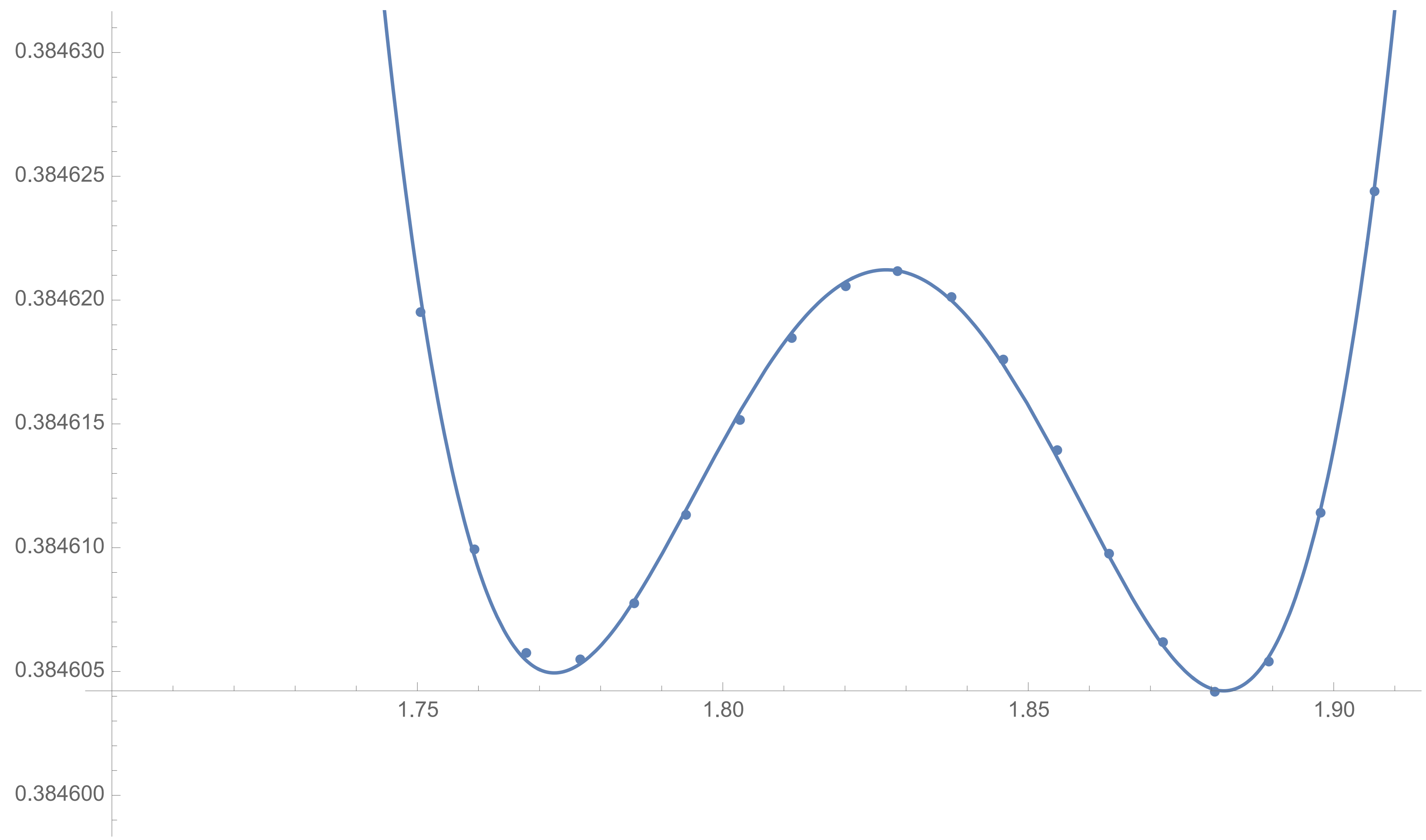
T_3 in d=2.6



The potential becomes very flat,
since three extrema become very close.

Numerically T_3 continues to exist up to N=∞.
Why the Large-N limit is not captured by conventional Large-N analysis??

Close view of the flat part of the potential for N=1500



- Fitting with quartic functions, we identify the positions of the three extrema.

Scaling behavior inside the boundary layer

- For very large N, the distances between the three extrema are proportional to $\epsilon \equiv N^{-1/2}$.

- $\bar{U}''(\bar{\phi})$ at the three extrema approach constant values. $\cdots$The third and higher order derivatives become singular.

- We can expect a scaling $\bar{U}''\left(\bar{\phi}\right) \simeq f\left(N^{1/2}\left(\bar{\phi} - \bar{\phi}_0\right)\right)$.

- We can identify the position of the boundary layer as $\bar{\phi}_0 \simeq \sqrt{2/(d-2)}$, from numerical solutions and boundary layer analysis

Boundary layer analysis

- To simplify the notation we employ Wilson-Polchinski version of LPA FP eq.

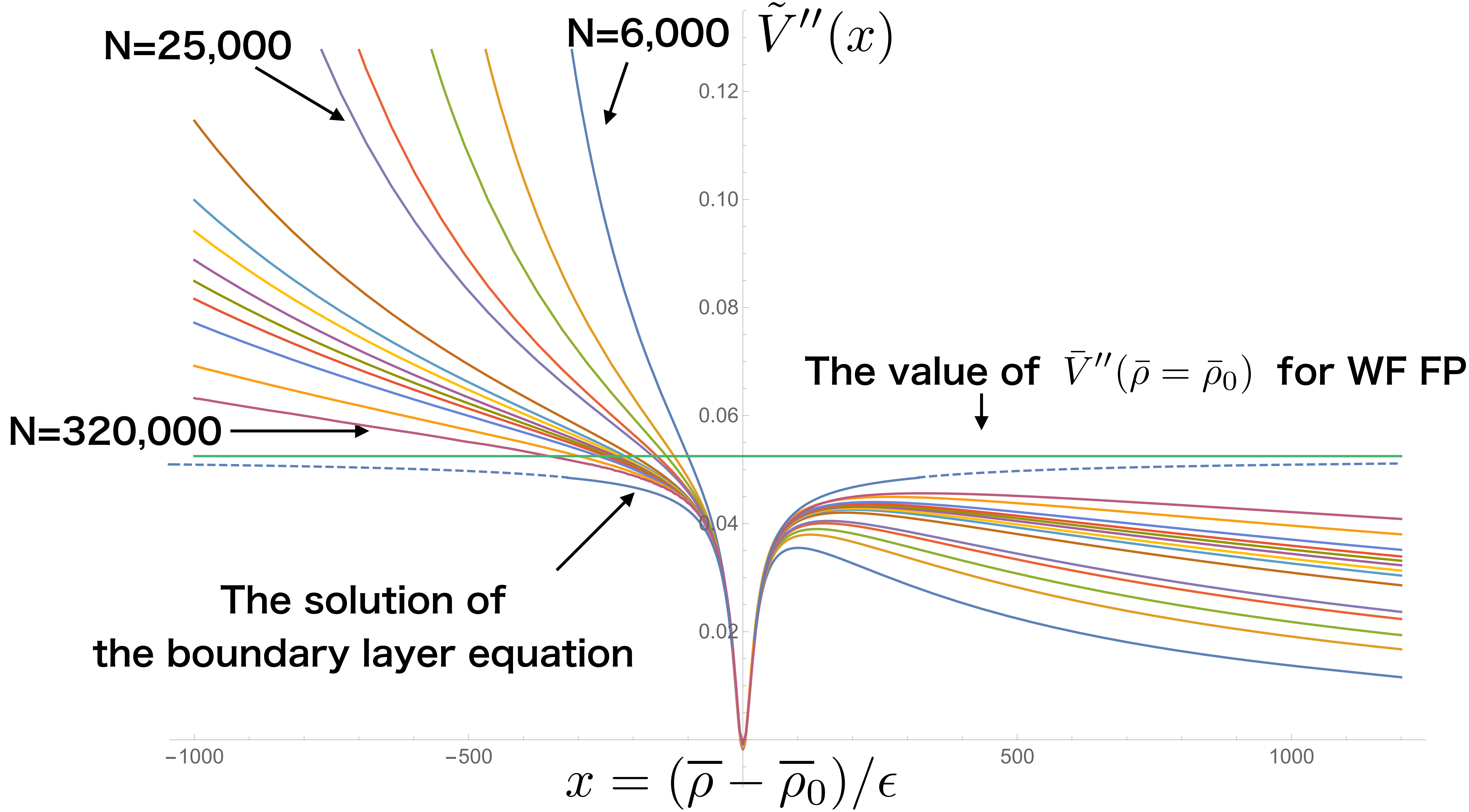
$$0 = d\bar{V} + (d-2)\bar{\varrho}\bar{V}' + 4\bar{\varrho}\bar{V}'^2 - 2\bar{V}' - \frac{4}{N}\bar{\varrho}\bar{V}''$$

- Around $\overline{\rho_0} = 2/(d-2)$ ($\bar{\phi}_0 \simeq \sqrt{2/(d-2)}$) ,
we introduce a scaled variable $x = (\bar{\rho} - \overline{\rho_0})/\epsilon$,
and the potential is scaled as $\bar{V}(\epsilon x + \overline{\rho_0}) = \epsilon^2 \tilde{V}(x)$.

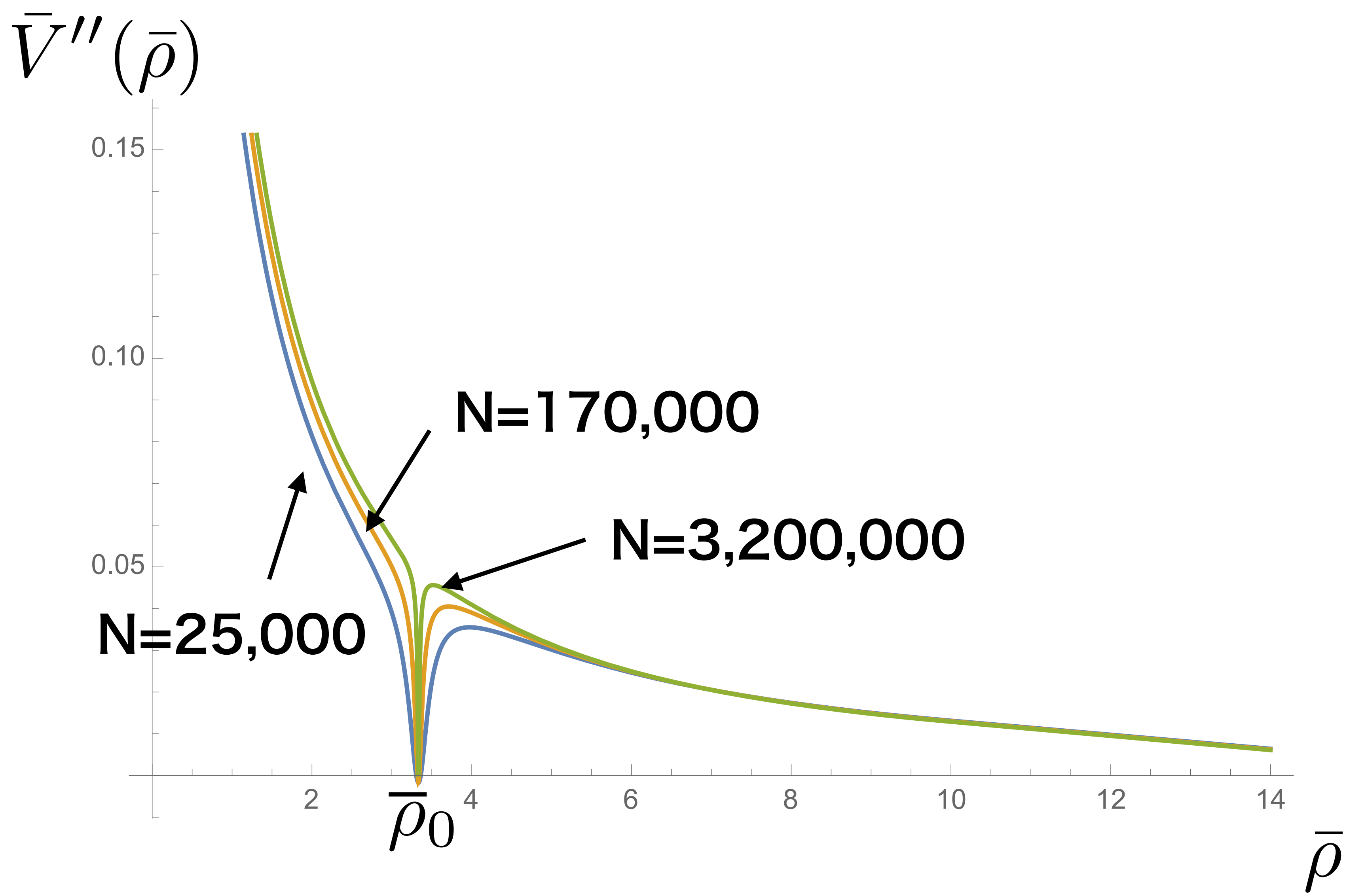
- The $O(\epsilon)$ contribution of FP eq. vanishes if we set $\rho_0 = 2/d-2$, and the $O(\epsilon^2)$ contribution is

$$\left(-\frac{8\tilde{V}''(x)}{d-2} + \frac{8\tilde{V}'(x)^2}{d-2} + (d-2)x\tilde{V}'(x) - d\tilde{V}(x) \right) = 0$$

Scaled boundary layer for finite but very large N



Global plot of the second derivative of the potential



It converges to that of WF FP except at $\bar{\rho} = \bar{\rho}_0$

Summary

- We followed tetracritical FP T_3 in O(N) models increasing N with LPA.
- It seems that T_3 continues to exist up to N=∞. The third and higher order derivatives become singular at $\bar{\rho} = \bar{\rho}_0$.
  This is why it is not seen in the usual N=∞ approach that relies on analyticity of the FP potential
- The potential converges to that of WF FP except at $\bar{\rho} = \bar{\rho}_0$.
- Can we conjecture that a similar scenario holds for T_n with odd n?