Vanishing regulators

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based on two publications with Alessio Baldazzi and Roberto Percacci (SISSA)
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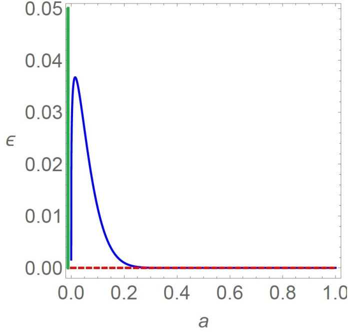
Overview

The parametric limit $R \to 0$

- can be taken in infinitely many ways (we mainly focus on two)
- which do not always result in the same RG scheme
- might be based on **analytic continuation**
  (i.e. not equivalent to a Wilsonian cutoff)
- can reproduce **dimensional regularization + $\overline{\text{MS}}$**
  as a special case
- appears to be well suited for the **preservation of symmetries**
  which are broken at $R \neq 0$ **(OUR MAIN MOTIVATION)**

Overview

Our two case studies:



a **)** VL: Vanishing limit of the Litim regulator (red horizontal axis)

$$R_k(z) = a(k^2 - z)\theta(k^2 - z)\,, \qquad a \to 0$$

$\boldsymbol{\epsilon}$ **)** $\overline{\text{MS}}$ pseudo regulator (green vertical axis)*

$$R_k(z) = z\left[\left(\frac{zk^2}{\mu^4}\right)^{\epsilon} - 1\right]\,, \qquad \epsilon \to 0$$

*and generalizations thereof

Outline

VL

- comparison with Callan-Symanzik (CS) reg. + analytic continuation
- split/shift symmetry restoration
- preserving symmetry in nonlinear O(N+1) models

$\overline{\text{MS}}$

- definition in LPA and at $O(\partial^2)$
- multi-critical models in $d = 2$
- preserving symmetry in nonlinear O(N+1) models
- two loops

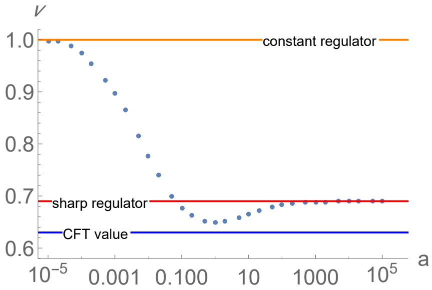
VL: 3D Ising

3D Ising in the LPA

$$V_k(\phi) = \sum_{n=0}^{\infty} \frac{\lambda_{2n}}{(2n)!}\phi^{2n}$$

$$\tilde{\lambda}_2^* \underset{a\to 0}{\sim} -\frac{2a}{5}$$

$$\tilde{\lambda}_4^* \underset{a\to 0}{\sim} \frac{16\pi\sqrt{a}}{3}$$



$$\tilde{\lambda}_{2n}^* = k^{-d+n(d-2)}\lambda_{2n}^* \underset{a\to 0}{\sim} a^{\frac{d-n(d-2)}{2}}\,\hat{\lambda}_{2n}^*$$

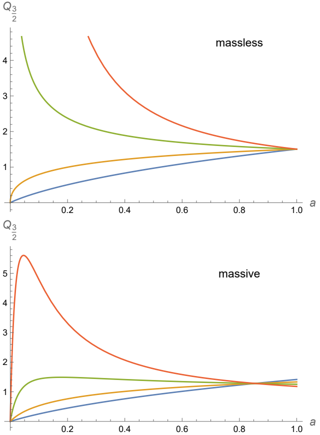
A natural variable $\quad k_{\text{eff}} = \sqrt{a}\,k$

the scale at which $\quad q^2 < R_k(q^2) \leftrightarrow q^2 < k_{\text{eff}}^2$

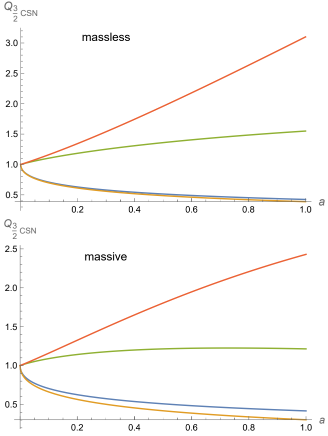
VL: 3D Ising

$$Q_n\left[\frac{\partial_t R_k}{(P_k+m^2)^\ell}\right] = \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\,\frac{\partial_t R_k(z)}{(P_k(z)+m^2)^\ell}$$

at fixed k



at fixed k_{eff}
CS normalized



VL: the same as CS?

General arguments:

- **'passive'** transformation: rescaling k **(Litim 2002)**

$$R_k(z) = bk^2 a\, r_1(z/(bk^2))\,, \qquad b = 1/a$$

- **'active'** transformation: rescaling ϕ and v

$$\tilde{\phi} = a^{(d-2)/4}\hat{\phi}$$
$$v(\tilde{\phi}) = a^{d/2}\hat{v}(\hat{\phi}) + a\frac{1}{(d-2)(4\pi)^{d/2}\Gamma(1+d/2)}$$

they work fine **for Ising in $d=3$:** **VL = CS**

VL: 3D Ising + Background

Background field $\phi = \phi_B + \varphi$

split/shift symmetry

$$\phi_B \mapsto \phi_B + \epsilon$$
$$\varphi \mapsto \varphi - \epsilon$$

background-dependent coarse graining **(Bridle, Dietz, Morris 2015)**

$$R_k(z) = (k^2 - k^2 h(\tilde{\phi}_B) - z)\theta(k^2 - k^2 h(\tilde{\phi}_B) - z)$$

modified Ward identity (mWI)

$$\frac{\delta\Gamma_k}{\delta\phi_B} - \frac{\delta\Gamma_k}{\delta\varphi} = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2\Gamma_k}{\delta\varphi\delta\varphi} + R_k\right)^{-1}\frac{\delta R_k}{\delta\phi_B}\right]$$

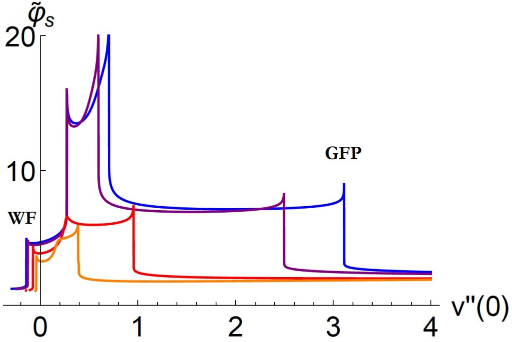
single-field approx $\varphi = 0$

Messes up the Wilson-Fischer FP!

VL: 3D Ising + Background

For instance $\quad h(\tilde{\phi}_B) = -2(\tilde{\phi}_B)^2$

Start with $a = 1/2$ and approach $a \to 0$



spurious FPs disappear

VL: 3D Ising + Background

Solving the mWI for VL

1. Rescale ...

$$\tilde{\varphi} = a^{(d-2)/4}\,\hat{\varphi}$$
$$v(\tilde{\varphi}) = a^{d/2}\,\hat{v}(\hat{\varphi}) + a\frac{1}{(4\pi)^{d/2}(d-2)\,\Gamma\left(\frac{d}{2}+1\right)}$$
$$h = a^{\gamma}\,\hat{h}$$

2. Let $a \to 0$ ...

$$\partial_{\hat{\varphi}}\hat{v} - \partial_{\hat{\phi}_B}\hat{v} = \frac{a^{\gamma+1-d/2}}{d(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\,\hat{h}' + \cdots$$

3. **first order PDE** ... solve it!

If $\gamma > \frac{d}{2}-1$: $\quad \hat{v}(\hat{\varphi},\hat{\phi}_B) = \hat{v}(\hat{\varphi}+\hat{\phi}_B)$

If $\gamma = \frac{d}{2}-1$: $\quad \hat{v}(\hat{\varphi},\hat{\phi}_B) = \hat{v}(\hat{\varphi}+\hat{\phi}_B) - \frac{1}{d(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\hat{h}(\hat{\phi}_B)$

VL: the same as CS?

General arguments:

- **'passive'** transformation: rescaling k **(Litim 2002)**

$$R_k(z) = bk^2 a\, r_1(z/(bk^2))\,, \qquad b = 1/a$$

they work fine **for Ising in $d = 3$:** **VL = CS**

but for other models or d's ?

caution: while the loop integrals converge for $a > 0$
they might diverge at $a = 0$

VL: the same as CS?

1st counter-example: **4D LPA**

$$\hat{\lambda}_2 = a^{-1}\tilde{\lambda}_2 \ ,$$
$$\hat{\lambda}_4 = \log(a)\tilde{\lambda}_4 \ ,$$
$$\hat{\lambda}_{2n} = a^{n-2}(\log a)^n\tilde{\lambda}_{2n}, \quad n > 2 \ .$$

$\hat{\lambda}_{2n}$ fixed for $a \to 0$

$$\partial_t\hat{\lambda}_2 = -2\hat{\lambda}_2 + \frac{\hat{\lambda}_4}{16\pi^2}\left[1 + \frac{1+\log\left(1+\hat{\lambda}_2\right)}{\log a}\right],$$

$$\partial_t\hat{\lambda}_4 = \frac{1}{\log a}\left[\frac{3}{16\pi^2}\frac{\hat{\lambda}_4^2}{1+\hat{\lambda}_2} + \frac{1}{16\pi^2}\hat{\lambda}_6\right],$$

$$\partial_t\hat{\lambda}_6 = 2\hat{\lambda}_6 - \frac{15}{16\pi^2}\frac{\hat{\lambda}_4^3}{(1+\hat{\lambda}_2)^2} + \frac{1}{16\pi^2}\hat{\lambda}_8$$
$$+ \frac{\hat{\lambda}_8}{16\pi^2}\frac{1+\log\left(1+\hat{\lambda}_2\right)}{\log a} + \frac{15}{16\pi^2}\frac{\hat{\lambda}_4\hat{\lambda}_6}{(1+\hat{\lambda}_2)\log a} \ ,$$

VL: the same as CS?

2^{nd} counter-example: **2D NLSM**

truncation: linear $O(N)$ at $O(\partial^2)$

$$\Gamma_k[\phi] = \int d^2x \left[U_k(\rho) + \frac{1}{2} Z_k(\rho)\partial_\mu \phi_a \partial^\mu \phi^a + \frac{1}{4} Y_k(\rho)\partial_\mu \rho \partial^\mu \rho\right]$$

special ansatz: the nonlinear $O(N+1)$ model

$$\Gamma_k[\phi] = \int d^dx \frac{Z_k}{2g_k^2}\left(\delta_{ab} + \frac{\phi_a\phi_b}{\frac{1}{Z_k} - 2\rho}\right)\partial_\mu \phi^a \partial^\mu \phi^b$$

Does the ERGE **preserve** this?

	CS	VL
$U_k = 0$	X	✓
$Z_k \& \tilde{Z}_k$	✓	✓

$$\partial_t g_k = -\frac{(N-1)g_k^3}{4\pi + g_k^2}$$

$$\eta_k = -\partial_t \log Z_k = \frac{2Ng_k^2}{4\pi + g_k^2}$$

$\overline{\text{MS}}$ in the LPA

In the LPA, $\overline{\text{MS}}$ beta functions are reproduced if $\exists R_k$ such that

$$Q_n\left[\frac{\partial_t R_k}{(P_k+\lambda_2)^\ell}\right] = \frac{2(-1)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\lambda_2^{n-\ell+1}$$

a pseudo-regulator achieving this for $\epsilon \to 0$

$$R_k(z) = \left(\frac{k^2}{\mu^4}\right)^\epsilon z^{1+\epsilon} - z$$

for instance in **even d**

$$\partial_t \tilde{V}_k = -d\tilde{V}_k + \left(\frac{d}{2}-1\right)\tilde{\phi}\tilde{V}_k' + c_d\left(-\tilde{V}_k''\right)^{\frac{d}{2}}$$

$\overline{\text{MS}}$ at $O(\partial^2)$

At $O(\partial^2)$, we can generalize

$$R_k(z) = Z_0 Z_k^{\sigma\epsilon}\left[\left(\frac{k^2}{\mu^4}\right)^{\epsilon} z^{1+\epsilon} - z\right]$$

we choose σ constant for $\epsilon \to 0 \longleftrightarrow$ finite η_k contribution
for instance in the LPA′

$$Q_n\left[G_k^{\ell}\partial_t R_k\right] = \frac{Z_k^{-n}\left(-V_k''\right)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\left(2 - \sigma\eta_k(1+H_0)\right)$$

where

$$H_0(n, Z_k, Z_0) = -\frac{n}{n+1}\left(\frac{Z_k}{Z_0}\right)^n {}_2F_1\left(n+1, n+1, n+2; 1-\frac{Z_k}{Z_0}\right)$$

autonomous flows only for

- $\sigma \to 0$ or $Z_0 \to \infty$ for which no RG improvement through η
- $Z_0 \to 0$ for which $H_0 \to 0$

$\overline{\text{MS}}$ at $O(\partial^2)$

1st application: **2D multicritical scalar models**

FP for a real scalar field

$$v_* = -\frac{2-\sigma\eta}{16\pi}\tilde{m}^2 \cos\left(\frac{2}{\sqrt{\eta}} \arctan\sqrt{\frac{\Phi^2}{1-\Phi^2}}\right)$$

$$\zeta_* = \zeta_0\left(1-\Phi^2\right)^{-1}, \qquad \Phi = \sqrt{\frac{4\pi\eta\zeta_0}{2-\sigma\eta}}\,\tilde{\phi}$$

$\eta = \frac{1}{p^2}, \quad p \in \mathbb{Z}$

$\nu = \frac{1}{2-2\eta}$

	$\overline{\text{MS}}$	opt. reg.	hom. reg.	exact
η_2	0.25	0.2132	0.309	0.25
ν_2	0.666667	...	0.863	1
η_3	0.111111	0.1310	0.200	0.15
ν_3	0.5625	...	0.566	0.556
η_4	0.0625	0.0910	0.131	0.1
ν_4	0.533333	...	0.545	0.536
η_5	0.04	0.0679	0.0920	0.0714
ν_5	0.520833	...	0.531	0.525
η_6	0.0277778	0.0522	0.0679	0.0535714
ν_6	0.514286	...	0.523	0.519
η_7	0.0204082	...	0.0521	0.0416667
ν_7	0.510417	...	0.517	0.514
η_8	0.015625	...	0.0412	0.0333333
ν_8	0.507937	...	0.514	0.511
η_9	0.0123457	...	0.0334	0.0272727
ν_9	0.50625	...	0.511	0.509
η_{10}	0.01	...	0.0277	0.0227273
ν_{10}	0.505051	...	0.509	0.508
η_{11}	0.00826446	...	0.0233	0.0192308
ν_{11}	0.504167	...	0.508	0.506

$\overline{\text{MS}}$ at $O(\partial^2)$

2nd application: **2D NLSM**

truncation: linear O(N) at $O(\partial^2)$

$$\Gamma_k[\phi] = \int d^d x \left[U_k(\rho) + \frac{1}{2} Z_k(\rho)\partial_\mu\phi_a\partial^\mu\phi^a + \frac{1}{4} Y_k(\rho)\partial_\mu\rho\partial^\mu\rho\right]$$

special ansatz: the nonlinear O(N+1) model

$$\Gamma_k[\phi] = \int d^d x \,\frac{Z_k}{2g_k^2}\left(\delta_{ab} + \frac{\phi_a\phi_b}{\frac{1}{Z_k} - 2\rho}\right)\partial_\mu\phi^a\partial^\mu\phi^b - h_k\sqrt{Z_k^{-1} - 2\rho}$$

Does the ERGE **preserve** this?

	CS	VL	$\overline{\text{MS}}$
$U_k \neq 0$	X	X	✓
$U_k = 0$	X	✓	✓
$Z_k \& \tilde{Z}_k$	✓	✓	✓

$$\partial_t g_k = -\frac{(N-1)g_k^3}{4\pi + \sigma g_k^2}$$

$$\eta_k = -\partial_t \log Z_k = \frac{2N g_k^2}{4\pi + \sigma g_k^2}$$

$\overline{\text{MS}}$ at two loops

All $\overline{\text{MS}}$ FRG equations so far look like:

- one loop + η (or m^2)-induced RG resummations
- no threshold effects (inevitably)

does the $\overline{\text{MS}}$ limit kill the nonperturbative nature of the ERGE?

Test: can we reproduce two-loop contributions?

(Papenbrock, Wetterich 1995)

Massive O(N) model in $d = 4$

$$\beta_\lambda = \frac{N+8}{16\pi^2}\lambda^2 - \frac{2(5N+22)}{(16\pi^2)^2}\lambda^3 + 2\eta\lambda$$

$$\eta = \frac{(N+2)}{2(16\pi^2)^2}\lambda^2$$

$$\partial_t \log m^2 = \frac{(N+2)}{16\pi^2}\lambda - \frac{5(N+2)}{2(16\pi^2)^2}\lambda^2$$

lessons:

- do not take $\epsilon \to 0$ too early
- RG resummations through coupling-dependence of R_k are good

Summary

The parametric limit $R \to 0$

- can be taken in infinitely many ways (we mainly focused on two)
- which do not always result in the same RG scheme
- might be based on **analytic continuation** (i.e. not equivalent to a Wilsonian cutoff)
- can reproduce **dimensional regularization + $\overline{\text{MS}}$**
- appears to be well suited for the **preservation of symmetries** which are broken at $R \neq 0$