Essential Renormalisation Group

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Introduction

- Based on *Essential renormalisation group* (to appear in Sci-Post) with Alessio Baldazzi and Riccardo Ben Alì Zinati.
- We have proposed a new scheme for the Exact Renormalisation Group (ERG) motivated by the desire of reducing the complexity of practical computations inspired by works of Giovanni Jona-Lasinio, Franz Wegner and Steven Weinberg in the 70's.
- The key idea: Physical systems can be described by many different mathematical models due to our freedom to make a change of variables.
- Mathematical models therefore fall into equivalence classes with each member of a class describing the same physics written in terms of different variables.
- Weinberg '79: There are two types of coupling constants, *inessential couplings* that change as we move within an equivalence class and *essential couplings* that are invariants of the class and enter expressions for observables.

Classical frame transformation

- For a *classical* (non-stochastic) model based on an action $S[\phi]$ an infinitesimal *active frame transformation* has the form

$$S[\phi] \to S'[\phi] = S[\phi + \epsilon \Phi[\phi]] = S[\phi] + \epsilon \int_x \Phi[\phi](x) \frac{\delta}{\delta \phi(x)} S[\phi] \,,$$

where $\Phi[\phi]$ is a local function of the field e.g.

$$\Phi(x) = A\,\phi(x) + B\,\phi^3(x) + C\,\partial^2 \phi(x) + D\phi(x)(\partial_\mu \phi(x))^2 \ldots$$

- The transformation can be recognised as a Lie derivative of a scalar on configuration space. If we expand $S[\phi]$ in some basis $S[\phi] = c_n O_n[\phi]$ the couplings c_n will change $c_n \to c'_n(c)$

Classical inessential couplings and redundant operators

- A coupling $g_i = g_i(c)$ has a conjugate operator $\mathcal{O}_i$ where

$$\frac{\partial}{\partial g_i} S[\phi] = \mathcal{O}_i[\phi]$$

- By definition an inessential coupling is one where the conjugate operator is of the form of an active frame transformation

$$\frac{\partial}{\partial \zeta_\alpha} S[\phi] = \int_x \Phi_\alpha[\phi](x) \frac{\delta}{\delta\phi(x)} S[\phi] \,,$$

  e.g.

$$\Phi_1(x) = \phi(x) \,, \quad \Phi_3(x) = \phi^3(x) \,, \quad \Phi_{1,2}(x) = \partial^2 \phi$$

- We call the operator conjugate to an inessential coupling a *redundant operator*.

Frame transformations of the microscopic action.

- In quantum field theory (QFT) we are interested in computing expectation values of operators by averaging over all values of the field

$$\langle \mathcal{O}[\hat{\phi}] \rangle \equiv \mathcal{N} \int_{\mathcal{M}} \prod_x \mathrm{d}\hat{\phi}(x) e^{-S[\hat{\phi}]} \mathcal{O}[\hat{\phi}] \,,$$

where $\mathcal{N}^{-1} = \int (\mathrm{d}\hat{\chi}) e^{-S_{\hat{\chi}}[\hat{\chi}]}$.

- We can make a change of variables which keeps expectation values unchanged

$$\langle \mathcal{O}[\hat{\phi}] \rangle = \int_{\mathcal{M}} \prod_x \mathrm{d}(\hat{\phi}(x) + \epsilon \hat{\Phi}[\hat{\phi}](x)) e^{-S[\hat{\phi} + \epsilon \hat{\Phi}[\hat{\phi}]]} \mathcal{O}[\hat{\phi} + \epsilon \hat{\Phi}[\hat{\phi}]]$$

Inessential couplings of the microscopic action.

- Observables $\mathcal{O}[\hat{\phi}]$ transform as scalars on configuration space
- To transform $S[\hat{\phi}]$ we must take into account that the measure also transforms (i.e. $\mathrm{e}^{-S[\hat{\phi}]}$ is a density)
- Inessential couplings in QFT

$$\frac{\partial}{\partial \zeta} \mathrm{e}^{-S[\hat{\phi}]} = \int_x \frac{\delta}{\delta \hat{\phi}(x)} \left(\hat{\Phi}[\hat{\phi}](x) \mathrm{e}^{-S[\hat{\phi}]} \right)$$

Equivalently (Wegner '74)

$$\frac{\partial}{\partial \zeta} S[\hat{\phi}] = \int_x \hat{\Phi}[\hat{\phi}](x) \frac{\delta}{\delta \hat{\phi}(x)} S[\hat{\phi}] - \int_x \frac{\delta \hat{\Phi}[\hat{\phi}](x)}{\delta \hat{\phi}(x)}$$

the operator on the RHS is a redundant operator of a microscopic action (e.g. the Wilsonian effective action).

Redundant operators of the effective average action

- The Effective Average Action is defined by

$$e^{-\Gamma_k[\phi]} = \int d\hat{\phi}\, e^{-S[\hat{\phi}]} e^{(\hat{\phi}-\phi)\cdot \frac{\delta \Gamma_k}{\delta \phi} - \frac{1}{2}(\hat{\phi}-\phi)\cdot R_k \cdot (\hat{\phi}-\phi)}$$

- We can take a derivative with respect to an inessential coupling

$$\frac{\partial}{\partial \zeta} e^{-\Gamma_k[\phi]} = \int d\hat{\phi}\, \frac{\delta}{\delta \hat{\phi}} \cdot \left(\hat{\Phi}[\hat{\phi}] e^{-S[\hat{\phi}]} \right) e^{(\hat{\phi}-\phi)\cdot \frac{\delta}{\delta \phi}\Gamma_k[\phi] - \frac{1}{2}(\hat{\phi}-\phi)\cdot R_k \cdot (\hat{\phi}-\phi)}$$

- One then finds that (Baldazzi, Ben Alì Zinati, KF '21)

$$\frac{\partial}{\partial \zeta}\Gamma_k[\phi] = \Phi_k[\phi] \cdot \frac{\delta}{\delta \phi}\Gamma_k[\phi] - \mathrm{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \cdot \frac{\delta}{\delta \phi}\Phi_k[\phi] \cdot R_k$$

where $\Phi_k[\phi] = \langle \hat{\Phi}[\hat{\phi}] \rangle_{\phi,k}$

Redundant operators for the effective average action

- Instead of changing an inessential coupling in the microscopic action we can keep S fixed and modify the source and regulator terms.
- Let's consider a frame for the microscopic action where the field is $\hat{\chi}$ but we couple the sources to a composite operator $\hat{\phi}[\hat{\chi}]$

$$e^{-\Gamma_{\hat{\phi}}[\phi,R_k]} = \int d\hat{\chi}\, e^{-S[\hat{\chi}]} e^{(\hat{\phi}[\hat{\chi}]-\phi)\cdot\frac{\delta\Gamma_k}{\delta\phi} - \frac{1}{2}(\hat{\phi}[\hat{\chi}]-\phi)\cdot R_k\cdot(\hat{\phi}[\hat{\chi}]-\phi)}$$

- If we change the functional form of $\hat{\phi}[\hat{\chi}] \to \hat{\phi}[\hat{\chi}] - \epsilon\hat{\Phi}[\hat{\chi}]$ then

$$\Gamma_{\hat{\phi}}[\phi,R_k] \to \Gamma_{\hat{\phi}}[\phi,R_k] + \epsilon\Phi_k[\phi]\cdot\frac{\delta}{\delta\phi}\Gamma_{\hat{\phi}}[\phi,R_k] - \epsilon\mathrm{Tr}\frac{1}{\Gamma^{(2)}_{\hat{\phi}}[\phi,R_k]+R_k}\cdot\frac{\delta}{\delta\phi}\Phi_k[\phi]\cdot R_k$$

Generalised flow equation

- Adopting the *principle of frame invariance*, we can exploit the freedom to change frames along the RG flow by letting $\hat{\phi}_k[\hat{\chi}]$, the generalised EAA is then given by

$$\Gamma_k[\phi] = \Gamma_{\hat{\phi}_k}[\phi, R_k]$$

and obeys the Pawlowski's generalised flow equation

$$\left(\partial_t + \Psi_k[\phi] \cdot \frac{\delta}{\delta\phi}\right)\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\,\frac{1}{\Gamma_k^{(2)}[\phi] + R_k}\left(\partial_t + 2 \cdot \frac{\delta}{\delta\phi}\Psi_k[\phi]\right) \cdot R_k \,,$$

where $t = \log(k/k_0)$ is the RG time and

$$\Psi_k[\phi] = \langle \partial_t \hat{\phi}_k[\hat{\chi}] \rangle_{\phi, R_k}$$

which we call the *RG kernel*.

Essential schemes

- By construction the freedom to choose $\Psi_k[\phi]$ is exactly the freedom to choose the flow of the inessential couplings.
- We can think of the flow equation as an equation of motion in a gauge theory where frame transformations are the gauge transformations.
- Conditions which fix the inessential couplings are the analogy of gauge fixing conditions.
- Essential schemes: We allow all possible terms in $\Psi_k[\phi]$ and fix all inessential couplings.

Minimal essential scheme

- To generalise the standard scheme we can include all possible terms in ψ_t

$$\Psi_k[\varphi] = \sum_\alpha \gamma_\alpha(k)\Phi_\alpha[\varphi]\,,$$

where $\Phi_\alpha[\varphi]$ form a basis of linearly independent local operators.

- We can then apply renormalisation conditions to fix every inessential coupling ζ_α
- There is not a unique way to do this whether a coupling is inessential or essential depends on which part of theory space we are in.

Minimal essential scheme

- We can fix the inessential couplings at the Gaussian fixed point by setting to zero terms of the form

$$-\Phi_\alpha \cdot \partial^2 \phi$$

- This will fix inessential couplings provided $\Upsilon_{\alpha\beta}$ is invertible where

$$\Phi_\alpha[\phi] \cdot \frac{\delta}{\delta\phi}\Gamma_k[\phi] - \mathrm{Tr}\frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \cdot \frac{\delta}{\delta\phi}\Phi_\alpha[\phi] \cdot R_k = -\sum_\beta \Upsilon_{\alpha\beta}\Phi_\beta \cdot \partial^2\phi + \sum_a v_{\alpha a} O_a[\phi]$$

Properties of the minimal essential scheme

- The minimal essential scheme puts a restriction on which physical theories we can have access to.
- However, it is intuitively clear that this restriction has a physical meaning: we can have access to theories which share the kinematics of the Gaussian fixed point (see also Diego Buccio , Roberto Percacci 2207.10596).
- This is seen clearly since when evaluated on any constant field configuration the propagator has the form

$$\frac{1}{\Gamma_k^{(2)}[\tilde{\phi}] + R(p^2)} = \frac{1}{p^2 + V_k^{(2)}(\tilde{\phi}) + R_k(p^2)},$$

- One should bear in mind that this is not the propagator for the physical field χ however.
- What the minimal essential scheme assumes is that the propagator can be brought into this form by a change of variables.

Order ∂^2

- At order ∂^2 in the *standard scheme* we have

$$\Gamma_k = \int \mathrm{d}^d x \left\{ V_k(\phi) + \frac{1}{2} z_k(\phi) \partial_\mu \phi \partial_\mu \phi \right\}$$

  and $\Psi_k = -\frac{1}{2}\eta_k \phi$. We can fix one renormalisation condition e.g. $z_k(0) = 1$.

- In the minimal essential scheme we can eliminate all inessential couplings by setting $z_k(\phi) = 1$ for all values of the field. So the action is given by

$$\Gamma_k = \int \mathrm{d}^d x \left\{ V_k(\phi) + \frac{1}{2} \partial_\mu \phi \partial_\mu \phi \right\}$$

- To close the equations we take $\Psi_k = F_k(\phi)$.
- While the number of equations that one has to solve is the same the equation is linear in F_k in the essential scheme.
- So we trade a non-linear dependence on z_k for a linear one F_k.

Application to the 3D Ising model

- As a first application of the essential scheme we have studied the Wilson Fisher fixed point in $d = 3$ dimensions.
- To find a fixed point we go to dimensionless variables in units of k and then look for k independent solutions. So $F_k = k^{1/2} f_t(\varphi)$ and $V_k = k^3 v_t(\varphi)$ where $\varphi = \phi/k^{1/2}$.
- For the cutoff $R(p^2/k^2) = (1 - p^2/k^2)\Theta(1 - p^2/k^2)$ we then have the differential equations

$$3v(\varphi) - \frac{1}{2}\varphi v^{(1)}(\varphi) + f(\varphi)v^{(1)}(\varphi) = \frac{1 + \frac{2}{5} f^{(1)}(\varphi)}{1 + v^{(2)}(\varphi)}\,, \tag{1a}$$

$$-f^{(1)}(\varphi) = \frac{1}{2}\frac{\left[v^{(3)}(\varphi)\right]^2}{\left[1 + v^{(2)}(\varphi)\right]^4}\,.$$

- To find physical fixed points of the Ising model we impose the symmetry $\varphi \to -\varphi$ and look for solutions which exist for all values of the field
- We find just two solutions: The free fixed point where $v = \mathrm{const.}$ and the interacting Wilson Fisher fixed point.

Wilson Fisher Fixed point

- The potential at the Wilson Fisher Fixed point can be found numerically

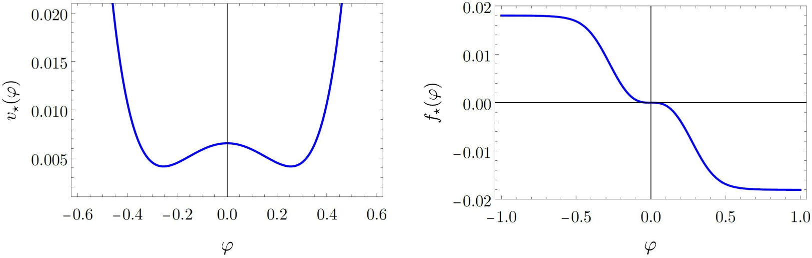

Figure: Potential and RG kernel at the Wilson Fisher Fixed point

Wilson Fisher Fixed point

- The critical exponent $\theta = 1/\nu$ is identified with the sole relevant exponent associated to an even perturbation.
- The critical exponent $\theta = (5 - \eta)/2$ is identified with the sole relevant exponent associated to an odd perturbation.
- We obtain

$$\nu = 0.6271\,, \quad \eta = 0.0470$$

- At the same order $(s = 2)$ the standard scheme obtains (after applying the principle of minimum sensitivity (PMS)) Canet et al. 2003 10.1103/PhysRevD.67.065004

$$\nu = 0.6260\,, \quad \eta = 0.0470$$

(The best estimates from the conformal bootstrap El-Showk et al. (2014) are $\nu = 0.629971$ and $\eta = 0.036298$.)

Application to the Ising model: Summary

- Using the essential scheme there is a large reduction in the complexity of the calculation at order $s = 2$ in comparison to the standard scheme .
- The values of the universal exponents are comparable to standard scheme
- Going to higher orders in the derivative expansion there are further reductions in the number of “potentials” that appear in the action
- At order $s = 4$ for example the essential form of the action is

$$\Gamma_k = \int_x \left\{ V_k(\phi) + \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + W_k(\phi)\left(\partial_\mu\phi\partial_\mu\phi\right)^2 \right\} ,$$

  which involves only two potentials while in the standard scheme we would have five.
- To close the equations we set

$$\Psi_k(x) = F_0(\phi) - F_{2,a}(\phi)\partial^2\phi + \phi F_{2,b}(\phi)\left(\partial_\mu\phi\,\partial_\mu\phi\right).$$

	standard	essential
LPA	1	1
∂^2	2	1
∂^4	5	2
∂^6	13	4
$\vdots$	$\vdots$	$\vdots$

Generalisability

- The ideas are not limited to scalar field theories: We can use essential schemes in any application of the FRG since there will always be inessential couplings.
- The minimal essential scheme for quantum gravity can have some profound implications for the asymptotic safety scenario: we can ensure that there are no ghost poles in the propagator (Alessio Baldazzi, KF 2107.00671;Alessio Baldazzi, KF, Renata Ferrero 2112.02118; Benjamin Knorr 2204.08564)
- Talk by Oleg Melichev today.

Conclusions

- Using the freedom to perform general changes of variables one can apply renormalisation conditions that fix the values of inessential couplings.
- This reduces the complexity of calculations allowing access to physical quantities with less effort.
- Message: Since inessential couplings can take quite arbitrary values we can specify their values instead of computing their flow.
- Try it!