The Effective Average Action with Smooth Cutoffs

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July 26, 2022

Outline

A Quick Recap

The Effective Average Action

- $Z_k(J) = \int \exp\left[\int J\phi - S_{\text{eucl}}(\phi) - \frac{1}{2}F_k(\phi)(\phi)\right]\mathcal{D}\phi$
- $W_k = \ln \circ Z$
- $\Gamma_k(\phi) = \sup_J\left[\int J\phi - Z_k(J)\right] - \frac{1}{2}F_k(\phi)(\phi)$

The Flow Equation

- For nice enough F_k, $\lim_{k\to\infty}\Gamma_k(\phi) = S_{\text{eucl}}(\phi)$ and

$$\partial_k\Gamma_k(\phi) = \frac{1}{2}\text{Tr}\left\{(\partial_k F_k)\left[D^2_\phi\Gamma_k + F_k\right]^{-1}\right\}$$

How is This Derived?

Inverting the Derivatives

- Supremum in definition of Γ_k should be attained
- Extreme Point should be unique
- The derivative DW_k should be invertible

$$\Gamma_k(\phi) = (DW_k)^{-1}(\phi)(\phi) - W_k\left((DW_k)^{-1}(\phi)\right) - \frac{1}{2}F_k(\phi)(\phi)$$

Boundary Condition

- F_k should diverge sufficiently fast for $k \to \infty$

Mathematical Tasks

TODO List

- Rigorously define the path integral
- Verify assumptions about W_k and DW_k
- Find strong enough condition for F_k
- Prove everything

The Path Integral of a Free Scalar Field

Understanding the Path Integral

Let $\phi \in \mathcal{S}$ be an arbitrary test function. Is there a measure μ on some space X such that

$$\int_X T(\phi)\, T(\phi)\, \mathrm{d}\mu\,(T) = \int_{\mathbb{R}^d} \tilde{\phi}\,(-p)\, \frac{1}{p^2 + m^2} \tilde{\phi}\,(p) \tag{1}$$

and the higher correlators also match those of the Euclidean free scalar field? This precisely corresponds to making sense of

$$\int_X T(\phi)\, T(\phi) \exp\left[-\frac{m^2}{2} \int T^2 - \frac{1}{2} \int (\partial_\nu T)\,(\partial_\nu T)\right] \mathrm{d}T. \tag{2}$$

The Path Integral of a Free Scalar Field

The Gaussian Measure

- Such a measure exists
- $X = \mathcal{S}'_\beta$ (tempered distributions, dual space of $\mathcal{S}$).
- Note, T^2 is not generally well-defined for $T \in \mathcal{S}'_\beta$

$\Rightarrow$ Use as starting point for interactions $\sim \int T^4$, etc.

The Path Integral of a Free Scalar Field

Regularisation

- T^4 does not make sense for $T \in \mathcal{S}'_\beta$
- Define regulator as linear map $\mathcal{R} : \mathcal{S}'_\beta \to \mathcal{S}$

  $\Rightarrow$ smoothen the distributions (cutoff momenta above Λ_{UV})

  $\Rightarrow$ make them decay at infinity (cutoff momenta below Λ_{IR})
- Consider the pushforward measure $\nu = \mathcal{R}_* \mu$ on $\mathcal{S}$

  $\Rightarrow$ Regularise the propagator

The Path Integral of a Free Scalar Field

Regularisation - An Example

$$\mathcal{R}T = \chi \cdot (\xi * T) \tag{3}$$

with

$$\begin{aligned}\chi(x) &= \exp\left[-\frac{\Lambda_{\mathrm{IR}}^2}{2}x^2\right] \\ \xi(x) &= \left(\frac{\Lambda_{\mathrm{UV}}^2}{2\pi}\right)^{d/2} \exp\left[-\frac{\Lambda_{\mathrm{UV}}^2}{2}x^2\right]\end{aligned} \tag{4}$$

Note, $w - \lim_{\Lambda_{\mathrm{UV}}\to\infty,\Lambda_{\mathrm{IR}}\to 0} \mathcal{R}T = T$ for all $T \in \mathcal{S}'_\beta$.

Yet Another Scale...

A Quadratic Addition

Define the partition function

$$Z_k(T) = \frac{\int_{\mathcal{S}} \exp\left[T(\psi) - \frac{1}{2}F_k(\psi,\psi) - S^{\mathrm{int}}(\psi)\right] \mathrm{d}\nu(\psi)}{\int_{\mathcal{S}} \exp\left[-\frac{1}{2}F_k(\psi,\psi) - S^{\mathrm{int}}(\psi)\right] \mathrm{d}\nu(\psi)} \quad (5)$$

for all $k \geq 0$. (Call denominator N_k)

- Demand $\exp[S^{int}] \in L^q(\nu)$ for some $q > 1$
- $-S^{int}$ should be bounded by some squared seminorm
- F_k nonnegative, symmetric, bilinear form, e.g $k^2 \int \psi^2$
- F_k should diverge roughly as k^2 pointwise + technicalities

The Conjugate Problem

The Effective Average Action

- Let $W_k(T) = \ln Z_k(T)$

  $\Rightarrow W_k$ is convex ('trivial' from measure theory)

  $\Rightarrow \mathrm{D}W_k$ is injective (almost 'trivial' - Gaußian measures)

  $\Rightarrow W_k$ extends to closure of $\mathcal{S}'_\beta$ in $L^2(\nu)$ ('trivial' - Gaußian measures)

  $\Rightarrow \mathrm{D}W_k$ is surjective on this space! (tricky)

  $\Rightarrow$ range of $\mathrm{D}W_k$ is $H(\nu)$ - the 'Cameron-Martin space' of ν

The Effective Average Action

Some Properties

- $H(\nu)$ is a separable real Hilbert space
- $H(\nu)$ is dense in $\mathcal{S}$

Finally, set

$$\Gamma_k(\phi) = (DW_k)^{-1}(\phi)(\phi) - W_k\left((DW_k)^{-1}(\phi)\right) - \frac{1}{2}F_k(\phi)(\phi)$$

for all $\phi \in H(\nu)$.

FRGE

The Flow Equation

Usually, $F_k \to R_k$ with R_k function in Fourier space. Then,

$$\partial_k \Gamma_k(\phi) = \frac{1}{2} \mathrm{Tr}\left\{ (\partial_k R_k) \left[\mathcal{F} \circ \mathrm{D}^2_\phi \Gamma_k \circ \mathcal{F}^{-1} + R_k \right]^{-1} \right\} + \partial_k \ln N_k \quad (6)$$

and

$$\lim_{k \to \infty} \Gamma_k(\phi) = \frac{1}{2} \langle \phi, \phi \rangle_{H(\nu)} + S^{\text{int}}(\phi) \;. \quad (7)$$

- N_k from normalisation
- arXiv:2106.09466

The Assumptions

ϕ^4 Theory

- Consider interaction $\lambda \int_{\mathbb{R}^4} \phi^4$ in $d = 4$

  $\Rightarrow$ Counterterm $-\Delta m^2 \int_{\mathbb{R}^4} \phi^2$

  $\Rightarrow -S^{\text{int}}$ blows up!

  $\Leftarrow \lambda \int_{\mathbb{R}^4} \phi^4$ cannot control $-\Delta m^2 \int_{\mathbb{R}^4} \phi^2$

- $$\int_{\mathcal{S}} \exp\left[-\lambda \int_{\mathbb{R}^4} \phi^4 + \Delta m^2 \int_{\mathbb{R}^4} \phi^2\right] \mathrm{d}\nu(\phi) < \infty$$

  for all $\Delta m^2 > 0$?

General Math Problem

Question of Integrability

- X a locally convex space
- μ a Gaußian measure on X
- p, q seminorms on X
- Conditions for

$$\int_X \exp\left[-p(x)^4 + \alpha q(x)^2\right] \mathrm{d}\mu(x) < \infty$$

for all $\alpha > 0$?

- Rather general solution in arXiv:2205.09189 (JZ, Benjamin Hinrichs, Dann Willem Janssen)

Outlook

Eliminating Cutoffs? Asymptotic Safety?

- Does the equation survive the limit of vanishing cutoffs?

  $\Leftarrow$ Quantum triviality?

  $\Rightarrow$ Perhaps $\Lambda_{\mathrm{IR}} \to 0$

Further Directions

- Application to nonrelativistic problems (in progress)
- Fermions?
- Coupled systems?