Ultraviolet properties of Lifshitz-type field theories

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Based on D. Zappalà, Eur. Phys. J. C 82 (2022) 341.

Outline :

* Horava-Lifshitz Gravity-inspired Higher Derivative Field Theories with space-time anisotropy.

* Smoothening of Ultraviolet divergences.
  Source of Lorentz Violation.

* Restriction to a consistent scalar and fermionic theory.
  Extension to gauge fields.

* Ultraviolet completion and effects on hierarchy problem.

Action including space derivatives of order 2z

$$S = \int d^3x\, dt \left(\frac{1}{2} \left(\partial_t \phi\right)^2 - \frac{\widehat{a}_z}{2} \left(\partial_i^z \phi\right)^2 - \frac{\widehat{a}_{z-1}}{2} \left(\partial_i^{z-1} \phi\right)^2 - \ldots - \frac{\widehat{a}_1}{2} \left(\partial_i \phi\right)^2 - V \right)$$

$$V = \sum_{n=2}^{\overline{n}} \frac{\widehat{g}_n}{n!} \phi^n$$

* Built in the spirit of Horava-Lifshitz gravity to improve UV behavior.
* Higher time derivative are excluded to preserve unitarity.

K.S. Stelle, Phys. Rev. D **16**, 953–969 (1977).
S.Chada, H.B.Nielsen, Nucl. Phys. B217 (1983) 125.
P. Horava, Phys. Rev. D 79, 084008(2009); Phys. Rev. Lett. 102, 161301(2009).
R.P.Woodard, Lect. Notes Phys. 720 (2007) 403.
D.Anselmi, M. Halat, Phys. Rev. D 76 (2007) 125011.

Scaling in the UV region regulated by the presence of anisotropic Lifshitz points

Under scale transformations with $b > 1$

$$x^i \to b\, x^i$$

$$t \to b^z\, t$$

scaling dimensions are

$$[x^i]_s = -1$$

$$[t]_s = -z$$

$$[\widehat{a}_z]_s = -\eta$$

and Anisotropic Lifshitz Points appear.

R. Hornreich, M. Luban, S. Shtrikman, Phys. Rev. Lett. 35 (1975) 1678.
R. Hornreich, J. Magnetism and Magnetic Materials 15 (1980) 387

Anisotropic Lifshitz Points (ALP)

Euclidean *d* - dimensional system with *d_s* "spacelike" and (*d* - *d_s*) "timelike" coordinates

$$[x^i]_s = -1 \qquad [t]_s = -z$$

Field scaling dimension $$[\phi]_s = [z\,(d - d_s - 2) + d_s + \eta]/2$$

Coupling scaling dimension $$[\widehat{g}_n]_s = \left(1 - \frac{n}{2}\right)[\,z(d - d_s) + d_s\,] + nz$$

Range of existence of non-trivial ALP (if $\eta = 0$) :

R. Hornreich, M. Luban, S. Shtrikman, Phys. Rev. Lett. 35 (1975) 1678.

Upper critical dim. lines from $[\widehat{g}_4]_s = 0$

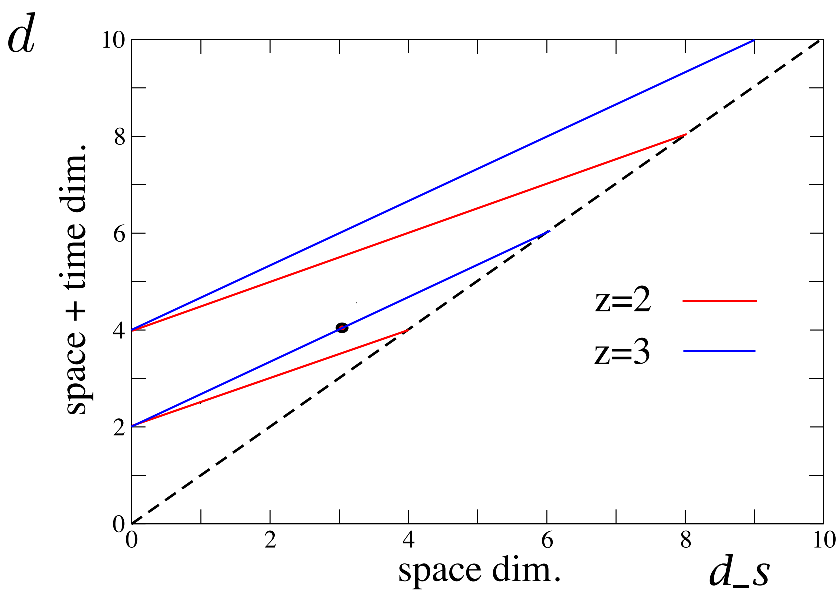
Lower critical dim. lines from $[\phi]_s = 0$

Physical point with $d = 3 + 1$

sits on the lower line for z=3 and

only the Gaussian-like ALP is expected :

$$S_{FP} = \int d^3x\, dt \left(\frac{1}{2}\,(\partial_t \phi)^2 - \frac{\widehat{a}_z}{2}\,(\partial_i^z \phi)^2\right)$$



Lorentz symmetry violation (LV)

Maximum z with non-negative field scaling dimension —> $z=3$ and $[\phi]_s = 0$

$$S = \int d^3x\, dt \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{\widehat{a}_3}{2} \left(\partial^2 \partial_i \phi \partial^2 \partial_i \phi \right) - \frac{\widehat{a}_2}{2} \left(\partial^2 \phi \partial^2 \phi \right) - \frac{\widehat{a}_1}{2} \left(\partial_i \phi \partial_i \phi \right) - V(\phi) \right]$$

$$V = \sum_{n=2}^{\overline{n}} \frac{\widehat{g}_n}{n!} \phi^n$$ and in principle $\overline{n} \to \infty$

BUT the field ϕ and the couplings a and g have non-zero canonical dimension

Introduce one reference scale M to normalize all dimensional variables

$$\widehat{a}_s = \frac{a_s}{M^{2(s-1)}} \Bigg|_{s=1,2,3} \qquad \widehat{g}_n = \frac{g_n}{M^{n-4}} \Bigg|_{n=2,3,4,\ldots}$$

The dispersion relation reads

$$E^2 = \vec{k}^{\,2} \left[a_1 + a_2 \left(\frac{k}{M} \right)^2 + a_3 \left(\frac{k}{M} \right)^4 \right] + g_2 M^2$$

M plays the role of Lorentz Symmetry Violating scale :
Two different regimes occur above and below M

Pattern of divergences above the scale M

Propagator —> $\dfrac{1}{\left[k_0^2 - A^2 + i\epsilon\right]}$ with $A = \sqrt{\widehat{a}_3\,\vec{k}^{\,6} + \widehat{a}_2\,\vec{k}^{\,4} + \widehat{a}_1\,\vec{k}^{\,2} + \widehat{g}_2}$

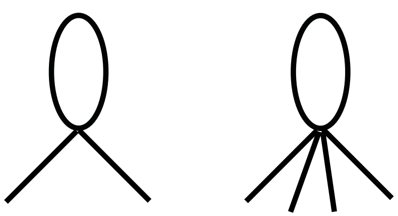
Primitive degree of divergence

$$D_\Lambda = [3L - 2z\,(I - L) - z\,L]\,|_{z=3} \;=\; 6(L - I) \;=\; 6\left(1 - \sum_n V_n\right)$$

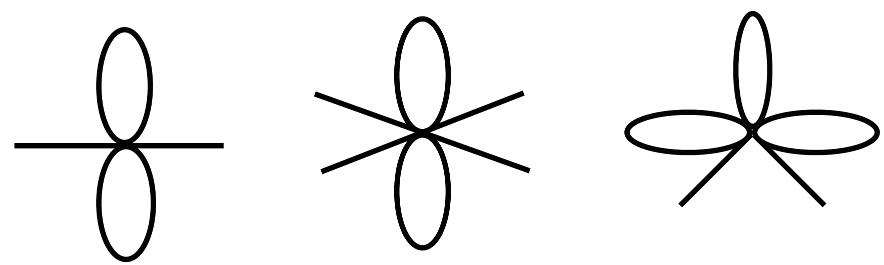
Only the number of vertices Vn is present.

NO dependence on E = # of external lines NOR on n = # of lines linked to Vn.

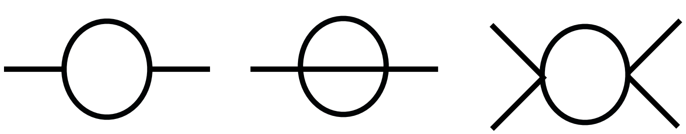
Only diagrams with ONE vertex diverge logarithmically.



and multiloop



Finite diagrams



a1, a2, a3 get only finite corrections

$$\widehat{I}_1 = \int \frac{d^3k\,dk_0}{(2\pi)^4}\,\frac{i}{\left(k_0^2 - \widehat{a}_3\vec{k}^{\,6} - \widehat{a}_2\,\vec{k}^{\,4} - \widehat{a}_1\vec{k}^{\,2} - \widehat{g}_2 + i\epsilon\right)}$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{\widehat{a}_3\vec{k}^{\,6} + \widehat{a}_2\,\vec{k}^{\,4} + \widehat{a}_1\vec{k}^{\,2} + \widehat{g}_2}}$$

Selection of the renormalizable Action

D. Zappala, Eur. Phys. J. C 82 (2022) 341.

$$S = \int d^3x\, dt \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{\widehat{a}_3}{2} (\partial^2 \partial_i \phi \partial^2 \partial_i \phi) - \frac{\widehat{a}_2}{2} (\partial^2 \phi \partial^2 \phi) - \frac{\widehat{a}_1}{2} (\partial_i \phi \partial_i \phi) - V(\phi) \right] \qquad V = \sum_{n=2}^{\overline{n}} \frac{\widehat{g}_n}{n!} \phi^n$$

$$\widehat{a}_s = \left. \frac{a_s}{M^{2(s-1)}} \right|_{s=1,2,3}$$

Get only finite renormalization

According to scaling dimensional analysis, the renormalizable action should include :

R. Iengo, J.G Russo, M.Serone, , JHEP 11 (2009) 020.
J. Alexandre, Int.J. Mod.Phys. A 26 (2011), 4523.

$$w_{m,s}\ \phi^m \left(\partial_i^s \phi \partial_i^s \phi \right)$$

Momentum dependent vertices that alter the previous divergence analysis.

△ Dangerous consequences :

Logarithmic corrections prop. to $w_{m,s}$ modify the 'speed of light' a_1

Different interacting fields show a potentially detectable difference of their speed of light Δa_1

Experimentally forbidden —> Fine tuning on $w_{m,s}$ required !

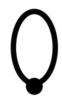
Our constraint $w_{m,s} = 0$ is consistent, as no divergent loop corrections to $w_{m,s}$ are generated

$$(p_i\ p_i')^s\ \mathcal{W}_{m,s} = (p_i\ p_i')^s \left[\left(\frac{\partial^2}{\partial p_j\ \partial p_j'} \right)^s \Gamma^{(m+2)}(p, p', 0, \ldots, 0) \right]_{p=p'=0}$$

and therefore it does not entail any unnatural fine tuning.

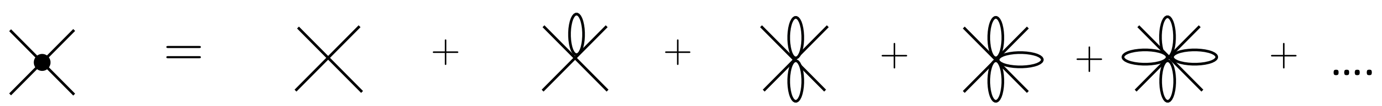
Divergences and properties in the UV range (1)

Integrate the tadpole between *M* (larger than other IR scales) and the 3-momentum UV cutoff Λ



$$\widehat{I}_1(\Lambda, M) = M^2\, I_1(\Lambda, M) = \frac{M^2}{(2\pi)^2}\ \ln\left(\frac{\Lambda}{M}\right) + O\left(\frac{M^4}{\Lambda^2}\right)$$

Divergent diagram series for the 4-point vertex. (It is the same for any n-point vertex)



$$V = \sum_{n=2}^{\overline{n}} \frac{\widehat{g}_n}{n!}\phi^n$$

$$\widehat{g}_n = \frac{g_n}{M^{n-4}}\bigg|_{n=2,3,4,\ldots}$$

Combinatorics produce truncated expon. series.

$$g_{2R} = g_2 + g_4\left(\frac{I_1}{2}\right) + \frac{g_6}{2!}\left(\frac{I_1}{2}\right)^2 + \frac{g_8}{3!}\left(\frac{I_1}{2}\right)^3 + \ldots + \frac{g_{\overline{n}}}{(\overline{n}/2-1)!}\left(\frac{I_1}{2}\right)^{(\overline{n}/2-1)}$$

$$g_{4R} = g_4 + g_6\left(\frac{I_1}{2}\right) + \frac{g_8}{2!}\left(\frac{I_1}{2}\right)^2 + \frac{g_{10}}{3!}\left(\frac{I_1}{2}\right)^3 + \ldots + \frac{g_{\overline{n}}}{(\overline{n}/2-2)!}\left(\frac{I_1}{2}\right)^{(\overline{n}/2-2)}$$

* And so on for g_{6R}, g_{8R}, etc.

* Analogously for odd indexes.

1) If $\overline{n} \to \infty$ and $g_n = g > 0,\ \forall n$ then : $g_R = g\, e^{I_1/2}$ and $\lim_{\Lambda\to\infty} g = 0$

2) If $\overline{n} \to \infty,\ g_n > 0,\ \forall n,\ g = Sup\{g_n\}$ then: $g_{nR} \in \Re$ and $\lim_{\Lambda\to\infty} g_n = 0$

3) If $\overline{n}$ finite, then g_n grows as a power of logarithm ${I_1}^n$

Divergences and properties in the UV range (2)

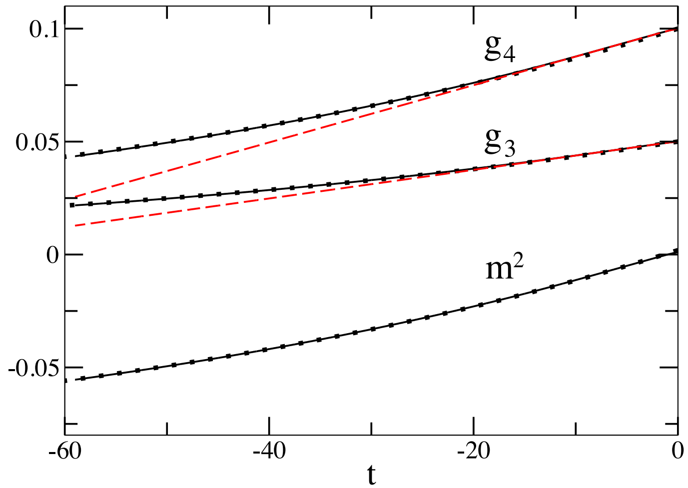
Running of g_4, g_3, $m^2 = g_2$ vs $t = \ln(M/\mu)$ and μ is the running scale. Boundaries fixed at M

Black dotted : exponentially vanishing flow of couplings

$$g_j(t) = g_j(t=0)\ \mathrm{e}^{\frac{t}{8\pi^2}}$$

Black solid: truncated series of couplings at $\overline{n} = 22$ divergent at very large t)

Red dashed : truncated series of couplings at $\overline{n} = 6$ (linearly divergent)



m^2 : case with two independent parameters $g_2(t)$ and $g(t) = g_4(t) = g_6(t) = g_8(t) = \ldots$

$$m^2(t) = m^2(0) + g(0)\left(e^{\frac{t}{8\pi^2}} - 1\right)$$

m^2 gets a negative finite correction in the UV limit $t \to -\infty$

Inclusion of Fermions (1)

Natural generalization of higher derivative structure

$$S_{ferm} = \int d^3x\, dt\, \bar{\psi} \left[i\gamma^0 \partial_0 - \left(a_1 + a_3 \frac{\partial_j \partial_j}{M^2} \right) (i\gamma^i \partial_i) - m_f \right] \psi \qquad [\psi]_s = \frac{3}{2}$$

2n-Fermion interaction $\mathcal{G}_n (\bar{\psi}\psi)^n$ renormalizable only if n=2. Replaced by Yukawa interaction via Hubbard-Stratonovich transf.

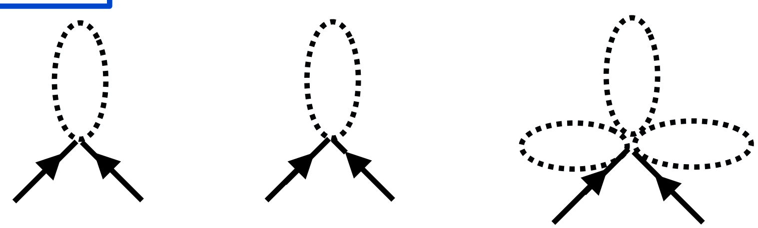
Scalar + Fermion $S = S_{scal} + S_{ferm} - \int d^3x\, dt\, V_Y$

Yukawa inter. $V_Y = \sum_i^{\overline{n}} y_n \bar{\psi} \psi \frac{\phi^n}{M^{n-1}}$

$$D_\Lambda = -3V - \frac{3}{2} E_f + 6$$ —> divergent diagrams only if: $V \leq 2 - \frac{E_f}{2}$

Only logarithmic divergences

$E_f = 2: \quad V = 1$

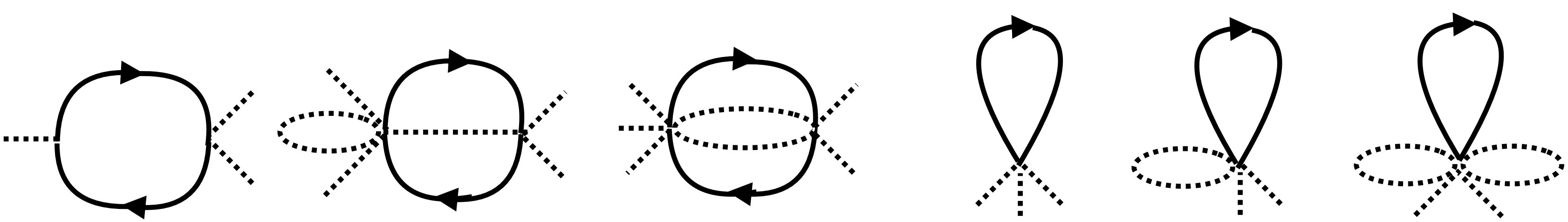


Finite corrections to fermion mass

$E_f = 0$

$V = 2$

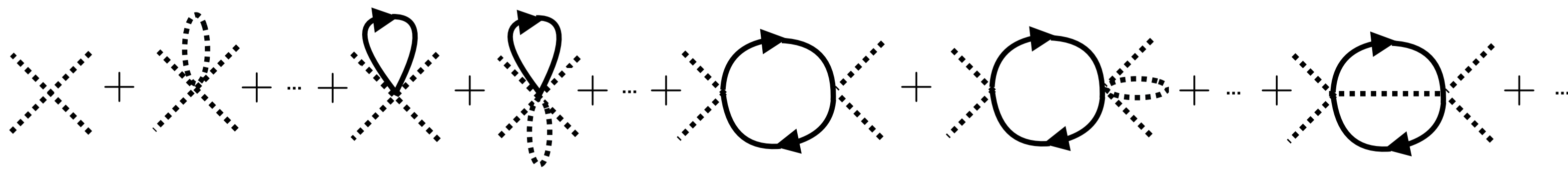
$V = 1$



Inclusion of Fermions (2)

Simple summable structure of divergences due to absence of momentum dependent vertices.

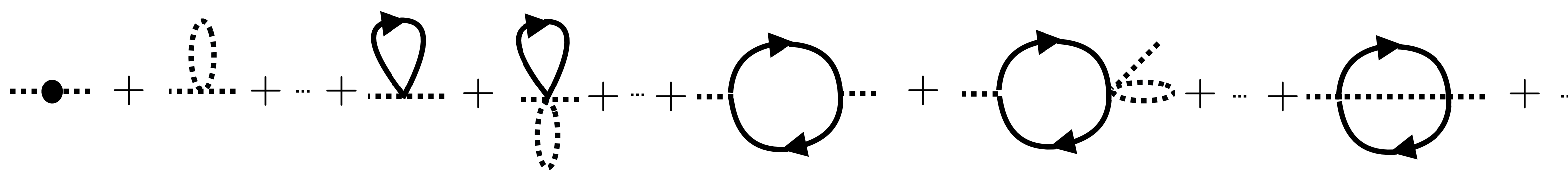
Elementary case with $\overline{n} \to \infty,\ g_n = g,\ y_n = y,\ \forall n$ except $g_2 = m^2$

Sum for g 

$$g(M^2) = gE - 4\left(\frac{m_f}{M}\right) yEI_1 - (yEI_1)^2\,(12 + 16E^2)$$

$I_1 = (1/4\pi^2)\log(\mu/M)$

$$E = \exp(I_1/2)$$

Sum for m² 

$$m^2(M^2) = m^2 - g + gE - 4\left(\frac{m_f}{M}\right) yEI_1 - 2(yE^2I_1)^2$$

From coefficients of the most divergent term ▲

$$g^{-1} \propto E$$
$$y^{-1} \propto E^2 I_1$$

1) g and m renormalization conditions are finite.

2) Both couplings are asymptotically free.

Corrections to the scalar mass from renormalization of g and m^2

In the limit $\mu >> M$ —> $m^2(M^2) - m^2(\mu^2) = g(M^2) + 14(yE^2 I_1)^2 > 0$

No boson-fermion cancellation if $g(M^2) > 0$

$$\widehat{m}^2(M^2) - \widehat{m}^2(\mu^2) \simeq O(M^2\, g(M^2))$$

Finite square mass corrections
BUT proportional to M^2

Corrections comparable to those generated in the IR region below M

Experimental indications on the Lorentz violating scale M

$$c_g(k) = 1 + \lambda \left(\frac{k}{M}\right)^{\alpha}$$

Bounds on M from time of flight difference
of high and low energy photons from GRB.
They depend on the form of the dispersion rel.

$$\alpha = 1 \qquad M \simeq 10^{17}\ GeV$$
$$\alpha = 2 \qquad M \simeq 10^{10}\ GeV$$

$$E^2 = \vec{k}^2 \left[a_1 + a_2 \left(\frac{k}{M}\right)^2 + a_3 \left(\frac{k}{M}\right)^4\right] + m^2$$

$$\alpha = 2 \qquad M \simeq 10^{10}\ GeV$$

J. Ellis, R.Konoplich, et al.
Phys. Rev. D 99 (2019) 083009.
B. Chen, Q.G: Huang, Phys. Lett. B
683, (2010) 108.
J. Ellis, N. Mavromatos, et al.
Astron.Astrophys.402 (2003) 409.

Extremely large scale if compared to the Higgs mass

Gauge fields

Plain construction of generalized covariant derivative

$$\partial'_\mu = \left\{ \partial_0, \ \left(a_1 + a_3 \frac{\partial_j \partial_j}{M^2} \right) \partial_i \right\} \quad \longrightarrow \quad \begin{aligned} D'_\mu &= \partial'_\mu - iqA_\mu \\ F'_{\mu\nu} &= \partial'_\mu A_\nu - \partial'_\nu A_\mu \end{aligned}$$

$$S_{SQED} = \int d^3x\, dt \left(D'_\mu \phi \left(D'^\mu \phi \right)^* - V(\phi) - \frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} \right)$$

Gauge invariance violated

Still, with Pauli-Villars regularisation, quantum corrections to the photon mass vanish and gauge non-invariant terms are suppressed at low energy as $(E/M)^2$.

Gauge invariant alternative with standard covariant derivative

$$S_{YM} = \int d^3x\, dt \left[\frac{1}{2} \mathrm{Tr}(E_i E_i) - \frac{1}{2} \mathrm{Tr} \left(D_h D_j F_{ik} \right)^2 \right]$$

Speed of light altered by logarithmic divergences through terms like $A_i A_i \left(\partial^2 A\, \partial^2 A \right)$

P. Horava, Phys. Lett. B 694, (2011), 172.
W. Chao, Commun. Theor.Phys. 65 (2016), 743.
J. Alexandre, Int.J. Mod.Phys. A 26 (2011), 4523

Modified covariant der. —> No Gauge inv. & No Lorentz symm. if E>>M (effectively recovered at E<<M)

No restraint to use 3-momentum cut-off regulator.

Then, the effect of Gauge fields is equivalent to adding further scalar d.o.f.
Above conclusions on scalar+fermion qualitatively unchanged.

Conclusions

* Anisotropic (z=3) higher derivative theories are renormalizable and in some cases have asymptotically free interactions.

* Phenomenology requires restriction of the Action general form. Still, above scale *M*, the theory is affected by Lorenz and Gauge violating effects.

* Quantum corrections to the scalar square mass are $O(gM^2)$: No progress with respect to a standard effective theory with UV cutoff = *M*.

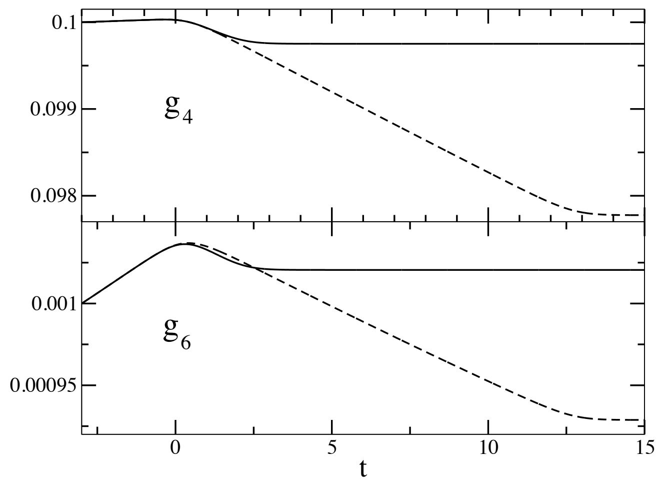
Crossover regime around M (1)

Boundaries fixed at t= - 3

In particular:

Solid lines: $m^2 = 0.007$

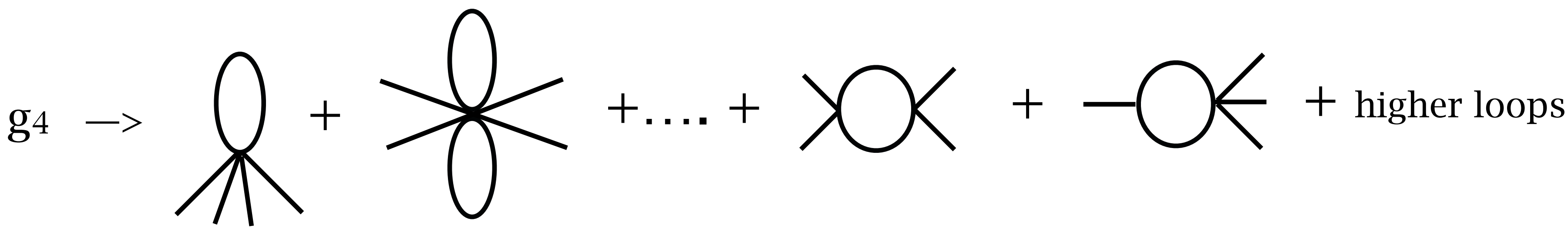
Dashed lines: $m^2 \simeq -0.004$



Running stops

Below the IR mass

Due to the modified propagator :



Dominant above and around M

Dominant below M

Change of slope around M due to the number of vertices involved

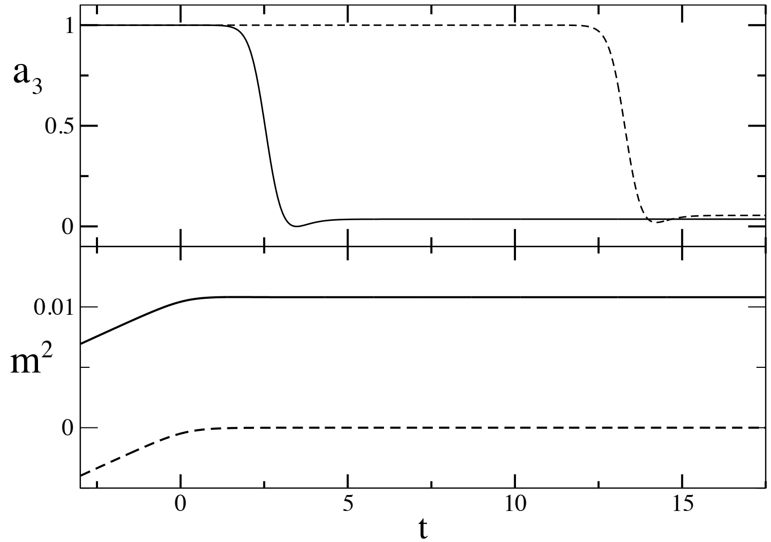
Crossover regime around M (2)

Boundaries fixed at t= - 3

In particular:

Solid lines: $m^2 = 0.007$

Dashed lines: $m^2 \simeq -0.004$



a2 gets much smaller correction.

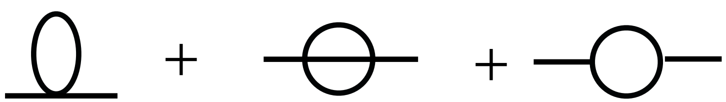
a1 extremely small correction.

$m^2 \simeq 10^{-2}$

$m^2 \simeq 10^{-12}$

IR masses in units of M^2

g_2 —>



+ higher loops

Dominant above and around M

Dominant below M

Fine tuning on mass boundary value to adjust its IR value

a1, a2, a3 only finite corrections. But if g2 < 0 then from the propagator
$$\frac{1}{\sqrt{\hat{a}_3\vec{k}^{\,6} + \hat{a}_2\vec{k}^{\,4} + \hat{a}_1\vec{k}^{\,2} + \hat{g}_2}}$$
a singularity could is developed unless g2 in fine tuned.

Regardless of the UV finite (or vanishing), or logarithmically (weakly) divergent behavior, below M^2 the quadratic growth of m^2 is unavoidable and requires fine tuning, which is also essential to avoid singularities in the propagator.