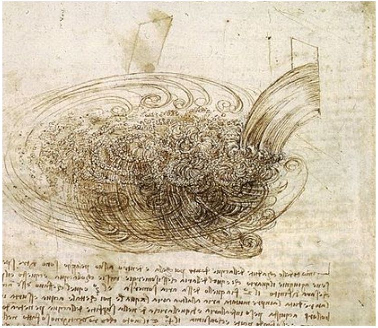
Université Grenoble Alpes

Turbulence and shell models of turbulence from Functional Renormalisation Group

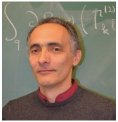


Leonardo da Vinci notebook

Léonie Canet

ERG Berlin 28/07/2022

In collaboration with ...



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References:

LC, B. Delamotte, N. Wschebor, Phys. Rev. E **93** (2016)
M. Tarpin, LC, N. Wschebor, Phys. Fluids **30** (2018)
A. Gorbunova, G. Balarac, LC, G. Eyink, V. Rossetto, Phys. Fluids **33** (2021)
C. Pagani, LC, Phys. Fluids **33** (2021)
A. Gorbunova, C. Pagani, G. Balarac, LC, V. Rossetto, Phys. Rev. F **6** (2021)
C. Fontaine, M. Tarpin, F. Bouchet, LC, ArXiv:2208.00225 (2022)
LC, arXiv:2205.01427, to appear in J. Fluid Mech. *Perspectives* (2022)

Presentation outline

1. Statistical theory of turbulence and Renormalisation Group

2. Time dependence of n-point correlation functions

3. Anomalous scaling in shell models of turbulence

Turbulence: ubiquitous and challenging phenomenon

- Presence of turbulence in every fluid flows

ocean

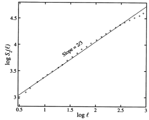
clouds

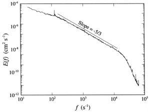
rivers

smoke

- Some characteristic features of turbulence

- chaos and impredictability
- extreme events



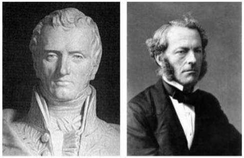


- scale invariance and universality

$$S_2 \sim r^{\zeta_2}$$

$$E(k) \sim k^{-5/3}$$

Navier-Stokes equation



Claude-Louis Navier 1785-1836 *Sir George Stokes 1819-1903*

- incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu \nabla^2 \mathbf{v} - \frac{1}{\rho} \nabla \pi + \mathbf{f}$$

$$\nabla \cdot \mathbf{v} = 0$$

200th anniversary this year

. . . yet still unsolved!

challenge: obtain statistical theory of turbulence

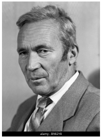
- spatial properties

  multi-fractal scaling of structure functions

- temporal properties

  sweeping effect in Eulerian correlation functions

Kolmogorov theory



Kolmogorov theory 1941 (K41)

A.N. Kolmogorov

Dokl.Akad.Nauk.SSSR **30, 31, 32** (1941)

dimensional analysis:

- eddy size $\ell \sim k^{-1}$
- energy flux ϵ

▶ structure functions

$$S_p(r) \equiv \left\langle |\mathbf{v}(\mathbf{r}_0 + \mathbf{r}) - \mathbf{v}(\mathbf{r}_0)|^p \right\rangle$$

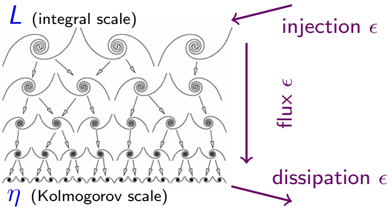
$$S_p(\ell) = C_p\, \epsilon^{p/3}\, \ell^{p/3} \qquad \text{(K41)}$$

$$S_3(\ell) = -\frac{4}{5}\, \epsilon\, \ell \qquad \text{(exact)}$$

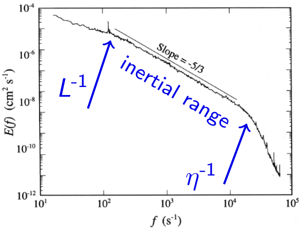
▶ kinetic energy spectrum

$$E(k) = C_K \epsilon^{2/3} k^{-5/3}$$

Richardson energy cascade

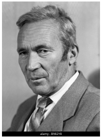


Nazarenko, Lecture Notes in Physics **825** (2011)



Maurer,Tabeling, Zocchi EPL **26** (1994)

Kolmogorov theory and intermittency of structure functions



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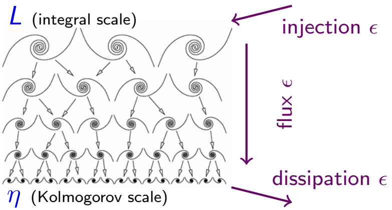


dimensional analysis:

- eddy size $\ell \sim k^{-1}$
- energy flux ϵ

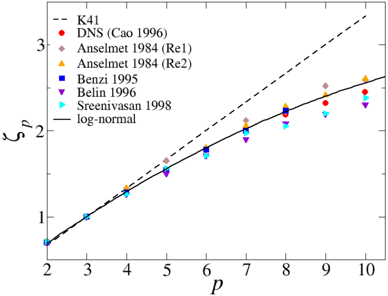
▶ structure functions

$$S_p(r) \sim r^{\zeta_p}$$

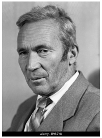
- numerical simulations and experiments

$$\zeta_p \neq p/3$$

▶ multi-fractality, intermittency



Kolmogorov theory



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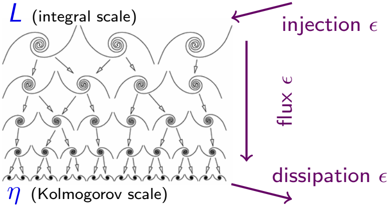
- energy flux ϵ

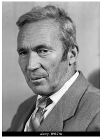
▶ decorrelation time scale

$$\tau_a \sim \epsilon^{-1/3} k^{-2/3} \quad \text{(K41)}$$

Richardson energy cascade



Kolmogorov theory and random sweeping effect



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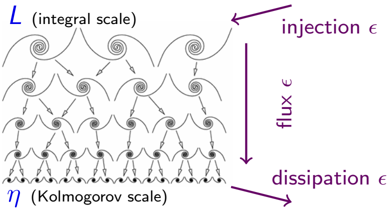
$$\tau_a \sim \epsilon^{-1/3} k^{-2/3} \quad \text{(K41)}$$

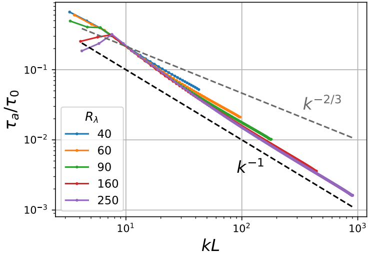
- numerical simulations and experiments

$$\tau_a \sim k^{-1}$$

▶ random sweeping effect

Richardson energy cascade





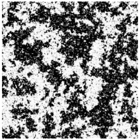
A. Gorbunova, *et al*, Phys. Rev. F **6** (2021)

Two challenges: spatial and temporal properties of turbulence

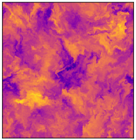
Statistical theory for turbulence, beyond K41 theory . . . "first principles" approaches

- fluid dynamics described by Navier-Stokes equations
  - many closure schemes (*e.g.* Direct Interaction Approximation by Kraichnan)
  - simplified models of turbulence – Burgers' equation, shell models
  - . . .

- many similarities between critical phenomena and turbulence
  - scale invariance, self-similarity
  - universality
  - anomalous critical exponents



second order phase transition



turbulence

$\Longrightarrow$ Renormalisation Group

Statistical theory for turbulence : Field theory for Navier-Stokes equation

- Stochastic Navier-Stokes equation

$$
\begin{aligned}
\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha \pi - \nu \nabla^2 v_\alpha &= f_\alpha \\
\partial_\alpha v_\alpha &= 0 \\
\big\langle f_\alpha(t,\mathbf{x}) f_\beta(t',\mathbf{x}')\big\rangle &= 2\delta_{\alpha\beta}\delta(t-t') N_L(|\mathbf{x}-\mathbf{x}'|).
\end{aligned}
$$

f_α Gaussian distributed, N_L peaked at the integral scale (energy injection)

- Martin-Siggia-Rose-Janssen-de Dominicis formalism

Martin, Siggia, Rose, PRA **8** (1973), Janssen, Z. Phys. B **23** (1976), de Dominicis, J. Phys. Paris **37** (1976)

$$
\mathcal{Z} = \int \mathcal{D}\mathbf{v}\,\mathcal{D}\bar{\mathbf{v}}\,\mathcal{D}\pi\,\mathcal{D}\bar{\pi}\; e^{-\mathcal{S}_{\rm NS} + \text{ source terms}}
$$

$$
\mathcal{S}_{\rm NS} = \int_{t,\mathbf{x}} \bar{v}_\alpha \left[\partial_t v_\alpha + v_\beta\partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha \pi - \nu\nabla^2 v_\alpha\right] + \bar{\pi}\left[\partial_\alpha v_\alpha\right] - \int_{t,\mathbf{x},\mathbf{x}'} \bar{v}_\alpha \left[N_L(|\mathbf{x}-\mathbf{x}'|)\right]\bar{v}_\alpha
$$

Statistical theory for turbulence : Perturbative Renormalisation Group

- Field theory for stochastic Navier-Stokes equation

$$\mathcal{S}_{\rm NS} = \int_{t,\mathbf{x}} \bar{v}_\alpha \left[\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha \pi - \nu \nabla^2 v_\alpha\right] + \bar{\pi}\left[\partial_\alpha v_\alpha\right] - \int_{t,\mathbf{x},\mathbf{x}'} \bar{v}_\alpha \left[N_L(|\mathbf{x}-\mathbf{x}'|)\right] \bar{v}_\alpha$$

- absence of a small expansion parameter
  - no upper critical dimension $d_c = \infty$
  - no small coupling (interaction = advection term)

Perturbative RG approaches

- introduction of a small parameter via forcing covariance $N_L(\mathbf{k}) \propto k^{d-\varepsilon}$

de Dominicis, Martin, PRA **19** (1979), Fournier, Frisch, PRA **28** (1983), Yakhot, Orszag, PRL **57** (1986)

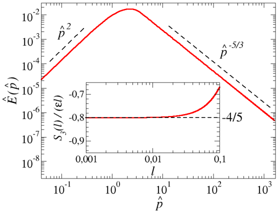
$\Longrightarrow$ 3D kinetic energy spectrum $E(k) \propto k^{1-2\varepsilon/3}$

- K41 scaling recovered in the limit $\varepsilon \to 4$!
- freezing mechanism invoked for $\varepsilon < 4$
- does it lead to the same universal behaviour ?

Statistical theory for turbulence : Functional Renormalisation Group

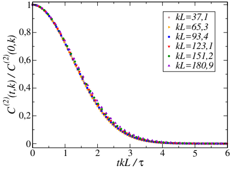
- ▶ RG fixed point for large-scale forcing
  - K41 scaling
  - 4/5th law preserved

Tomassini, Phys. Lett. B **411** (1997)
Mejía-Monasterio, Muratore-Ginnaneschi, PRE **86** (2012)
LC, Delamotte, Wschebor, PRE **93** (2016)



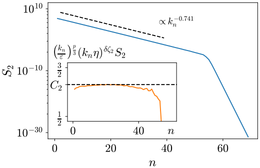
- ▶ time dependence of generic *n*-point correlation functions

LC, V. Rossetto, N. Wschebor, G. Balarac, Phys. Rev. E **95** (2017)
M. Tarpin, LC, N. Wschebor, Phys. Fluids **30**, 055102 (2018)
LC, arXiv:2205.01427, to appear in J. Fluid Mech. *Perspectives* (2022)



- ▶ anomalous scaling of structure functions in shell models of turbulence

C. Fontaine, M. Tarpin, F. Bouchet, LC, ArXiv:2208.00225 (2022)



Presentation outline

1. Statistical theory of turbulence and Renormalisation Group
2. Time dependence of *n*-point correlation functions
3. Anomalous scaling in shell models of turbulence

Functional renormalisation group for turbulence

- Field theory for stochastic Navier-Stokes equations + regulators

$$\mathcal{Z}_\kappa = \int \mathcal{D}\mathbf{v}\,\mathcal{D}\bar{\mathbf{v}}\,\mathcal{D}\pi\,\mathcal{D}\bar{\pi}\; e^{-\mathcal{S}_{\rm NS}-\Delta\mathcal{S}_\kappa + \text{source terms}}$$

$$\mathcal{S}_{\rm NS} = \int_{t,\mathbf{x}} \bar{v}_\alpha \left[\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha \pi - \nu \nabla^2 v_\alpha\right] + \bar{\pi}\left[\partial_\alpha v_\alpha\right]$$

$$\Delta\mathcal{S}_\kappa = -\int_{t,\mathbf{x},\mathbf{x}'} \left\{ \bar{v}_\alpha \left[N_\kappa(|\mathbf{x}-\mathbf{x}'|)\right] \bar{v}_\alpha + \bar{v}_\alpha \left[R_\kappa(|\mathbf{x}-\mathbf{x}'|)\right] v_\alpha \right\}$$

- exact flow equation for Γ_κ and $\mathcal{W}_\kappa = \ln \mathcal{Z}_\kappa$

Wetterich, Phys. Lett. B **301** (1993)

$$\partial_\kappa \Gamma_\kappa = \frac{1}{2}\mathrm{Tr} \int_{\mathbf{x},\mathbf{y}} \partial_\kappa \mathcal{R}_{\kappa_{ij}}(\mathbf{x}-\mathbf{y}) \left[{\Gamma_\kappa}^{(2)} + \mathcal{R}_\kappa\right]^{-1}_{ji}$$

$$\partial_\kappa \mathcal{W}_\kappa = -\frac{1}{2}\mathrm{Tr} \int_{\mathbf{x},\mathbf{y}} \partial_\kappa \mathcal{R}_{\kappa_{ij}}(\mathbf{x}-\mathbf{y}) \left\{ \frac{\delta^2 \mathcal{W}_\kappa}{\delta j_i(\mathbf{x})\delta j_j(\mathbf{y})} + \frac{\delta \mathcal{W}_\kappa}{\delta j_i(\mathbf{x})}\frac{\delta \mathcal{W}_\kappa}{\delta j_j(\mathbf{y})} \right\}$$

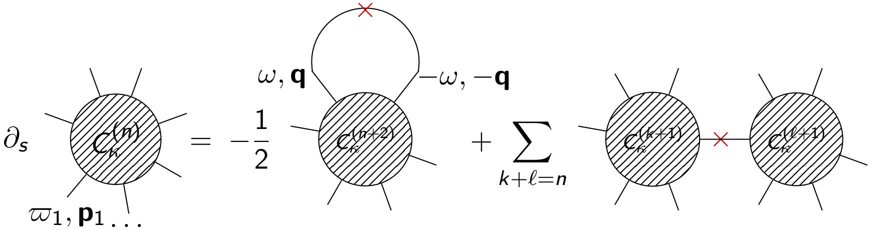
Time dependence of generic n-point correlation functions

- statistical theory for turbulence: n-point correlation functions

$$C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$$

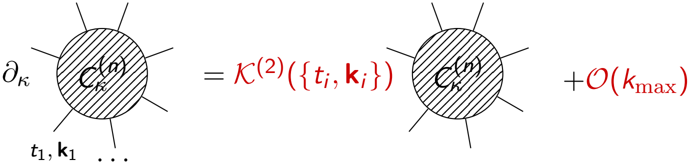
- within the FRG framework

$\Longrightarrow$ exact (but infinite hierarchy) of FRG flow equations



Time dependence of generic n-point correlation functions

exact closure for all $C^{(n)}(\{t_i, \mathbf{k}_i\})$ in the limit $|\mathbf{k}_i| \gg L^{-1}$



M. Tarpin, LC, N. Wschebor, Phys. Fluids **30** (2018)

- based on Ward identities from extended symmetries
  - time-gauged Galilean invariance: $\mathcal{G} = \begin{cases} x_\alpha \to x_\alpha + \epsilon_\alpha(t) \\ v_\alpha \to v_\alpha - \dot{\epsilon}_\alpha(t) \end{cases}$
  - time-gauged shift symmetry: $\mathcal{R} = \begin{cases} \delta \bar{v}_\alpha(t, \vec{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta \bar{p}(t, \vec{x}) & = v_\beta(t, \vec{x}) \bar{\epsilon}_\beta(t) \end{cases}$

LC, B. Delamotte, N. Wschebor, Phys. Rev. E **91** (2015)

Time dependence of generic n-point correlation functions

- can be solved analytically at the fixed point !

$$C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\vec{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$$
$$= \mathbf{K41} \times \textbf{dominant time scaling}$$

- existence of two time regimes:

$$C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{k}_i\}) \propto \begin{cases} \exp\left(-\alpha_0 \frac{L^2}{\tau_0^2}\,\big|\sum_\ell \mathbf{k}_\ell t_\ell\big|^2 + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t \ll \tau_0 \\ \exp\left(-\alpha_\infty \frac{L^2}{\tau_0}\,|t|\,\sum_{k\ell} \mathbf{k}_k \cdot \mathbf{k}_\ell + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t \gg \tau_0 \end{cases}$$

  - for $C^{(2)}$, at small times $\tau_a \propto k^{-1} \neq k^{-2/3} \Longrightarrow$ random sweeping
  - generalised for any n-point correlation
  - prediction of a new regime at large time
  - extensive checks with direct numerical simulations

Time dependence of generic n-point correlation functions

- can be solved analytically at the fixed point !

$$C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i,\vec{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$$
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  - but at $t=0$, dominant time scaling $=0$

$\Longrightarrow$ intermittency corrections to K41 scaling not accessible at this order

Passively advected scalars in turbulent flows

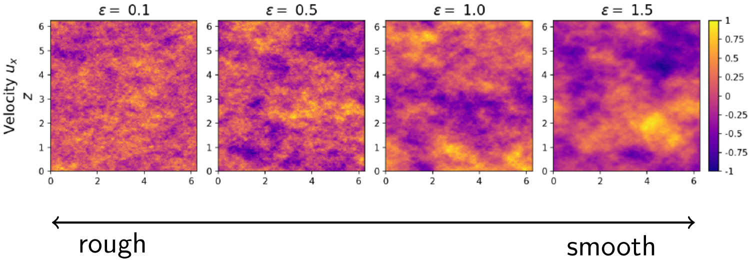
- diffusion-advection equation

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta - \kappa \nabla^2 \theta = f$$

v synthetic random field Kraichnan, Phys. Fluids **11** (1968)

$$\left\langle \hat{v}_i(t, \mathbf{k}) \hat{v}_j(t', -\mathbf{k}) \right\rangle = P_{ij}^{\perp}(\mathbf{k}) \frac{D_0}{k^{d+\varepsilon}} \delta(t - t')$$

©Walter Baxter



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©Walter Baxter

- correlation function from functional renormalisation group

$$C_\theta(t,\mathbf{k}) = C_\theta(0,\mathbf{k}) \exp\left(-\kappa_{\mathrm{ren}} k^2 |t|\right)$$

$$\kappa_{\mathrm{ren}} = \kappa + \underbrace{\frac{d-1}{2d} \int_{\mathbf{k}} \frac{D_0}{(k^2+m^2)^{(d+\varepsilon)/2}} d^d\mathbf{k}}_{\text{determined by velocity only}}$$

- exponential decay for all times
- exact expression for κ_{ren}

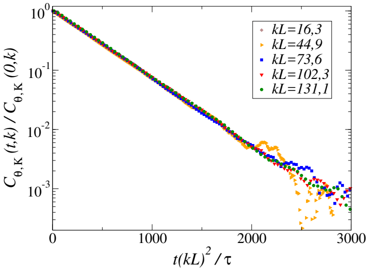
C. Pagani, LC, Phys. Fluids **33** (2021)

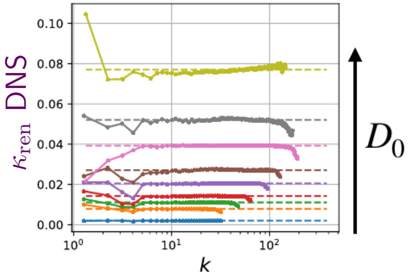
Correlations in 3D Kraichnan model

- result from functional renormalisation group (FRG)

$$C_\theta(t,\mathbf{k}) \propto \exp\left(-\kappa_{\rm ren} k^2 |t|\right) \qquad \kappa_{\rm ren} = \kappa + \frac{1}{3}\int_{\mathbf{k}} \frac{D_0}{(k^2+m^2)^{(3+\varepsilon)/2}} d^3\mathbf{k}$$

- results from direct numerical simulations (DNS)





A. Gorbunova, C. Pagani, G. Balarac, LC, V. Rossetto, Phys. Rev. F **6** (2021)

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$$C_\theta(t,\mathbf{k}) \propto \exp\left(-\kappa_{\rm ren} k^2 |t|\right) \qquad \kappa_{\rm ren} = \kappa + \frac{1}{3}\int_{\mathbf{k}} \frac{D_0}{(k^2+m^2)^{(3+\varepsilon)/2}} d^3\mathbf{k}$$

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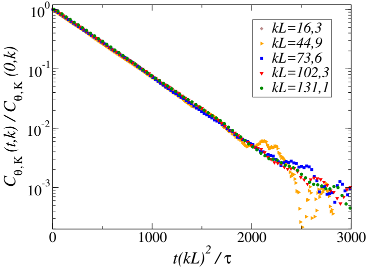

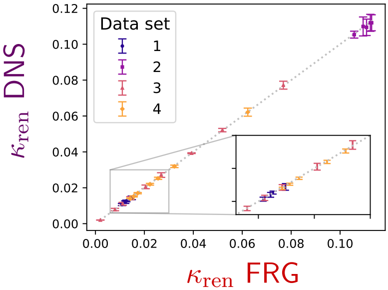

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Definition of shell models of turbulence

Navier-Stokes equations

- velocity field $\mathbf{v}(t, \mathbf{k}) \in \mathbb{R}^d$ on $\mathbf{k} \in \mathbb{R}^d$

- dynamics in spectral space

$$\begin{cases} \dfrac{\partial v_\alpha}{\partial t} = B_\alpha[\mathbf{v}] - \nu \mathbf{k}^2 v_\alpha + f_\alpha \\ B_\alpha[\mathbf{v}] = -ik_\beta \displaystyle\int_{\mathbb{R}^d} P^\perp_{\alpha\gamma} v_\beta(\mathbf{k}') v_\gamma(\mathbf{k} - \mathbf{k}') \end{cases}$$

- quadratic invariants:

$$E = \frac{1}{2} \int_{\mathbb{R}^d} |\mathbf{v}|^2 d^d\mathbf{x} \quad H = \frac{1}{2} \int_{\mathbb{R}^d} \mathbf{v} \cdot \nabla \mathbf{v} d^d\mathbf{x}$$

Sabra shell model

- velocity mode $v_n(t) \in \mathbb{R}$ or $\mathbb{C}$ on discrete shells $k_n = k_0 \lambda^n$

- dynamics

$$\begin{cases} \dfrac{dv_n}{dt} = B_n[v, v^*] - \nu k_n^2 v_n + f_n \\ B_n[v, v^*] = i\Big[a k_{n+1} v_{n+2} v^*_{n+1} + b k_n v_{n+1} v^*_{n-1} \\ \qquad - c k_{n-1} v_{n-1} v_{n-2} \Big] , \end{cases}$$

▶ only local interactions

- quadratic invariants:

$$E = \sum_n v_n v_n^* , \quad H = \sum_n \left(\frac{a}{c}\right)^n v_n v_n^*$$

L'vov, Podivilov, Pomyalov, Procaccia, Vandembroucq, PRE **58** (1998)

Biferale, Ann. Rev. Fluid **35** (2003), Ditlevsen, *Turbulence and shell models* (2011)

Structure functions in shell models

Navier-Stokes equations

- continuous symmetries
  - scale invariance

    $$(t, k, v) \to (\lambda^{1-h}, \lambda^{-1} k, \lambda^{h+d} v)$$

  - translation invariance

    $$\mathbf{v}(t, \mathbf{k}) \to e^{-i\mathbf{k}\cdot\mathbf{x}_0} \mathbf{v}(t, \mathbf{k})$$

- exact 4/5th law

  $$S_3(r) = -\frac{4}{5}\, \varepsilon\, r,$$

- structure functions

  $$S_p(r) \equiv \langle |\mathbf{v}(\mathbf{r}_0 + \mathbf{r}) - \mathbf{v}(\mathbf{r}_0)|^p \rangle \sim r^{\zeta_p}$$

▶ anomalous scaling $\zeta_p \neq \frac{p}{3}$

Sabra shell model

- discrete symmetries
  - scale invariance

    $$(t, k_n, v_n) \to (\lambda^{1-h}, k_m, \lambda^{h} v_m)$$

  - translation invariance

    $$v_n(t) \to e^{i\theta_n} v_n(t)\,, \quad \theta_{n+2} = \theta_{n+1} + \theta_n$$

- exact law

  $$S_3(k_n) = -\frac{\varepsilon}{2k_n(a-c)}\,,$$

- structure functions

  $$S_p(k_n) = \begin{cases} \langle |v_n|^p \rangle & \text{even } p \\ \mathrm{Im}\langle v_{n+1}^* v_n v_{n-1} |v_n|^{p-3} \rangle & \text{odd } p \end{cases} \sim k_n^{-\zeta_p}$$

▶ anomalous scaling $\zeta_p \neq \frac{p}{3}$

Anomalous scaling in Sabra shell models

Navier-Stokes equations

- structure functions

$$S_p(r) \equiv \langle |\mathbf{v}(\mathbf{r}_0 + \mathbf{r}) - \mathbf{v}(\mathbf{r}_0)|^p \rangle$$
$$\sim r^{\zeta_p}$$

Sabra shell model

- structure functions

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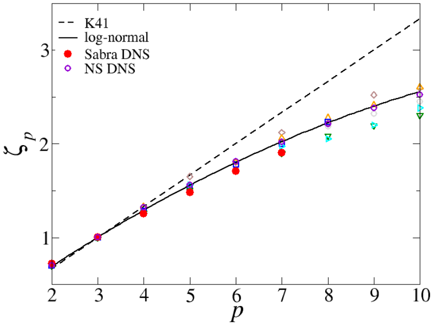
$$\sim k_n^{-\zeta_p}$$



L'vov *et al*

PRE **58** (1998)

Absence of sweeping effect in shell models

- Sabra model in the presence of a zero mode v_Ω

$$\begin{cases} \partial_t v_n = B_n[v, v^*] - i k_n v_\Omega v_n - \nu k_n^2 v_n + f_n \\ \partial_t v_\Omega = f_\Omega \,, \end{cases}$$

- invariance under Galilean transformation

$$v_n(t) \to e^{itVk_n} v_n(t)\,, \quad v_n^*(t) \to e^{-itVk_n} v_n^*(t)\,, \quad v_\Omega(t) \to v_\Omega(t) - V$$

$\Longrightarrow$ sweeping effect carried by the zero mode only

remove zero mode $\Longleftrightarrow$ eliminate sweeping effect

Absence of sweeping effect in shell models

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$\Longrightarrow$ sweeping effect carried by the zero mode only

remove zero mode $\Longleftrightarrow$ eliminate sweeping effect

- FRG to compute intermittency effect in structure function

Côme Fontaine, M. Tarpin, F. Bouchet, LC, ArXiv:2208.00225 (2022)

Côme's poster

Functional renormalisation group approach to shell models

▶ Leading order (LO) Ansatz for Γ_κ

$$\Gamma_\kappa{}^{\mathrm{LO}}[u, u^*, \bar{u}, \bar{u}^*] = \sum_n \int_t \left\{ \bar{u}_n^* \left[f_{\kappa,n}^{\lambda} \partial_t u_n + B_n[u, u^*] + f_{\kappa,n}^{\nu} u_n \right] - \frac{1}{2} \bar{u}^* f_{\kappa,n}^{D} \bar{u}_n + \mathrm{c.c.} \right\}$$

- initial condition: $f_{\kappa=\Lambda,n}^{\lambda} = 1, \quad f_{\kappa=\Lambda,n}^{\nu} = \nu k_n^2, \quad f_{\kappa=\Lambda,n}^{D} = 0$
- $\Gamma_\kappa^{(2)}$: functional dependence in n, bare frequency dependence
- only bare vertices (given by $B_n[u, u^*]$)

similar to Mejía-Monasterio, Muratore-Ginnaneschi, PRE **86** (2012), LC, Delamotte, Wschebor, PRE **93** (2016)

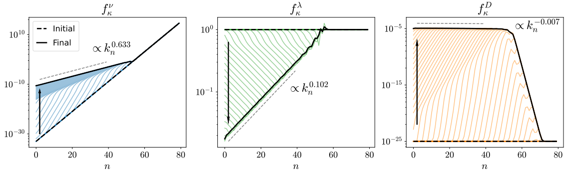
▶ Flow equations for $f_{\kappa,n}^{\nu}$, $f_{\kappa,n}^{D}$, $f_{\kappa,n}^{\lambda}$

$\Longrightarrow$ same simplification (as hydrodynamics vs shell models) occurs!

- set of ordinary coupled equations
- strictly local in wavenumber around κ !
- dimensionful integration $\longrightarrow$ fixed-point builds up as $\kappa \to 0$

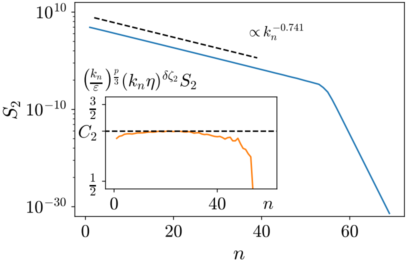
Fixed-point with anomalous scaling for the Sabra shell model

- fixed-point for $f^{\nu}_{\kappa,n}$, $f^{D}_{\kappa,n}$, $f^{\lambda}_{\kappa,n}$



$\Longrightarrow$ anomalous scaling for the three functions !

- second order structure function



$\zeta_2^{\mathrm{K41}} = 2/3$

$\zeta_2^{\mathrm{LO}} \simeq 0.74 \pm 0.02$

$\zeta_2^{\mathrm{DNS}} \simeq 0.720 \pm 0.008$

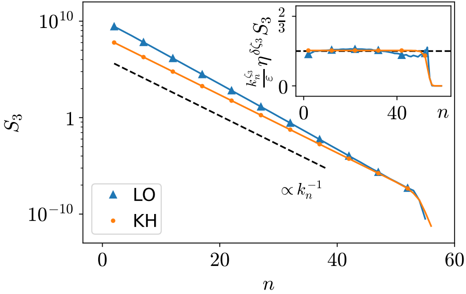
Fixed-point with anomalous scaling for the Sabra shell model

- third-order structure function
  - from direct calculation at LO

$$S_3(k_n) = \Im\mathrm{m}\langle v_{n-1} v_n v_{n+1}^* \rangle = \Im\mathrm{m}\left[\frac{\delta^3 \mathcal{W}_\kappa}{\delta j_{n-1}(t) j_n(t) j_{n+1}^*(t)} \right]\Bigg|_{\kappa \to 0}$$

  - from recursion relation using analogue of Kármán-Howarth relation and S_2 at LO

$$\nu k_n^2 S_2(k_n) = a k_{n+1} S_3(k_{n+1}) + b k_n S_3(k_n) + c k_{n-1} S_3(k_{n-1}) + N_n \, .$$

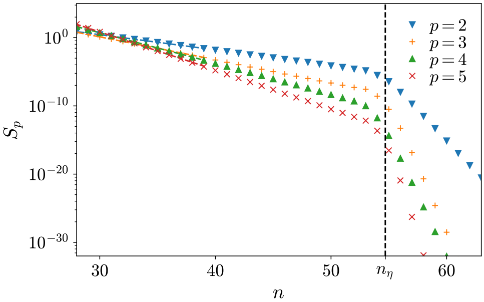


$\zeta_2^{\mathrm{LO}} \simeq 1.11 \pm 0.04$

$\zeta_2^{\mathrm{rec}} = 1$

Fixed-point with anomalous scaling for the Sabra shell model

- higher-order structure function from direct computation at LO



anomalous scaling for all S_p with $\zeta_p \neq p/3$, but ζ_p affine in p

- akin a β-model Frisch, sulem, Nelkin, JFM **87** (1978), anomalous but uni-fractal
- due to keeping only bare vertices at LO $\Longrightarrow$ to be improved ...

Effect of long-range forcing

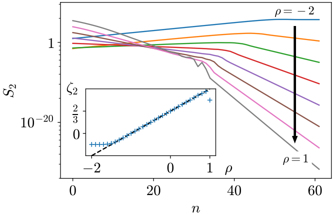
- two sources of forcing: LR power-law + SR large scale forcing

$$\langle f_n(t) f_{n'}(t') \rangle = 2\left[D (k_n L)^2 \exp(-(k_n L)^2) + D^{\mathrm{LR}} k_n^{-\rho} \right] \delta(t-t') \delta_{nn'} \,,$$

  - pure LR forcing, $D = 0$

$$\zeta_p = \frac{(1+\rho)p}{3}$$

$\Longrightarrow$ no anomalous scaling



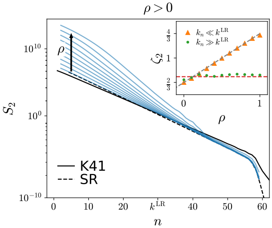
  - LR and SR forcing

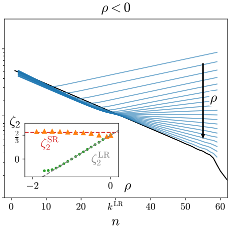
- for shells $k_n \ll k_n^{\mathrm{LR}}$:

LR dominates for $\rho > 0$

SR dominates for $\rho < 0$

- freezing for $\rho < 0$





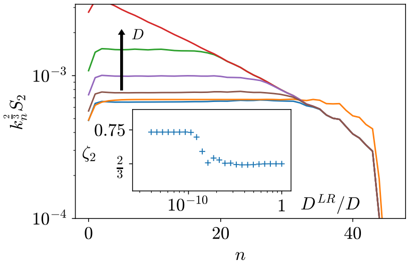
Effect of long-range forcing

- LR with $\rho = 0$, SR with $D \uparrow$

LR: $\zeta_2^{\mathrm{LR}} = 2/3$ (K41)

SR: $\zeta_2^{\mathrm{SR}} \simeq 0.74$ (anomalous)

$\Longrightarrow$ two distinct fixed-points



SR fixed-point intermittent, LR fixed-point non-intermittent

▶ anomalous scaling cannot be captured from the $\rho \to 0$ limit of the LR fixed-point

▶ $\rho \to 0$ in shell models $\Longleftrightarrow$ $\varepsilon \to 4$ in 3D Navier-Stokes likely to hold also in this case

Conclusions and perspectives

summary

- rigorous expression for time-dependence of n-point correlation functions in Navier-Stokes turbulence

  $\Longrightarrow$ sweeping effect

- calculation of structure functions in Sabra shell models

  $\Longrightarrow$ anomalous scaling but uni-fractal

perspectives

- improve determination of intermittency exponents in shell models
- compute structure functions in 3D Navier-Stokes turbulence
- use FRG for climate and meteo applications ?


Thank you for attention !