Glory and misery of the Derivative Expansion: ten years later.

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A long, long time ago...

- The ideas discussed in this talk were first presented in the Aussois ERG conference in 2012.
- Delamotte and co-workers had been implementing the Derivative Expansion (DE) in the Ising universality class up to order $\mathcal{O}(\partial^6)$ and he presented the first preliminary results for that.
- Tissier and I presented a two-parts talk reviewing the NPRG/FRG statistical mechanics results.
- In that talk I presented a first non-perturbative estimate of the radius of convergence of the DE and proposed that this was associated with a "small parameter" of the order of $\sim 1/4$.
- Indeed Tighe and Morris had analysed the question even before in the context of perturbation theory. [Morris, Tighe, JHEP 08 (1999)].
- The purpose of this talk is to review the progress made in methodological aspects of the DE applied to statistical mechanics since Aussois-ERG12.
- I do not discuss high orders of DE in other areas such gravity. See for example: [Knorr, SciPost Phys.Core 4 (2021)].

The Derivative Expansion status 10 years ago

Before Aussois:

- Broad empirical success of the DE in many applications mostly at order $\mathcal{O}(\partial^0)$ or $\mathcal{O}(\partial^2)$.
  [Berges, Tetradis, Wetterich, Phys.Rept. 363 (2002)]
- $\mathcal{O}(\partial^4)$ only for Ising exponents.
  [Canet, Delamotte, Mouhanna, Vidal, Phys.Rev.B 68 (2003)]
- The expansion parameter was supposed to be of order 1 (an expansion on q^2/k^2 with $q^2 \lesssim k^2$). Sometimes an expansion in η suggested.
- Empirically: Wetterich equation better than Polchinski's.
- Controversies about the choice of regulator
  [Litim, Phys.Lett.B 486 (2000); Canet, Delamotte, Mouhanna, Vidal, Phys.Rev.D 67 (2003)].

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- Broad empirical success of the DE in many applications mostly at order $\mathcal{O}(\partial^0)$ or $\mathcal{O}(\partial^2)$.

  [Berges, Tetradis, Wetterich, Phys.Rept. 363 (2002)]

- ~~$\mathcal{O}(\partial^4)$~~ $\to \mathcal{O}(\partial^6)$ only for Ising exponents.

  [Bertrand's talk]

- The expansion parameter was supposed to be of order ~~1~~ $\to 1/4$ (an expansion on ~~q^2/k^2~~ $\to q^2/(4k^2)$ with $q^2 \lesssim k^2$). ~~Sometimes an expansion in η suggested.~~

- Empirically: Wetterich equation better than Polchinski's.

  $\to$ The $1/4$ applies only to the 1PI equation!

- Controversies about the choice of regulator remains.

  [Litim, Phys.Lett.B 486 (2000); Canet, Delamotte, Mouhanna, Vidal, Phys.Rev.D 67 (2003)].

- Puzzling increasing regulator dependence ("conditional convergence").

Outline and main references

- First, we will review the main characteristics of the DE.
- Second, we will describe the main progress since Aussois:
  - The various preliminary ideas were confirmed.
  - The $\mathcal{O}(\partial^4)$ were implemented in $O(N)$ models.
  - Not only exponents were calculated: also universal amplitude ratios.
  - Controlled error bars were established.
  - The regulator dependence was clarified.
- This talk is the result of the work of several collaborations:
  - Convergence in the DE and application in the study of Ising critical exponents at order $\mathcal{O}(\partial^6)$:
    [Balog, Chaté, Delamotte, Marohnic, NW, PRL 123 (2019)]
  - Critical exponents at order $\mathcal{O}(\partial^4)$ in $O(N)$ models and error bars:
    [De Polsi, Balog, Tissier, NW, PRE 101 (2020)]
  - The role of conformal invariance at order $\mathcal{O}(\partial^4)$:
    [Balog, De Polsi, Tissier, NW, PRE 101 (2020)]
  - Universal amplitude ratios at order $\mathcal{O}(\partial^4)$ in $O(N)$ models:
    [De Polsi, Hernández, NW, PRE 104 (2021)]
  - Quantitative understanding of the regulator dependence: [De Polsi, NW, ArXiv: 2204.09170, accepted in PRE]

The Derivative Expansion (I)

- Let us first make a small review of the DE.
- Usually, DE is presented as an ansatz for Γ_k.
- Examples: In the Ising universality class:

$$\Gamma_k^{\partial^0}[\phi] = \int d^dx \Big[U_k(\phi) + \tfrac{1}{2}(\partial\phi)^2 \Big]. \tag{1}$$

$$\Gamma_k^{\partial^2}[\phi] = \int d^dx \Big[U_k(\phi) + \tfrac{1}{2} Z_k(\phi)(\partial\phi)^2 \Big]. \tag{2}$$

$$\begin{aligned}\Gamma_k^{\partial^4}[\phi] &= \int d^dx \Big[U_k(\phi) + \tfrac{1}{2} Z_k(\phi)(\partial\phi)^2 + \tfrac{1}{2} W_k^a(\phi)(\partial_\mu\partial_\nu\phi)^2 \\ &+ \tfrac{1}{2}\phi W_k^b(\phi)(\partial^2\phi)(\partial\phi)^2 + \tfrac{1}{2} W_k^c(\phi)\left((\partial\phi)^2\right)\big)^2 \Big].\end{aligned} \tag{3}$$

The Derivative Expansion (II)

- In the Ising universality class the order $\mathcal{O}(\partial^6)$ have been studied:

$$
\begin{aligned}
\Gamma_k[\phi] \; = & \int d^d x \Big[U_k(\phi) + \tfrac{1}{2} Z_k(\phi)(\partial\phi)^2 \\
& + \tfrac{1}{2} W_k^a(\phi)(\partial_\mu\partial_\nu\phi)^2 + \tfrac{1}{2}\phi W_k^b(\phi)(\partial^2\phi)(\partial\phi)^2 \\
& + \tfrac{1}{2} W_k^c(\phi)\left((\partial\phi)^2\right)^2 + \tfrac{1}{2}\tilde{X}_k^a(\phi)(\partial_\mu\partial_\nu\partial_\rho\phi)^2 \\
& + \tfrac{1}{2}\phi\tilde{X}_k^b(\phi)(\partial_\mu\partial_\nu\phi)(\partial_\nu\partial_\rho\phi)(\partial_\mu\partial_\rho\phi) \\
& + \tfrac{1}{2}\phi\tilde{X}_k^c(\phi)\left(\partial^2\phi\right)^3 + \tfrac{1}{2}\tilde{X}_k^d(\phi)\left(\partial^2\phi\right)^2(\partial\phi)^2 \\
& + \tfrac{1}{2}\tilde{X}_k^e(\phi)(\partial\phi)^2(\partial_\mu\phi)(\partial^2\partial_\mu\phi) + \tfrac{1}{2}\tilde{X}_k^f(\phi)(\partial\phi)^2(\partial_\mu\partial_\nu\phi)^2 \\
& + \tfrac{1}{2}\phi\tilde{X}_k^g(\phi)\left(\partial^2\phi\right)\left((\partial\phi)^2\right)^2 + \tfrac{1}{96}\tilde{X}_k^h(\phi)\left((\partial\phi)^2\right)^3 \Big]. \qquad (4)
\end{aligned}
$$

- Considering ansatze is useful for implementation of symmetries.
- However, it does not give clues about the range of validity of DE.
- That is why it is presented here in a somewhat different way.

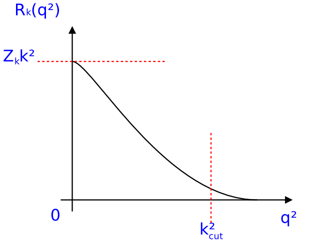
The Derivative Expansion (III)

- Consider a quantity which requires the knowledge of vertices (or derivatives of them) at zero momenta. Examples: critical exponents.
- That is, external momenta verify : $p_i^2 \ll k^2$
- Exact flow of Γ_k [C.Wetterich, Phys.Lett B301 (1993) 90.]:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_t R_k(x-y) \big(\Gamma_k^{(2)} + R_k \big)^{-1}_{x,y}$$

- The factor $\partial_t R_k$ makes internal momenta limited by :

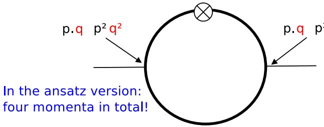
$q^2 \lesssim k_{cut}^2$ with $k_{cut} \sim k$.

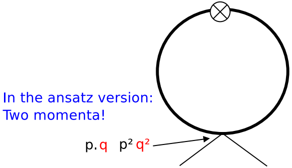


- Then, the sector $p_i^2, q^2 \lesssim k_{cut}^2$ is insensitive to other momenta. $\Rightarrow$ it makes sense to formulate an approximation for this sector alone.

Two versions of the Derivative Expansion

- One approximation very used : Expand vertices in this sector as a polynomial in momenta ⇔ a finite number of derivatives in direct space (Derivative Expansion).
- The expansion of any vertex at order s in momenta can be extracted from the ansatz of the Derivative Expansion $\Gamma_k^{\partial^s}[\phi]$.
- However: Many terms in FRG equations obtained in such a way are sub-leading in the momentum expansion.
- For this reason we have two different implementations of the DE:
  - Ansatz version: just put all the terms from the ansatz.
  - Strict version: Limit all the products of vertices to a given order of the momentum expansion. Example: at order $\mathcal{O}(\partial^2)$:





Results of DE in the Ising universality class (I)

- The DE can be employed to calculate many properties of the model.
- As an example, in order to test its quality we compare here the most relevant critical exponents.
- Those exponents are known for this universality class almost exactly from recent Conformal Boostrap methods.
- Without approximations: results should not depend on the regulator profile $R_k(q)$.
- However, within the DE, the results depends on the choice of $R_k(q)$.
- In practice, many profiles are used. Here we consider the families:

$$W_k(q^2) = \alpha Z_k k^2 \, y/(\exp(y) - 1)$$
$$\Theta_k^n(q^2) = \alpha Z_k k^2 \, (1 - y)^n \theta(1 - y) \quad n \in \mathbb{N}$$
$$E_k(q^2) = \alpha Z_k k^2 \, \exp(-y)$$

with $y = q^2/k^2$ and Z_k an appropriate running renormalization factor.

Results of DE in the Ising universality class (II)

- α is optimized according to the Principle of Minimal Sensitivity. That is: we choose α to be in an extrema for a given exponent.
- We compare the results of various orders of the strict DE.

Table: 3D Ising critical exponents for exponential regulator at various orders s.

	ν	η	ω
LPA	0.64956	0	0.654
$O(\partial^2)$	0.6308(27)	0.0387(55)	0.870(55)
$O(\partial^4)$	0.62989(25)	0.0362(12)	0.832(14)
$O(\partial^6)$	0.63012(16)	0.0361(11)	
Conf. Boots.	0.629971(4)	0.0362978(20)	0.82968(23)

Table: DE results from [Balog, Chaté, Delamotte, Bertrand, Marohnic and NW, PRL 123 (2019); De Polsi, Balog, Tissier, NW, PRE101 (2020)]. Results compared to CB [Kos, Poland and Simmons-Duffin (JHEP 1608, 2016); D.Simmons-Duffin, JHEP03, 086 (2017)].

Results of DE in the Ising universality class (III)

- The results seem to alternate around the “exact” value:

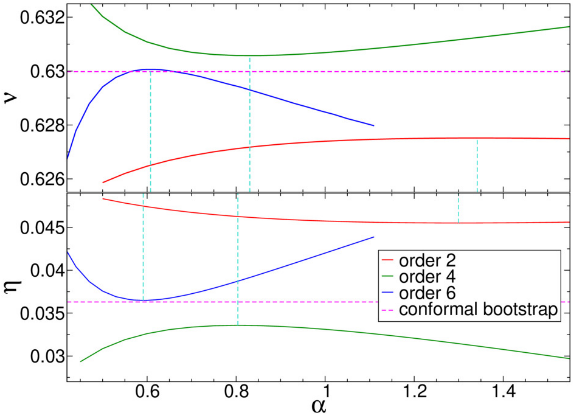


Figure: Exponent values $\nu(\alpha)$ and $\eta(\alpha)$ at different orders of the DE for exponential regulator. Vertical lines indicate α_{PMS}.

- This is another justification for using the Principle of Minimal Sensitivity.

Results for $O(N)$ models (I)

- Ising exponents are just an example.
- Not only exponents but also Universal Amplitude ratios have been calculated at order $\mathcal{O}(\partial^4)$ for $O(N)$ models.
- The employed truncation (in the strict sense!) is: [De Polsi, Balog, Tissier, NW, PRE 101 (2020)]

$$\Gamma_k^{\partial^4}[\phi] = \int_x \Big\{ U_k(\rho) + \frac{1}{2} Z_k(\rho)\big(\partial_\mu \phi^a\big)^2 + \frac{1}{4} Y_k(\rho)\big(\partial_\mu \rho\big)^2$$
$$+\frac{W_1(\rho)}{2}\big(\partial_\mu\partial_\nu\phi^a\big)^2 + \frac{W_2(\rho)}{2}\big(\phi^a\partial_\mu\partial_\nu\phi^a\big)^2$$
$$+W_3(\rho)\partial_\mu\rho\partial_\nu\phi^a\partial_\mu\partial_\nu\phi^a + \frac{W_4(\rho)}{2}\phi^b\partial_\mu\phi^a\partial_\nu\phi^a\partial_\mu\partial_\nu\phi^b$$
$$+\frac{W_5(\rho)}{2}\varphi^a\partial_\mu\rho\partial_\nu\rho\partial_\mu\partial_\nu\varphi^a + \frac{W_6(\rho)}{4}\Big(\big(\partial_\mu\varphi^a\big)^2\Big)^2$$
$$+\frac{W_7(\rho)}{4}\big(\partial_\mu\phi^a\partial_\nu\phi^a\big)^2 + \frac{W_8(\rho)}{2}\partial_\mu\phi^a\partial_\nu\varphi^a\partial_\mu\rho\partial_\nu\rho$$
$$+\frac{W_9(\rho)}{2}\big(\partial_\mu\varphi^a\big)^2\big(\partial_\nu\rho\big)^2 + \frac{W_{10}(\rho)}{4}\Big(\big(\partial_\mu\rho\big)^2\Big)^2 + \mathcal{O}(\partial^6)\Big\}.$$

Results for $O(N)$ models (II)

- For details and universal amplitude ratios: see De Polsi's talk!
- A second example: $O(4)$

  Table: Results for various methods with appropriate error bars for $O(4)$ in $d = 3$. Year: 2020

	ν	η	ω
LPA	0.805	0	0.737
$O(\partial^2)$	0.749(8)	0.0389(56)	0.731(34)
$O(\partial^4)$	0.7478(9)	0.0360(12)	0.761(12)
CB [1,2]	0.7472(87)	0.0378(32)	0.817(30)
MC [3,4]	0.7477(8)	0.0360(4)	0.765
6-loop $d = 3$ [5]	0.741(6)	0.0350(45)	0.774(20)
$\epsilon-$expansion, ϵ^5 [5]	0.737(8)	0.036(4)	0.795(30)
$\epsilon-$expansion, ϵ^6 [6]	0.7397(35)	0.0366(4)	0.794(9)

[1] Kos et al., JHEP 11 (2015). [2] Echeverri et al., JHEP 09 (2016). [3] Deng, PRE 73 (2006). [4] Hasenbusch, J.Phys.A 34 (2001). [5] Guida, Zinn-Justin, J. Phys. A31 (1998). [6] Kompaniets, PRD 96 (2017).

- DE gives the most precise field-theoretical results for ν and ω.

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MC [7] (new!!!)	0.74817(20)	0.03624(8)	0.755(5)

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- ~~DE~~ MC gives the most precise field-theoretical results.

Derivative Expansion: successful but why?

- The DE seems to have a large amount of successes [Dupuis et al., Phys.Rept. 910 (2021)]
  - It gives qualitatively good results in many tested situations;
  - Reasonable quantitative results are obtained in many models at order ∂^2;

  In $O(N)$ models:
  - Critical exponents known at order ∂^4 for many N.
  - Critical exponents known at order ∂^6 in the Ising (N=1) case.
  - Universal amplitude ratios known at order ∂^4 for many N.
  - Gives results that compete with best field theoretical estimates.

Question: Why Derivative Expansion should work?

The problem of the small parameter (I)

- A priori no small parameter:

  $p \leftrightsquigarrow$ order of magnitude of external momenta.

  $q \leftrightsquigarrow$ momenta circulating in the loop.

  $M_k \leftrightsquigarrow$ gap of the regulated theory.

- The gap has two components:
  - A direct contribution from the regulator $\sim k$.
  - A contribution $m_k^2 \propto \Gamma_k^{(2)}(p=0)$.

- If $p, m_k \ll M_k \sim k$, the expansion parameter is:

$$\lambda = \theta \frac{q^2}{M_k^2}$$

where:

- θ is a numerical coefficient a priori~ 1;
- k_{cut} is expected to be $\sim k$.

The problem of the small parameter (II)

- Given that $q \lesssim k_{cut} \sim k \sim M_k$ one expects naively $\lambda \sim 1$.
- But then why does it work?
- Useful: for momenta $p \lesssim k_{cut}$, the regulated critical theory looks as a massive theory with mass gap M_k

$$\Gamma_k^{(n)}(p, m_k) \sim \Gamma_{k=0}^{(n)}(p, M_k).$$

The massive scalar theory at small momenta (I).

- Before considering a critical model let us recall the behaviour of a massive theory with mass M at small momenta.
- Let's present the case of $\Gamma^{(2)}(p)$ (all $\Gamma^{(n)}(p)$ can be studied similarly).
- Consider the expansion

$$\frac{\Gamma^{(2)}(p)}{\Gamma^{(2)}(0)} = 1 + \frac{p^2}{M^2} + \sum_{n=2}^{\infty} c_n \Big(\frac{p^2}{M^2}\Big)^n$$

- The coefficients c_n are universal in the critical regime $M \ll \Lambda$.
- How are the coefficients c_n? $\rightarrow$ Already known! (with some precision...)

The massive scalar theory at small momenta (II).

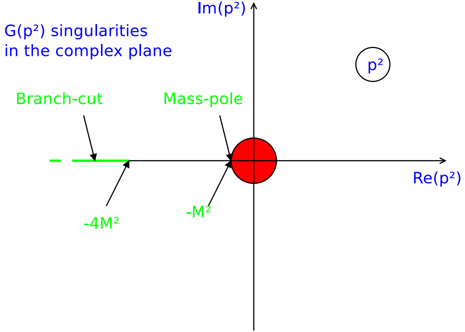
coefficient	$d=3,\ T>T_c$ HT / ϵ / fixed dim.	$d=3,\ T<T_c$ LT/ ϵ / fixed dim.
c_2	-(3.0 - 7.1)$\times 10^{-4}$	$\simeq -10^{-2}$
c_3	(0.5 - 1.3)$\times 10^{-5}$	$\simeq 4\times 10^{-3}$
c_4	-(0.3 - 0.6) $\times 10^{-6}$	

Table: Estimates for high-temperature phase for $N=1$. (from [Pelissetto, Vicari, Phys. Rept. 368, 549 (2002)]).

- Various observations:
  - The coefficients alternates.
  - $|c_2| \gg |c_3| \gg |c_4|$.
  - $|c_2| \ll 1$ (it is abnormally small!).
- One can estimate the radius of convergence of the series.
- It is at the first multi-particle singularity in the complex p^2 plane.

The massive scalar theory at small momenta (III).

- Let us represent the singularities in the complex plane of correlation functions.
- The Minkowskian extension of the massive model being unitary one knows the position of singularities.
- To simplify let us consider the connected correlation function $G^{(2)}(p^2)$:



- The circle of convergence is $|p^2| < M^2$.

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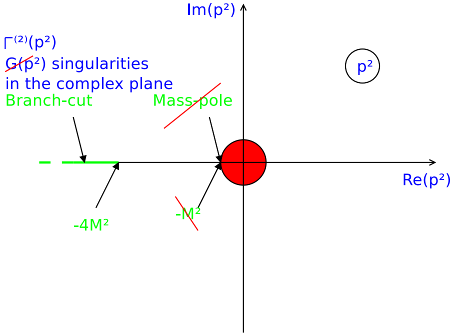

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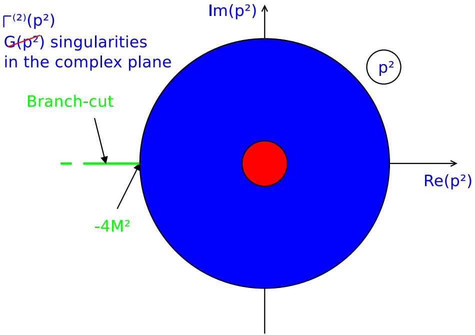
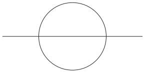


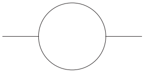
- The circle of convergence is $|p^2| < 4M^2$.

The massive scalar theory at small momenta (IV).

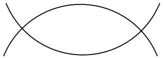
- In the symmetric phase, it is at $p^2 \simeq -9M^2$.
  ($\simeq 3M$ is the energy of the lightest 3-particle state).

at one loop $\rightarrow$ 

- Consequently: $c_{n+1} \simeq -\frac{1}{9} c_n$ for large n.
- Bad news: The $1/9$ expansion parameter is too optimistic:
  - In models with three-liner interactions and/or external field $\rightarrow$ the multi-particle threshold is at $p^2 \approx -4M^2$.

at one loop $\rightarrow$ 

  - Even with quartic interactions, higher vertices include thresholds at $p^2 \approx -4M^2$:

at one loop $\rightarrow$ 

- Good news: The $1/4$ is largely model independent.
- For the critical Ising universality class the parameter turns out to be in between $1/9$ and $1/4$. More good news: the leading coefficients are very small! ($\sim \eta$).

A small parameter for the DE (I)

- Let us now extend the previous results to the regularized theory.
- Before that, an important point: all good results presented for the DE are based on the PMS.
- For other values of α the DE behaves much worst.
- So, the extension from the convergence on the massive theory to the regularized critical theory depends strongly on α.
- Even if the PMS procedure is well founded phenomenologically we would like to avoid "external" elements to the DE.
- One way to justify the PMS is based on conformal invariance [Balog, De Polsi, Tissier, NW, PRE 101 (2020)]
- We now show that:
  - Successive orders of the DE are controlled by an expansion parameter.
  - This parameter is small ($\sim 1/4$) only in an interval of values of α for each regulator family.
  - PMS values of α tend to be in that interval when the order of the DE is increased.

A small parameter for the DE (II)

- We consider a regulator profile $R_k(q)$ smooth enough for the DE to be defined (up to a certain order).
- Let us admit that the regulated theory is well approximated by a massive theory for momenta $q < k_{cut}$. What is the mass M_k?
- Let's define $R_k(q^2) = \alpha Z_k \Big(k^2 - z q^2 + w q^4/k^2 + \dots \Big)$.
- In the critical regime

$$\Gamma^{(2)}(q) + R_k(q) = Z_k \Big(\alpha k^2 + q^2 - \alpha z q^2 + \mathcal{O}(q^4) \Big) \tag{5}$$

where we employed the property (usually correct) that

$$m_k^2 = |\Gamma^{(2)}(q=0)| \ll R_k(0) = \alpha Z_k k^2. \tag{6}$$

- As a consequence, M_k is well approximated by

$$M_k^2 = \frac{\alpha k^2}{1 - \alpha z}. \tag{7}$$

- That is, the mass gap depend on α!

A small parameter for the DE (III)

- One can go back to the expansion parameter which is:

$$\lambda_1 = \frac{k_{cut}^2}{4M_k^2} = \frac{1-\alpha z}{4\alpha\beta}, \tag{8}$$

where we defined $k_{cut}^2 = k^2/\beta$.

- That is:

A small parameter!

$$\theta \lesssim \frac{1}{4}$$

- Fixing precisely β is delicate.
- One criterion: to consider the the point where $\partial_t R_k(q)$ extrapolates to zero:

$$\partial_t R_k(q) = \alpha k^2 Z_k \Big(2 - \eta_k + \eta_k z \frac{q^2}{k^2} - w(2+\eta_k)\frac{q^4}{k^4} + \dots\Big). \tag{9}$$

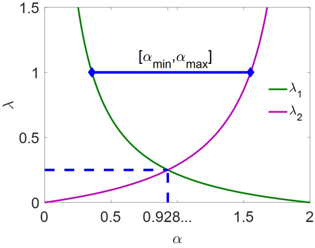
- That is, $k_{cut}^2 \approx \frac{k^2}{2\sqrt{w}} \Leftrightarrow \beta \approx \sqrt{w}$. (In fact, we employ $\beta = 2\sqrt{w}$)

A small parameter for the DE (IV)

- In order for the previous reasoning to be correct we need also that q^4-terms to be smaller than terms of order q^2.
- This gives rise to a second expansion parameter:

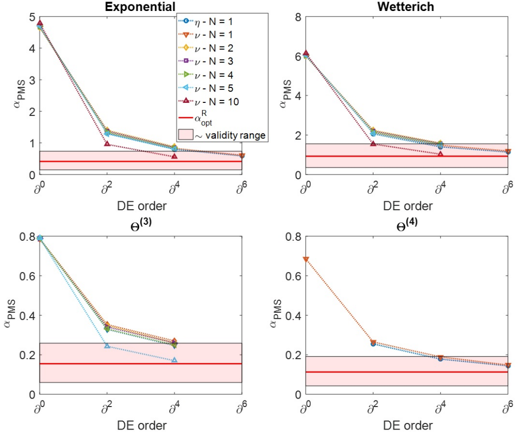
$$\lambda_2 = \frac{\alpha w k_{cut}^4/k^2}{(1-\alpha z)k_{cut}^2} = \frac{\alpha w}{\beta(1-\alpha z)}. \tag{10}$$

- Let us represent both parameters together (for Wetterich regulator W_k):



A small parameter for the DE (V)

- This fixes without assuming PMS the optimum choice of α at large orders of the DE.
- There is a relatively small interval of α where DE has a small parameter.
- How this estimate compares to PMS values?



A key progress: Error bars

- The small parameter $\sim 1/4$ allows the estimate of error bars.
- Shortly: the error is divided, at least, by a factor $1/4$ on successive orders.
- When a quantity alternates one improves the error bars. [Balog, De Polsi, Tissier, NW, PRE 101 (2020); Peli, PRE 103 (2021).]
- These error bars have been tested very carefully.
- They show accurate and precise for critical exponents and universal amplitude ratios in all tested $O(N)$ models.
  See De Polsi's talk for more details!
- Also tested in more involved model as the $\mathbb{Z}_4$ model. [Chlebicki, Sánchez, Jakubczyk, NW, ArXiv: 2204.02089]]
  See Sánchez talk for more details!

Error bars can and must
be included in our results.

Conclusions (I)

- In the last ten years a qualitative change in the results of the Derivative Expansion has taken place.
- For low momentum properties it has been shown to be a robust and accurate approximation scheme that, moreover, becomes precise at large enough orders.
- The existence of a small parameter makes it a controlled approximation for a very broad set of models. (error bars!!!).
- We presented here some examples of applications in Statistical Mechanics but the application is much broader.
- In many cases we became the most precise field-theoretical method.

Conclusions (II)

What is the program for the next ten years?

- DE only works for small momenta.
  $\rightarrow$ The same analysis can be extended for arbitrary momenta. In that case, one needs the more sophisticated BMW approximation.
  $\rightarrow$ The present analysis must be implemented in the BMW approximation scheme and corresponding error bars estimated.
- In the DE a huge set of applications can be implemented in the next few years with controlled error bars.
- Most of the results presented here are not rigorous. Can we made them mathematically rigorous?
- In particular, all the regime of momenta $q \gg k_{cut}$ was neglected. Is it important for the convergence?

A small parameter for the DE (V).

- Let us now compare to the results coming from the solution at DE6.
- In the NPRG the expansion for the $\Gamma_k^{(2)}(p,\phi)+R_k(0)$ is:

$$\frac{\Gamma_k^{(2)}(p,\phi)+R_k(0)}{\Gamma_k^{(2)}(0,\phi)+R_k(0)} = 1+\frac{Z_k p^2+W_k^a p^4+X_k^a p^6}{U_k''+R_k(0)}$$

$$\xrightarrow[k\to 0]{} 1+\frac{p^2}{m_{\text{eff}}^2}+\frac{w_a^* v^{*\prime\prime}}{z^{*2}}\frac{p^4}{m_{\text{eff}}^4}+\frac{x_a^* v^{*\prime\prime 2}}{z^{*3}}\frac{p^6}{m_{\text{eff}}^6}$$

- Here u^*, z^*, w_a^*, x_a^* are the dimensionless functions of $\tilde{\phi}$ at the fixed point.
- $m_{\text{eff}}^2 = k^2 v^{*\prime\prime}/z^*$ with $v^{*\prime\prime} = u^{*\prime\prime} + R_k(0)/Z_k^0 k^2$.
- The coefficients are now functions of the external field dimensionless $\tilde{\phi}$.

A small parameter for the DE (VI).

- Let us now compare successive orders:

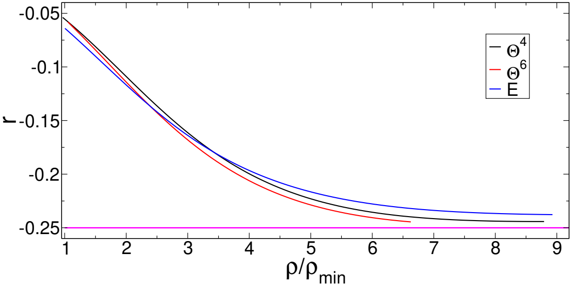


Figure: The ratio $r = x_a^* u^{*\prime\prime}/(w_a^* z^*)$ as a function of $\tilde{\rho} = \tilde{\phi}^2/2$. The line $r = 0.25$ is a guide for the eyes.

- At large fields the successive orders seem dominated by the $p^2 = -4m_{eff}^2$ pole.
- At lower fields the ratio of successive seems to verify $|r| < 1/4$.

The massive scalar theory at small momenta (II).

coefficient	$d = 3,\ T > T_c$ HT / ϵ / fixed dim.	$d = 2,\ T > T_c$ quasi-exact	$d = 3,\ T < T_c$ LT/ ϵ / fixed dim.
c_2	-(3.0 - 7.1)$\times 10^{-4}$	-7.936...$\times 10^{-4}$	$\simeq -10^{-2}$
c_3	(0.5 - 1.3)$\times 10^{-5}$	1.096...$\times 10^{-5}$	$\simeq 4 \times 10^{-3}$
c_4	-(0.3 - 0.6) $\times 10^{-6}$	-0.3127...$\times 10^{-6}$	
c_5		0.1267...$\times 10^{-7}$	
c_6		-0.6300...$\times 10^{-9}$	

Table: Estimates for high-temperature phase for $N = 1$. (from [Pelissetto, Vicari, Phys. Rept. 368, 549 (2002)]).

- One observes that $1 \gg |c_2| \gg |c_3| \gg |c_4| \gg |c_5| \gg |c_6|$.
- One can estimate the radius of convergence of the series.
- It is at the first multi-particle singularity in the complex p^2 plane.
- The Minkowskian extension of the massive model being unitary one knows the position of singularities.
- In the symmetric phase, it is at $p^2 \simeq -9M^2$.
  ($\simeq 3M$ is the energy of the lightest 3-particle state).

at one loop $\rightarrow$