Fixed Point Structure of Gradient Flow Exact Renormalization Group

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Based on arXiv:2201.04111 with Y.Abe(Kobe) and Y.Hamada(KEK)

Outline

- Introduction (2)
- Review of GFERG (7)
- Fixed point structure of GFERG for scalar field theories (8)
- Conclusion (2)

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Exact Renormalization Group (ERG)

- A framework to study physics under varying the energy scale

- The Wilson action S_τ is intuitively defined by integrating out higher momentum modes of the fields:

$$e^{-S_\tau} := \int D\phi_{p>\Lambda} e^{-S_0}$$

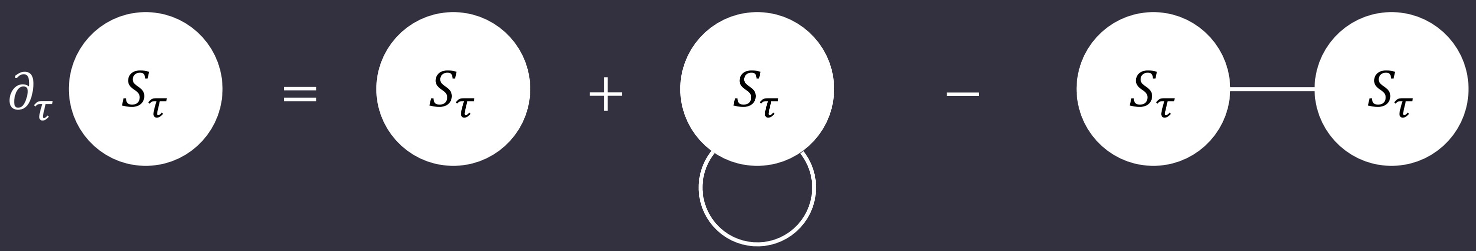
($\Lambda := \Lambda_0 e^{-\tau}$, Λ_0: cutoff)

- τ-dependence of the Wilson action S_τ is described by a differential equation → "ERG equation"

Wilson-Polchinski equation

- A typical example of the ERG equation:

$$\partial_\tau S_\tau = \int_p \left\{ \left[\left(2p^2 + \frac{D+2-\eta_\tau}{2} \right) + p_\mu \frac{\partial}{\partial p_\mu} \right] \phi_i(p) \frac{\delta S_\tau}{\delta \phi_i(p)} + \left(2p^2 + 1 - \frac{\eta_\tau}{2} \right) \left(\frac{\delta^2 S_\tau}{\delta \phi_i(p) \delta \phi_i(-p)} - \frac{\delta S_\tau}{\delta \phi_i(p)} \frac{\delta S_\tau}{\delta \phi_i(-p)} \right) \right\}$$



- This equation defines a renormalization procedure non-perturbatively

(η_τ: anomalous dimension) [J.Polchinski *Nucl. Phys. B* 231 (1984) 269-295]

(We work on the dimensionless framework and D-dimensional Euclidean space)

Gauge invariance in ERG

- The Wilson action with the naive UV cutoff is inconsistent with gauge invariance

- The gauge transformation mixes higher and lower momentum modes:

$$A_\mu^a(p) \to A_\mu^a(p) - p_\mu \omega^a(p) - i f^{abc} \int_q \omega^b(p-q) A^c(q)$$

- Can we define a Wilson action in a manifestly gauge invariant manner?

Renormalization and Diffusion

- The solution to the WP equation can be written in the following form:

$$e^{S_\tau[\phi]} = \hat{s}_\phi^{-1} \int D\phi' \prod_{x,i} \delta\left(\phi_i(x) - e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi_i'(t, xe^\tau)\right) \hat{s}_{\phi'} e^{S_{\tau=0}[\phi']}$$

($\hat{s}_\phi \coloneqq \exp\left(-\frac{1}{2}\int_x \frac{\delta^2}{\delta\phi_i(x)^2}\right)$: "scrambler")

- φ' is a solution to the diffusion equation:

$$\partial_t \varphi'(t,x) = \partial_x^2 \varphi'(t,x),\ \ \varphi'(0,x) = \phi'(x)$$

where $t \coloneqq e^{2\tau} - 1$

- This representation implies that the coarse-graining by the diffusion can be used to define an RG flow

Gradient Flow (GF)

- This one-parameter deformation of fields via the diffusion equation has been studied in the context of "gradient flow"

- The gradient flow is a method to construct composite operators without the equal-point singularity

- Correlation functions are UV finite with wave function renormalization:

$$Z_t^{-n/2}\langle\varphi(t,x_1)\varphi(t,x_2)\cdots\varphi(t,x_n)\rangle_\phi < \infty$$

  even for the equal point case (e.g. $x_1 = x_2$)

[F.Capponi, L.Debbio, S.Ehret, R.Pellegrini, A.Rago 1512.02851]

Gradient Flow ERG (GFERG)

- GF equation for gauge fields [M.Lüscher, P.Weisz 1101.0963]

$$\partial_t B'_\mu = \underbrace{D'_\nu G'_{\nu\mu}}_{\text{"diffusion"}} + \alpha_0 \underbrace{D'_\mu \partial_\nu B'_\nu}_{\text{"gauge fixing"}}$$

with $B'_\mu(0,x) = A'_\mu(x)$

- S_τ for gauge fields can be defined via the GF equation [H.Sonoda and H.Suzuki 2012.03568]

$$e^{-S_\tau[A^a_\mu]} := \hat{s}_A^{-1} \int [DA'^a_\mu] \prod_{x',a,\mu} \delta\left(A^a_\mu(x) - e^{\tau(D-4)} B'^a_\mu(t, x' e^\tau)\right) \hat{s}_{A'} e^{-S_{\tau=0}[A'^a_\mu]}$$

($\hat{s}_A := \exp\left[-\frac{1}{2}\int_x \frac{\delta^2}{\delta A^a_\mu(x)\delta A^a_\mu(x)}\right]$: "scrambler" for gauge fields)

- GFERG defines an RG flow in a gauge-invariant way!

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Our motivation

- The appropriate gradient flow eq. highly depends on details of the theory, such as its symmetry and interactions

- The GFERG flow depends on the form of the GF eq., and then becomes different for each theory

- This means that the GFERG eq. for general scalar field theories is no longer given by the WP eq.

- Is GFERG consistent with the conventional ERG?

Our motivation

- The gradient flow equation for the $O(N)$ non-linear sigma model

$$\partial_t \varphi_i = \partial_x^2 \varphi_i - (\varphi_j \partial_x^2 \varphi_j)\varphi_i$$

[H.Makino, H.Suzuki 1410.7538]

- But the continuum limits of this model and the linear sigma model in three dimension are characterized by the same fixed point

- "Does GFERG give the same prediction as the conventional ERG for the IR behavior of a theory?"

- These facts strongly motivate us to study the fixed points and the critical exponents in GFERG

General gradient flow equation

- General form of gradient flow equations

$$\partial_\tau \varphi_i = \underbrace{\partial_x^2 \varphi_i}_{\text{WP part}} + \sum_{n=n_{\min}}^{\infty} \int_{x_1,\dots,x_n} \underbrace{f_i^{i_1,\dots,i_n}(x; x_1,\dots,x_n; \partial_{x_1},\dots,\partial_{x_n}) \varphi_{i_1}(\tau,x_1)\cdots\varphi_{i_n}(\tau,x_n)}_{\text{extra terms}}$$

- Counterpart of WP eq. in GFERG (GFERG eq.)

$$\partial_\tau e^{-S_\tau[\Phi]} = (\text{WP part})$$
$$+ \sum_{n=n_{\min}}^{\infty} \lambda^{n-1}(\tau) \int_{x,x_1,\dots,x_n} \frac{\delta}{\delta\phi_i(x)} \left\{ f_i^{i_1,\dots,i_n} \left(\phi_{i_1}(x_1) + \frac{\delta}{\delta\phi_{i_1}(x_1)} \right) \times \cdots \right.$$
$$\left. \times \left(\phi_{i_n}(x_n) + \frac{\delta}{\delta\phi_{i_n}(x_n)} \right) \right\} e^{-S_\tau[\Phi]}$$

where $\boxed{\lambda(\tau) := e^{-\tau(D-2)/2} Z_\tau^{-1/2}}$

Fixed point action

- The fixed point action S^* is defined as

$$\partial_\tau S^* = 0 \text{ and } S^* := \lim_{\tau\to\infty} S_\tau$$

- S^* satisfies

$$0 = (\text{WP part}) + \sum_{n=n_{\min}}^{\infty} \lambda^{n-1}(\infty) \int_{x,x_1,\ldots,x_n} \frac{\delta}{\delta\phi_i(x)} \left\{ f_i^{i_1,\ldots,i_n} \left(\phi_{i_1}(x_1) + \frac{\delta}{\delta\phi_{i_1}(x_1)} \right) \times \cdots \times \left(\phi_{i_n}(x_n) + \frac{\delta}{\delta\phi_{i_n}(x_n)} \right) \right\} e^{-S^*[\Phi]}$$

- Asymptotic behavior of $\lambda(\tau)$

$$\lambda(\tau) \sim \exp(-\tau(D-2+\eta)/2) \qquad \frac{\eta}{2} := \frac{d}{d\tau} \log Z_\tau \Big|_{\tau=\infty}$$

- $\lambda(\infty)$ should vanish from the cluster decomposition principle at the fixed point theory $(\langle\phi(x)\phi(0)\rangle \sim x^{-(D-2+\eta)})$

$\Rightarrow S^*$ satisfies the fixed point condition of the WP eq.

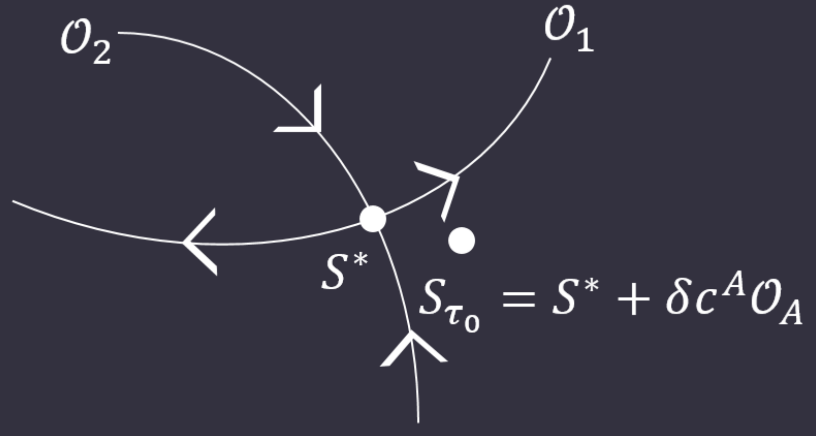
RG flow around fixed point

- Let us study the τ-dependence of S_τ after a long time $\tau = \tau_0 \gg 1$ so that $\lambda(\tau_0) \sim e^{-\tau_0(D-2+\eta)/2} \ll 1$
- Consider perturbating S_τ from a fixed point S^* at $\tau = \tau_0$ as

$$S_{\tau_0} = S^* + \sum_A \delta c^A \mathcal{O}_A$$

$|\delta c^A| \ll 1$

$\mathcal{O}_A$: a complete set of operators
(defined in the next slide)



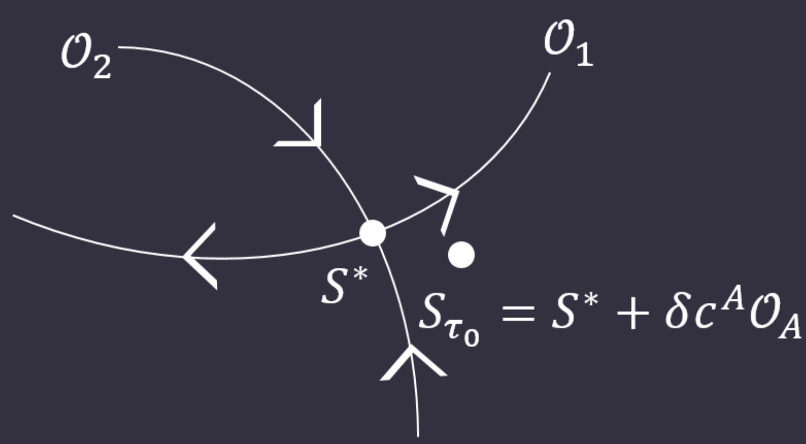
Solution in the $\tau \to \infty$ limit

- We have two small quantities: $\delta c^A,\ \lambda_0 := e^{-\tau_0(D-2+\eta)/2}$

- Solution to the GFERG eq. up to the leading order

$$S_\tau = S^* + \sum_A \left(\delta c^A e^{x_A \tau'} - \lambda_0^{n_{\min}-1}\left(e^{-(n_{\min}-1)(D-2+\eta)\tau'/2} - e^{x_A \tau'}\right) h^A\right) \mathcal{O}_A$$

where $\tau = \tau_0 + \tau'$



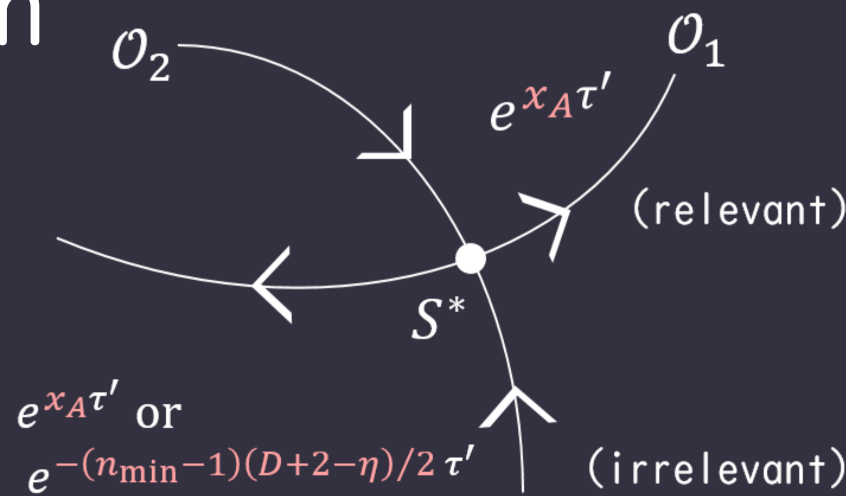
$\mathcal{O}_A$: eigenoperators of the linearized WP eq. around S^*

x_A: eigenvalue of $\mathcal{O}_A$

$n_{\min}$: minimum order of the non-linear terms in the GF eq.

h^A: some constant

Scaling dimension



- Scaling Dimension d_A

$$d_A = \max(x_A, -(n_m - 1)(D - 2 + \eta)/2)$$

$x_A \geq 0$ (relevant or marignal) $\Longrightarrow$ $d_A = x_A$

$x_A < 0$ (irrelevant) $\Longrightarrow$ $d_A = -\min(|x_A|, (n_m - 1)(D - 2 + \eta)/2)$

- d_A of relevant or marginal operators are x_A
  $\Rightarrow$ GFREG gives the same prediction as the WP equation for the IR behavior of a theory.

Outline

Conclusion

- Studied the fixed point structure of the GFERG equation associated with a general gradient flow equation for scalar field theories
- Showed that the fixed points are the same as those of the Wilson-Polchinski equation in general
- Discussed that the GFERG equation has a similar RG flow structure around a fixed point to the WP equation
- GFERG gives the same prediction as the conventional ERG for the IR behavior of a theory

Thank you!

Back up

Future Direction

- (Non-abelian) gauge theory
- Gravity and asymptotic safety
  “Wilson action with manifest diffeomorphism invariance”
- Scalar field theories with a non-trivial target space in two dimensions
  e.g.) $O(N)$ NL sigma model, $\boldsymbol{CP}^{N-1}$ model
- Effects of topological terms in QFT

GFERG is ERG?

- GFERG is NOT considered to be a kind of ERG but an alternative framework to study the low-energy physics in the Wilsonian sense
- Reasons:

1. The scaling dimensions are different from those of the WP equation
   (Different renormalization schemes should provide the same scaling dimensions within ERG by a field redefinition)
2. The effective couplings do not follow the scaling law around fixed points
   (They should scales as $\lambda^A \to e^{x_A \tau} \lambda^A$ under the coarse-graining of $x \to x e^{\tau}$)

Loophole

- Our arguments do not cover the case where $\lambda(\tau)$ and then the GFERG equation become τ-independent

- η_τ satisfies

$$\eta_\tau = 2 - D$$

- Examples: the pure Yang-Mills theory with $D = 4$,
  the $O(N)$ non-linear sigma model with $D = 2$
  (Note: the counterpart of $\lambda(\tau)$ in the YM theory is $e^{-\tau(D-4)/2} Z_\tau^{1/2}$)

- The GFERG equation can be regarded as a ERG equation in this case

GF eq. of O(N) NL sigma

- Gradient flow equation

$$\partial_t \varphi_i = \partial_x^2 \varphi_i - \left(\varphi_j \partial_x^2 \varphi_j\right)\varphi_i$$

- It obviously preserves the $O(N)$ symmetry $(\varphi_i \to O_{ij}\varphi_j)$
- The constraint of the fields $(\varphi_i^2 = 1)$ is respected, i.e., if $\varphi_i^2(0,x) = \phi_i^2(x) = 1$, $\varphi_i^2(t,x) = 1$ for $\forall t$
- The correlation functions of φ_i are UV finite without any additional wave function renormalization in two dimensions

- If we solve the constraint $\left(\phi_N = \pm\left(1 - \sum_a^{N-1} \phi_a^2\right)^{\frac{1}{2}}\right)$, the appropriate gradient flow equation is

$$\partial_t \varphi_a = \partial_x^2 \varphi_a + \left(\partial_\mu \varphi_b \partial_\mu \varphi_b + \frac{\left(\varphi_b \partial_\mu \varphi_b\right)^2}{1 - \varphi_c^2}\right)\varphi_a$$

General non-linear sigma model

- Lagrangian

$$\mathcal{L} = \frac{1}{2} G^{nm}(\phi)\partial_\mu \phi_n \partial^\mu \phi_m$$

- The Wilson action of GFERG is not covariant under target space diffeomorphism of the fields

$$e^{S_\tau[\phi]} = \hat{s}_\phi^{-1} \int D\phi' \prod_x \delta(\phi_i(x) - e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi_i'(t, xe^\tau))\, \hat{s}_{\phi'} e^{S_{\tau=0}[\phi']}$$

$$\hat{s}_\phi := \exp\left(-\int_x \frac{\delta^2}{\delta\phi_i(x)^2}\right)$$

- This expression should be interpreted as one for the models embedded in higher-dimensional Euclidean space with some constraints for the fields
- The appropriate GF equation is required to preserve those constraints instead

Constraints

- Consider a theory whose fields are constrained as $F[\phi] = 0$
  e.g.) the $O(N)$ NL sigma model: $F = \phi_i^2 - 1$

- S_τ preserves the constraint in the sense that $F[\lambda(\tau)\phi] = 0$ holds in the (modified) correlation functions:

$$\ll F[\lambda(\tau)\phi]\phi_{i_1} \cdots \phi_{i_n} \gg_{S_\tau} = 0$$

(Recall:

$$e^{S_\tau[\phi]} = \hat{s}_\phi^{-1} \int D\phi' \prod_x \delta(\phi_i(x) - e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi_i'(t, xe^\tau))\, \hat{s}_{\phi'} e^{S_{\tau=0}[\phi']}$$)

Scaling theorem

- S_τ of GFERG satisfies the scaling theorem:

[H.Sonoda, H.Suzuki 1901.05169]

$$\ll \phi_{i_1}(p_1 e^\tau)\cdots\phi_{i_n}(p_n e^\tau)\gg_{S_\tau}$$

$$= e^{-\tau n(D+2)/2} Z_\tau^{n/2} \prod_{i=1}^{n} \frac{K(p_i e^\tau)}{K(p_i)} \ll \phi_{i_1}(p_1)\cdots\phi_{i_n}(p_n)\gg_{S_{\tau=0}}$$

$\ll \mathcal{O} \gg_S := \int D\phi\, \hat{s}_\phi \mathcal{O}\, e^{S[\phi]}$: the modified correlation function

$K(p_i) := e^{-p^2}$: the cutoff function

- The solution to the WP equation holds this property
- S_τ describes the response of the system to the scale transformation (the coarse-graining)

RG invariance of Z

- Partition function

$$Z := \int D\phi \, e^{S_{\tau=0}[\phi]}$$

- The GFERG flow preserves the partition function:

$$\int D\phi \, e^{S_\tau} = \int D\phi \, e^{S_{\tau=0}} = Z$$

- We call S_τ of GFERG the “Wilson” action in the sense that it holds this property and satisfies the scaling theorem

Details of Solution to GFERG eq.

- Solution to the GFERG equation in the $\tau \to \infty$ limit

$$S_\tau = S^* + \sum_A \left(\delta c^A e^{x_A \tau'} - \lambda_0^{n_{\min}-1} \left(e^{-(n_{\min}-1)(D-2+\eta)\tau'/2} - e^{x_A \tau'} \right) h^A \right) \mathcal{O}_A$$

$$h^A = \frac{g^A}{x_A + D - 2 + \eta}$$

$$\sum_A g^A \mathcal{O}_A := e^{S^*} \int_{x, x_1, \dots, x_n} \frac{\delta}{\delta \phi_i(x)} \left\{ f_a^{i_1, \dots, i_{n_{\min}}} \left(\phi_{i_1}(x_1) + \int_{y_1} \mathcal{D}(x_1 - y_1) \frac{\delta}{\delta \phi_{i_1}(y_1)} \right)^{n_{\min}} e^{-S^*} \right.$$

Inclusion of matter fermions

[Y.Miyakawa and H.Suzuki 2106.11142]

- Wilson action for gauge fields and matter fermions

$$e^{S_\tau[A,\psi,\bar{\psi}]} \coloneqq \hat{s}_A^{-1} s_\psi^{-1} \int DA' D\psi' D\overline{\psi'} \prod_{x'',\nu,b} \delta\left(A_\nu^b(x'') - e^\tau g_\tau^{-1} B_\nu'^b(t, x'' e^\tau)\right)$$

$$\times\, \delta\left(\psi(x'') - e^{\tau(D-1)/2} Z_\tau^{1/2} \chi'(t, x'' e^\tau)\right) \delta\left(\bar{\psi}(x'') - e^{\tau(D-1)/2} Z_\tau^{1/2} \bar{\chi}'(t, x'' e^\tau)\right)$$

$$\times\, \hat{s}_{A'} \hat{s}_{\psi'} e^{S_{\tau=0}[A',\psi',\bar{\psi}']}$$

$\hat{s}_\psi \coloneqq \exp\left(-\int_x \frac{\delta_L}{\delta\psi(x)} \frac{\delta_L}{\delta\bar{\psi}(x)}\right)$: scrambler for fermions

- $\chi'(t,x)$ is the solution to the following GF equation:

$$\partial_t \chi'(t,x) = \left[D_\mu D_\mu - \alpha_0 \partial_\mu B_\mu'^{\,a}(t,x) T^a\right] \chi'(t,x), \quad \chi'(0,x) = \psi'(x)$$

Chiral symmetry

- S_τ is NOT invariant under the chiral transformation
$$\psi(x) \to (1 + i\alpha\gamma_5)\psi(x)$$
due to the scrambler $\hat{s}_\psi \coloneqq \exp\left(-\int_x \frac{\delta_L}{\delta\psi(x)} \frac{\delta_L}{\delta\bar{\psi}(x)}\right)$

- It is invariant under the modified chiral symmetry generated by
$$\hat{\Gamma}_5 \coloneqq \hat{s}_\psi^{-1} \hat{\gamma}_5 \hat{s}_\psi,$$
where $\hat{\gamma}_5$ is the generator of the ordinary one:
$$\hat{\gamma}_5 \coloneqq \int_x \gamma_5 \psi(x) \frac{\delta_L}{\delta\psi(x)} + \bar{\psi}(x)\gamma_5 \frac{\delta_L}{\delta\bar{\psi}(x)}$$

Modified chiral symmetry

- $\hat{\Gamma}_5$ acts on S_τ as

$$e^{-S_\tau}\hat{\Gamma}_5 e^{S_\tau} = \int_x \frac{\delta_R S_\tau}{\delta\psi}\gamma_5\psi + \bar{\psi}\gamma_5\frac{\delta_L S_\tau}{\delta\bar{\psi}} - 2\frac{\delta_R S_\tau}{\delta\psi}\gamma_5\frac{\delta_L S_\tau}{\delta\bar{\psi}} + 2\mathrm{tr}\left[\gamma_5\frac{\delta_L}{\delta\psi}\frac{\delta_R S_\tau}{\delta\bar{\psi}}\right]$$

- When S_τ is the bilinear in the fermion field as

$$S_\tau = -\int_x \bar{\psi}(x)D\psi(x)\cdots,$$

the modified chiral symmetry ($\hat{\Gamma}_5 e^{S_\tau} = 0$) implies the Ginsparg-Wilson relation:

$$\gamma_5 D + D\gamma_5 + 2D\gamma_5 D = 0$$

(Recall that we work on the dimensionless framework)

Gauge fixing in QED

[Y. Miyakawa, H. Sonoda and H. Suzuki 2111.15529]

- When perform a perturbative calculation, we should consider the gauge fixing and the ghosts
- If we introduce the ghosts $\bar{c}(x), c(x)$ and suppose they decouples from A_μ and ψ, the GFERG equation requires

  where $S_\tau = S[A, \psi] + S_{gh}$

$$S_{gh} = \int_x \bar{c}(x) \frac{\partial^2}{E(-e^{-2\tau}\partial^2) - \partial^2} c(x)$$

- The Ward-Takahashi identity requires

  should be invariant the modified gauge transformation

$$S_\tau^{inv} := S_\tau + \frac{1}{2}\int_x \frac{1}{\xi E(-e^{-2\tau}\partial^2)e^{2\partial^2} - \partial^2}\left(\partial_\mu A_\mu(x)\right)^2$$

ξ: the gauge fixing parameter

$E(x)$: an arbitrary function analytic at $x = 0$

Modified gauge invariance

- S_τ^{inv} is required to be invariant under the modified gauge transformation:

$$\delta A_\mu(x) = \frac{\xi E\left(-e^{-2\tau\partial^2}\partial^2\right)e^{2\partial^2} - \partial^2}{\xi E(-e^{-2\tau}\partial^2)e^{2\partial^2}}\partial_\mu\omega(x),$$

$$\delta\psi(x) = ie\omega(x)\psi(x),$$

$$\delta\bar{\psi}(x) = -ie\omega(x)\bar{\psi}(x)$$

- S_τ is required to transform under it as

$$\delta S_\tau = \int_x A_\mu(x)\frac{\partial^2}{\xi E(-e^{-2\tau}\partial^2)e^{2\partial^2}}\partial_\mu\omega(x)$$

1PI effective action

[H. Sonoda and H. Suzuki 2201.04448]

- The generating functional $\Gamma[\mathcal{A}_\mu, \Psi, \bar{\Psi}]$ for one-particle irreducible diagrams is defined as

$$\Gamma_\tau[\mathcal{A}_\mu, \Psi, \bar{\Psi}] \coloneqq S_\tau[A_\mu, \psi, \bar{\psi}] + \int_x \left[\frac{1}{2}\left(A_\mu - \mathcal{A}_\mu\right)^2 + i\left(\bar{\psi} - \bar{\Psi}\right)(\psi - \Psi)\right],$$

where A_μ, ψ and $\bar{\psi}$ are set to be the solutions to

$$\mathcal{A}_\mu = A_\mu + \frac{\delta S_\tau}{\delta A_\mu}, \Psi = \psi + i\frac{\delta_L S_\tau}{\delta \psi}, \bar{\Psi} = \bar{\psi} + i\frac{\delta_R S_\tau}{\delta \bar{\psi}}$$

- Γ_τ is invariant under the ordinary gauge transformation, while S_τ^{inv} is invariant under the modified one

Unused Slides

Summary of today's talk

- We study the fixed point structure of "Gradient Flow Exact Renormalization Group" (GFERG), a new framework to define the Wilson action via a gradient flow equation
- We consider the GFERG equation associated with a general gradient flow equation for scalar field theories
- We show that the fixed points are the same as those of the Wilson-Polchinski equation in general
- Furthermore, we discuss that the GFERG equation has a similar RG flow structure around a fixed point to the WP equation

Our work

- We study the fixed points of the general GFERG eq. for scalar fields
  → The fixed points of the WP eq. appear in the $\tau \to \infty$ limit along the GFERG flow

- We show the GFERG eq. has a similar RG flow structure around a fixed point
  → Scaling dimensions of relevant or marginal operators are the same, while those of irrelevant operators can be different

- GFERG gives the same prediction as the conventional ERG for the IR behavior of a theory

Gradient Flow (GF)

- This one-parameter deformation of fields via the diffusion equation has been studied in the context of "gradient flow"

- The gradient flow is a method to construct composite operators without the equal-point singularity

- Correlation functions are UV finite with wave function renormalization:

$$Z_t^{-n/2}\langle \varphi(t_1,x_1)\varphi(t_2,x_2)\cdots\varphi(t_n,x_n)\rangle_\phi < \infty$$

  even for the equal point case (e.g. $x_1 = x_2$)

[F.Capponi, L.Debbio, S.Ehret, R.Pellegrini, A.Rago 1512.02851]

GFERG equation

- We get the counterpart of the WP equation by differentiating S_τ with respect to τ:

$$\partial_\tau e^{-S_\tau} =$$

$$\int_x \frac{\delta}{\delta A_\mu^a(x)} \left[-2D_\nu F_{\nu\mu}^a - 2\alpha_0 D_\mu \partial_\nu A_\nu^a - \left(\frac{D-2+\eta_\tau}{2} + x^\mu \frac{\partial}{\partial x^\mu} \right) A_\mu^a \right]_{A \to A + \delta/\delta A} e^{-S_\tau}$$

- This equation is called "the GFERG equation"
- It includes higher-functional derivatives more than two. (The WP equation includes up to a second derivative)

Recent Studies

- Inclusion of matter fermions, the chiral symmetry and the axial anomaly

  [Y.Miyakawa and H.Suzuki 2106.11142]

  [Y.Miyakawa 2201.08181]

- Gauge fixing, ghosts and perturbative analysis around the Gaussian fixed point in QED

  [Y.Miyakawa, H.Sonoda and H.Suzuki 2111.15529]

- GFERG equation for the 1PI effective action and gauge symmetry in QED

  [H.Sonoda and H.Suzuki 2201.04448]

Vanishing λ

- The connected two point function at the fixed point theory
$$\langle \phi(x)\phi(0)\rangle_{\text{connected}} \propto \frac{1}{x^{D-2+\eta}}$$

- It should become zero when $|x| \to \infty$
  → $D-2+\eta > 0$ is required
  → $\lambda(\infty)$ ($\lambda(\tau) \sim e^{-\tau(D-2+\eta)/2}$) and the extra terms vanish and only the WP part survives
  ⇒ S^* satisfies the fixed point condition of the WP eq.

- The fixed points of the WP eq. appear in the $\tau \to \infty$ limit along the GFERG flow

$O(N)$ non-linear sigma model

- Let us illustrate our results with the $O(N)$ non-linear sigma model in $4-\epsilon$ dimensions
- Lagrangian

$$\mathcal{L} = \frac{1}{2g}\left(\partial_\mu \phi_i\right)^2, \quad (\phi_i)^2 = 1$$

- Its continuum theory is characterized by Wilson-Fisher fixed point in $4-\epsilon$ dimensions

$$S^*_{\mathrm{WF}} = \int_x \frac{1}{2}\left(\partial_\mu \phi_i\right)^2 + \frac{m_*^2}{2}\phi_i^2 + \frac{\lambda_*}{8}\left(\phi_i^2\right)^2 + O(\epsilon^2)$$

$$m_*^2 := -\frac{\epsilon}{4}\frac{N+2}{N+8}, \; \lambda^* := \epsilon\frac{8\pi^2}{N+8}$$

[K.Wilson, M.Fisher *Phys.Rev.Lett.* 28 (1972) 240-243]
[S.Dutta, B.Sathiapalan, H.Sonoda 2003.02773]

GFERG of $O(N)$ NL sigma model

- Gradient flow equation

$$\partial_\tau \varphi_i = \partial_x^2 \varphi_i - \left(\varphi_j \partial_x^2 \varphi_j\right)\varphi_i$$

[H.Makino, H.Suzuki 1410.7538]

- GFERG equation

$$\partial_\tau e^{-S_\tau[\Phi]} = (\text{WP part})$$
$$-\lambda^2(\tau) \int_x \frac{\delta}{\delta\phi_i(x)}\left(\left(\phi_j(x) + \frac{\delta}{\delta\phi_j(x)}\right)\right.$$
$$\left.\times\, \partial_x^2 \left(\phi_j(x) + \frac{\delta}{\delta\phi_j(x)}\right)\left(\phi_i(x) + \frac{\delta}{\delta\phi_i(x)}\right)\right) e^{-S_\tau[\Phi]}$$

Fixed point, Scaling dimension

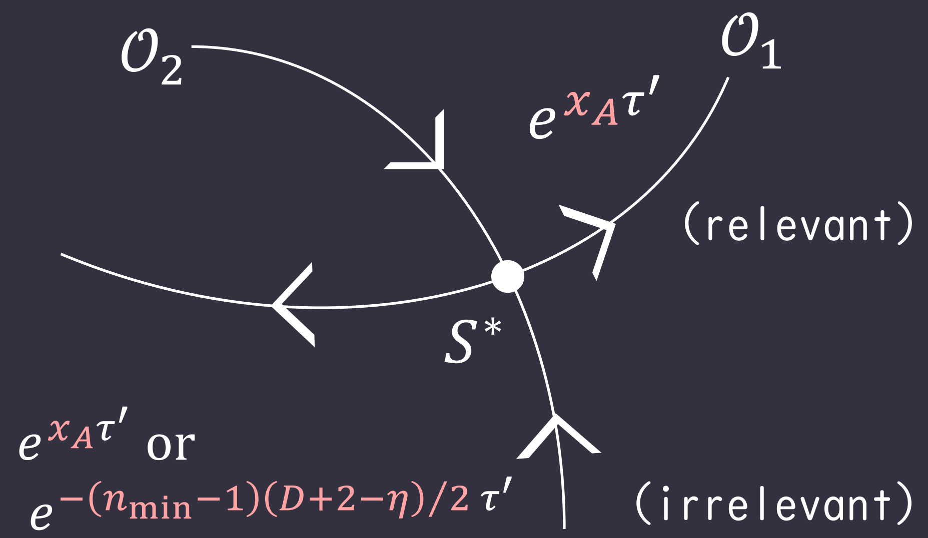
- Anomalous dimension at WF fixed point

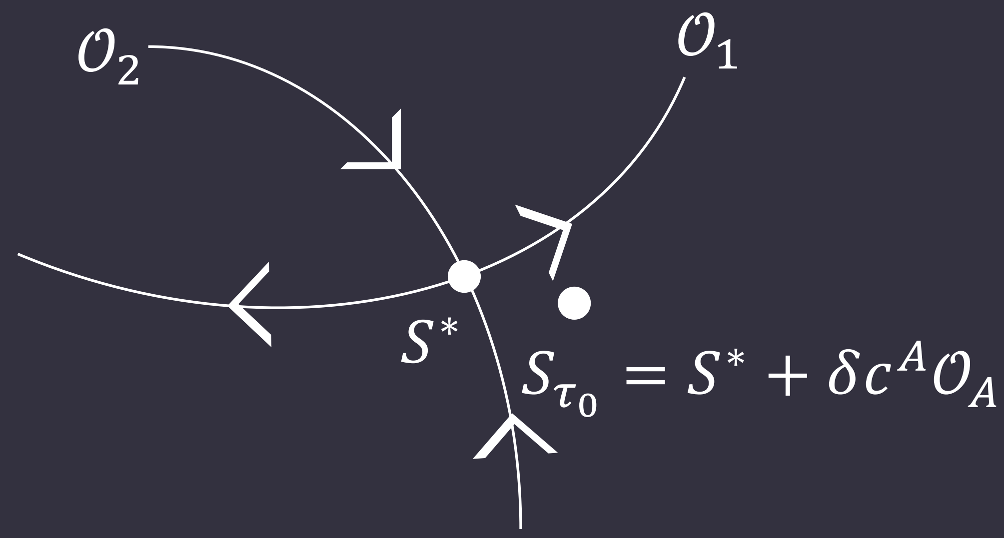
$$\frac{\eta}{2} = \frac{N+2}{(N+8)^2}\frac{\epsilon^2}{2} + O(\epsilon^3)$$

$\rightarrow$ $$\lambda(\tau) \sim \exp\left(-\frac{\tau}{2}\left(2-\epsilon+\frac{N+2}{(N+8)^2}\epsilon^2\right)\right) \to 0 \quad (\tau \to \infty)$$

- Scaling dimensions
  - $\mathcal{O}_1 = \frac{1}{2}\phi_i^2 \to d_1 = \max\left(2-\frac{N+2}{N+8}\epsilon, -(D-2+\eta)\right) = 2-\frac{N+2}{N+8}\epsilon > 0$
    (critical exponent $\nu^{-1} := d_1$)
  - $\mathcal{O}_2 = \frac{1}{8}(\phi_i^2)^2 - \frac{N+2}{64\pi^2}\phi_i^2 \to d_2 = \max(-\epsilon, -(D-2+\eta)) = -\epsilon < 0$

Material


$\mathcal{O}_2$
$\mathcal{O}_1$
$e^{x_A \tau'}$
(relevant)
S^*
$e^{x_A \tau'}$ or
$e^{-(n_{\min}-1)(D+2-\eta)/2\,\tau'}$
(irrelevant)


$\mathcal{O}_2$
$\mathcal{O}_1$
S^*
$S_{\tau_0} = S^* + \delta c^A \mathcal{O}_A$