ERG 2022, Berlin 25-29 July 2022

Mathematics of the Climate Crisis: From Smooth Response to Critical Transitions

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Thanks:

T. Bodai, M. Chekroun, H. Dijkstra, M. Ghil, T. Grafke, A. Gritsun, A. Laio, V. Lembo, F. Lunkeit, G. Margazoglou, F. Ragone, M. Santos Gutierrez, A Tantet, J. Wouters

CriticalEarth

TiPES TIPPING POINTS IN THE EARTH SYSTEM

CMPE UNIVERSITY OF READING

University of Reading

The Climate System

Subsystems

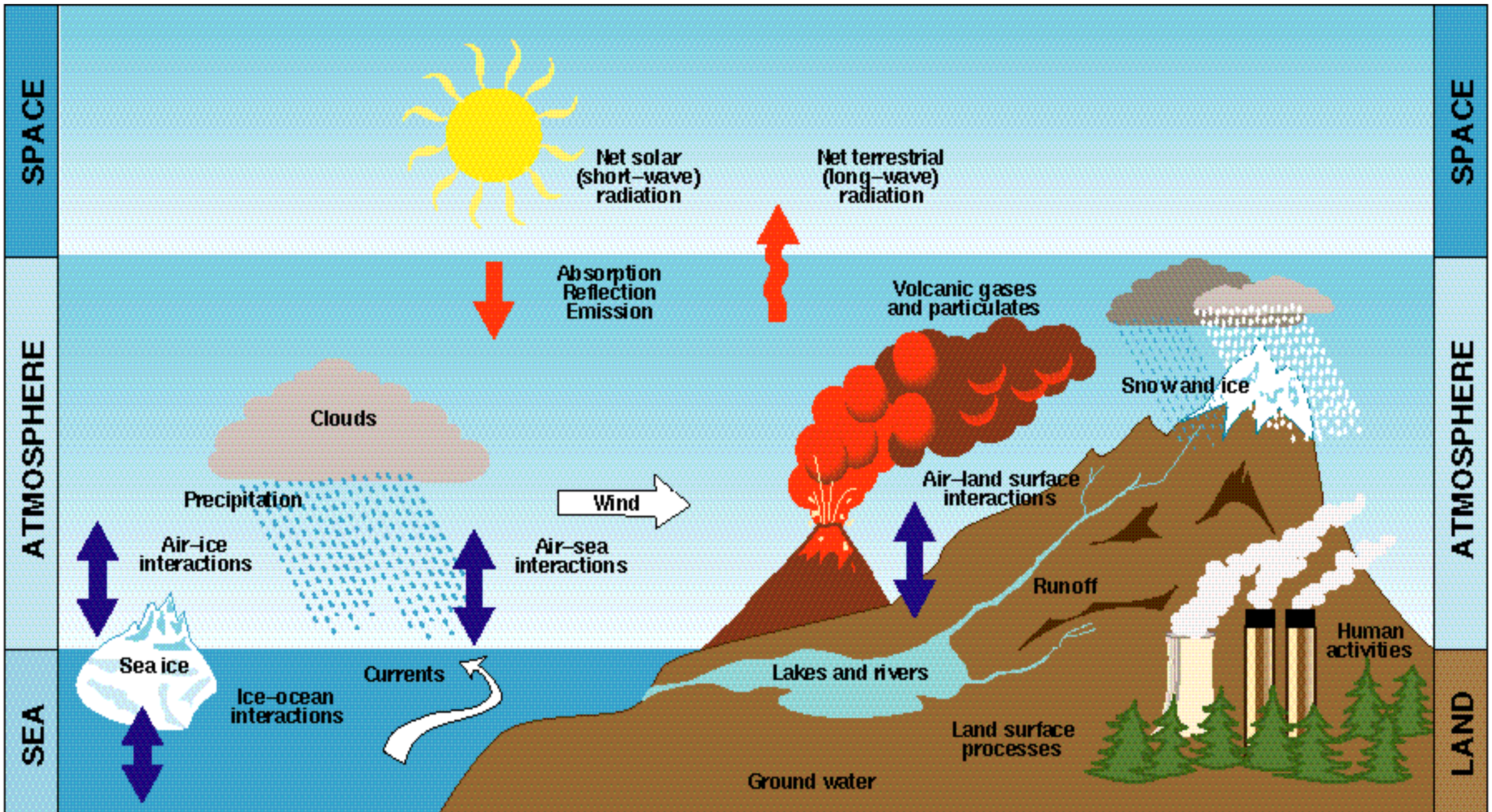
Atmosphere

Land

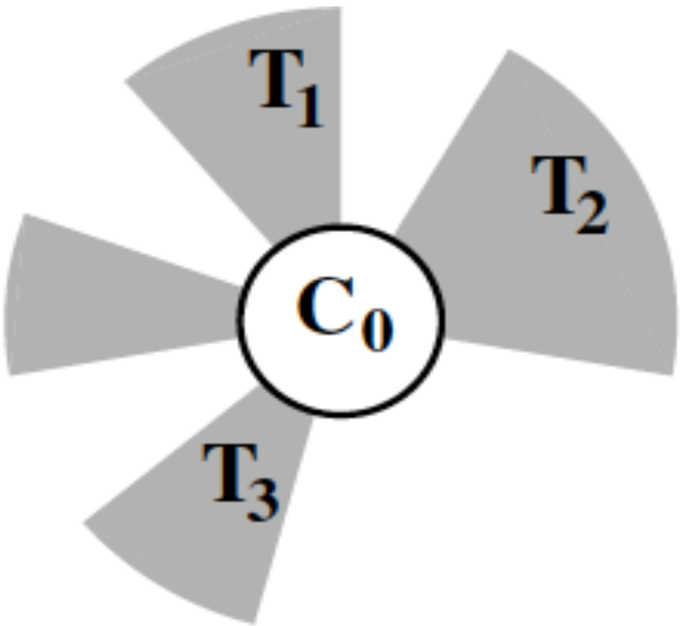
Cryosphere

Hydrosphere

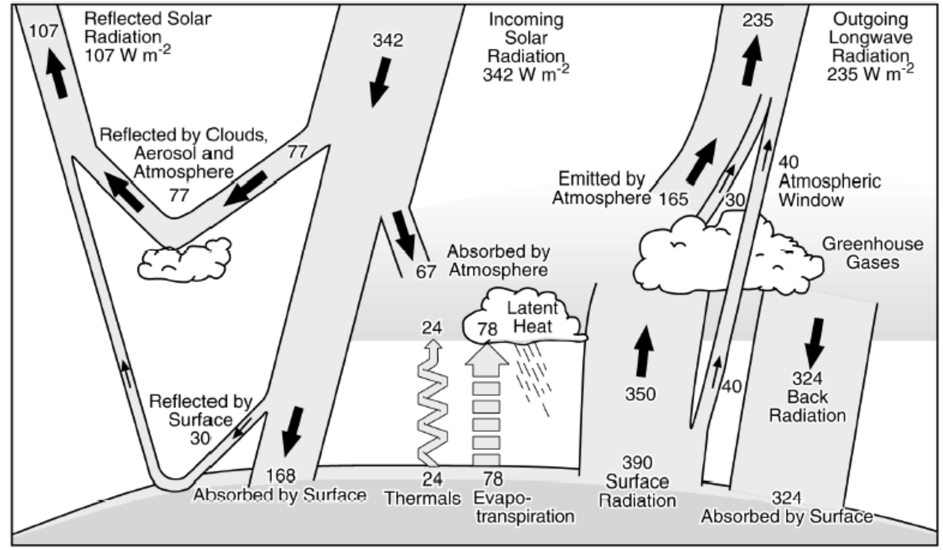
Biosphere



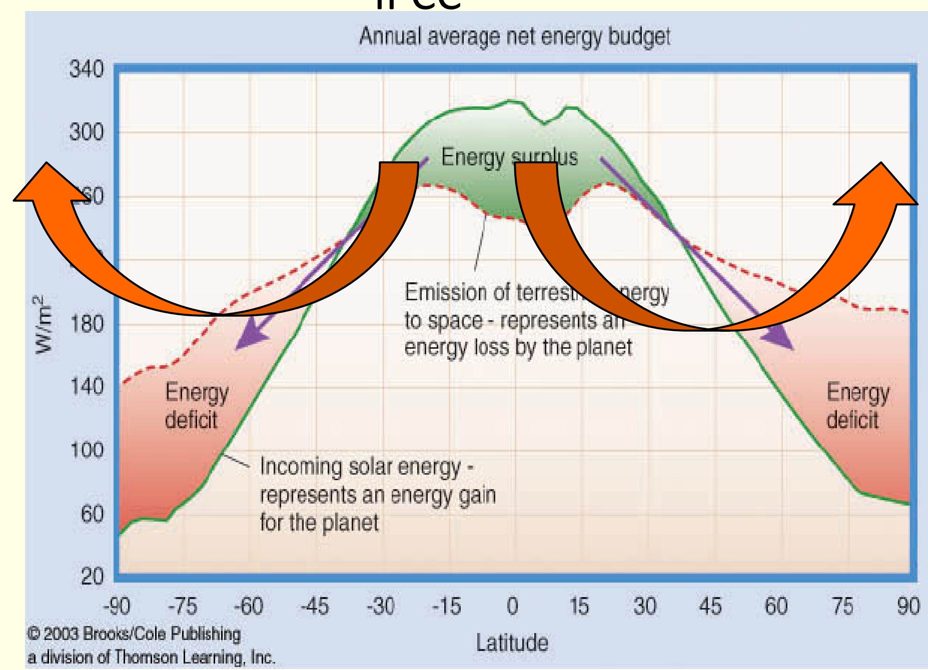
Climate as a nonequilibrium system …



Gallavotti 2014



IPCC



Zonally averaged radiation balance in the atmosphere.

Reviews of Geophysics

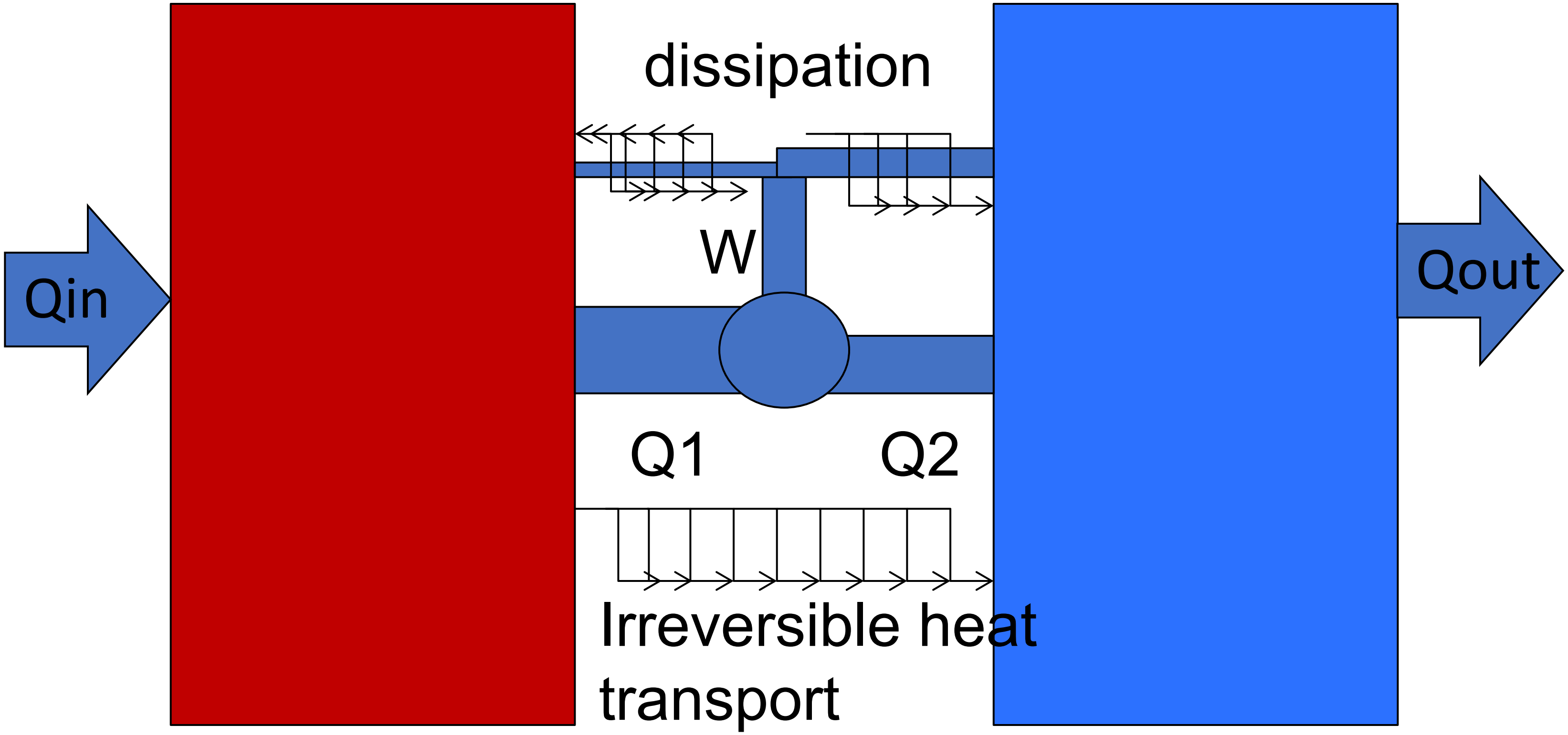
REVIEW ARTICLE
10.1002/2013RG000446

Mathematical and physical ideas for climate science

Valerio Lucarini[1,2,3], Richard Blender[1], Corentin Herbert[4], Francesco Ragone[1,5], Salvatore Pascale[1], and Jeroen Wouters[1,6]

Key Points:

Climate as an extremely non-ideal engine …



- Energy transformation – Lorenz energy cycle
- Baroclinic and barotropic instability
- Entropy Production (dissipation, mixing, hydrological cycle)
- Thermodynamic Route to Climate Dynamics

PHYSICAL REVIEW E 80, 021118 (2009)

Thermodynamic efficiency and entropy production in the climate system

Valerio Lucarini*

Geosci. Model Dev., 12, 3805–3834, 2019
https://doi.org/10.5194/gmd-12-3805-2019

Geoscientific Model Development

TheDiaTo (v1.0) – a new diagnostic tool for water, energy and entropy budgets in climate models

Valerio Lembo[1], Frank Lunkeit[1], and Valerio Lucarini[1,2,3]

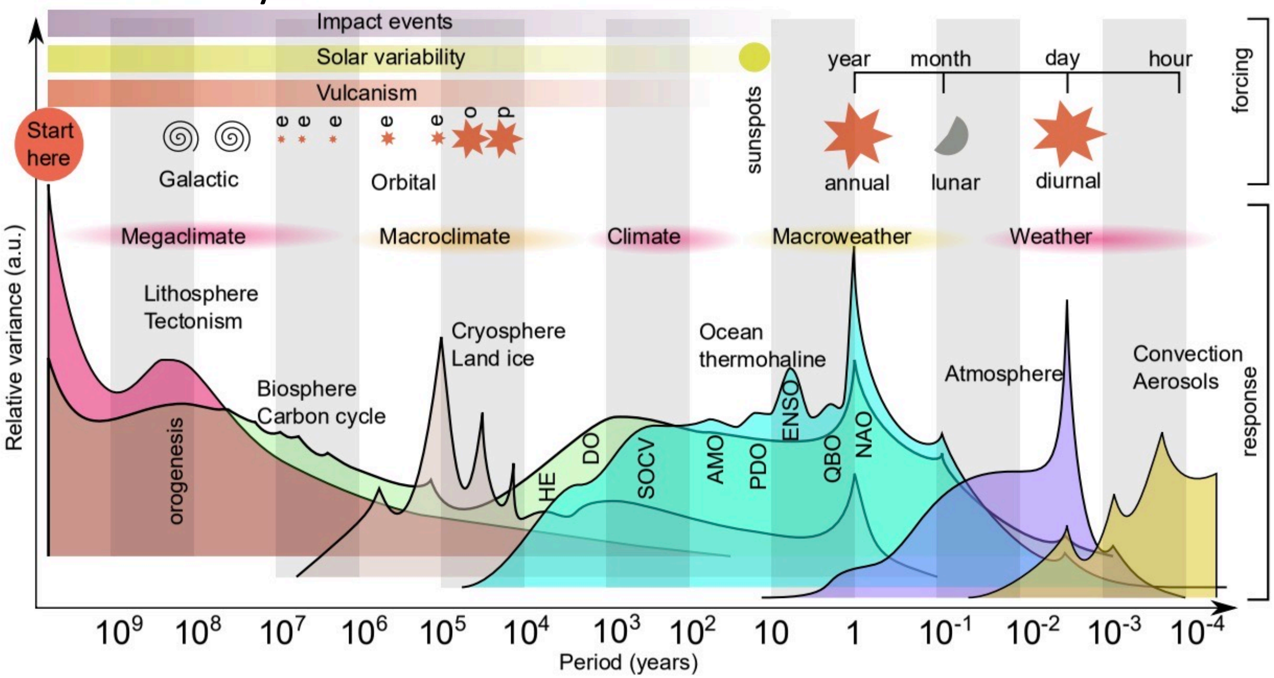
Climate as a multiscale system …

www.nature.com/scientificreports
SCIENTIFIC REPORTS
natureresearch
OPEN

Beyond Forcing Scenarios: Predicting Climate Change through Response Operators in a Coupled General Circulation Model

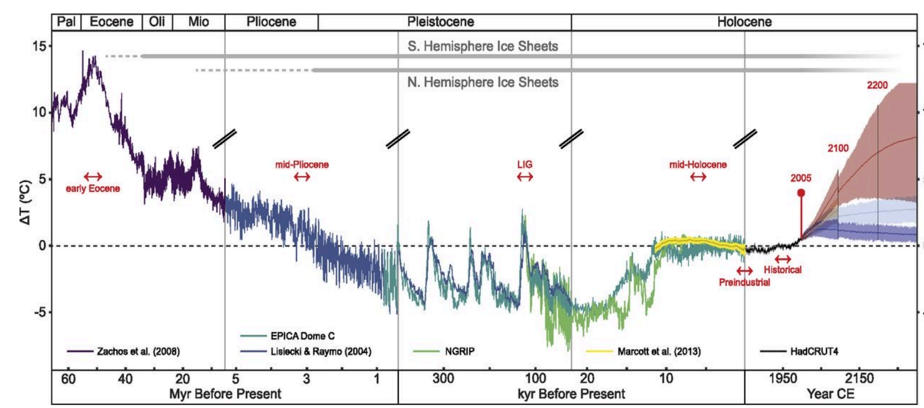
Valerio Lembo, Valerio Lucarini & Francesco Ragone

Von der Heydt et al. 2021



- Broad-band continuum spectrum, despite (quasi)-periodic forcing
- Different components dominant in different range of frequencies

- Internally-generated noise, multiple scales
- Climate variability and climate change



REVIEWS OF MODERN PHYSICS, VOLUME 92, JULY–SEPTEMBER 2020

The physics of climate variability and climate change

Climate as a metastable system

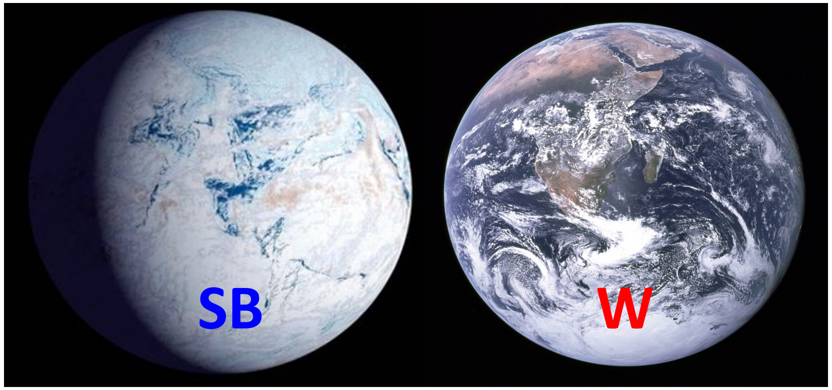
PROCEEDINGS A

royalsocietypublishing.org/journal/rspa

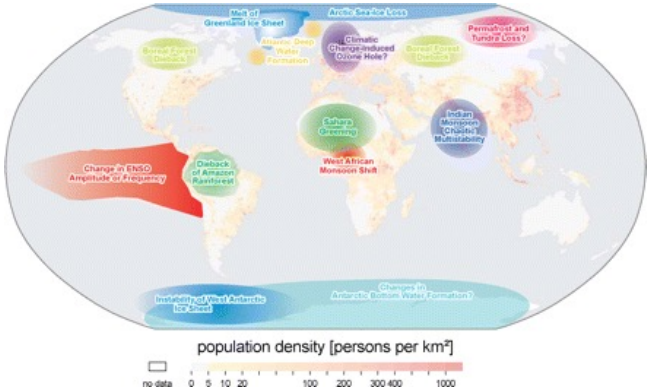
Research

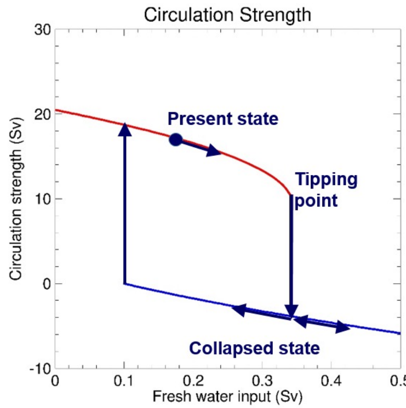
Dynamical landscape and multistability of a climate model

Georgios Margazoglou[1,2], Tobias Grafke[3], Alessandro Laio[4] and Valerio Lucarini[1,2]



- Competing Warm and Snowball States (we came out of it about 600 Mya; then multicellular life)
- Tipping Points often associated with competing steady states
- Example: Atlantic Meridional Overturning Circulation
- Parameter.-, rate- and noise-induced transitions



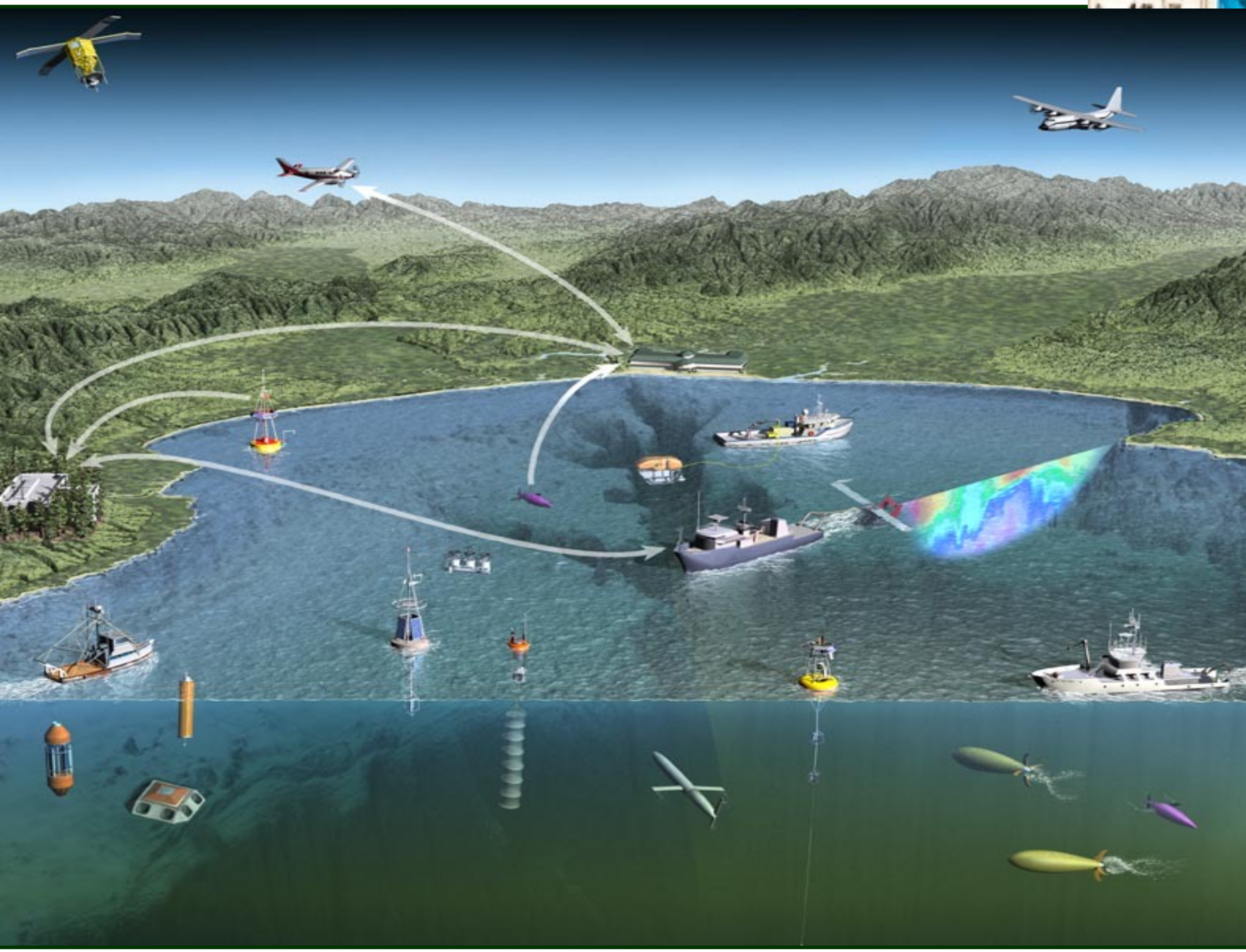


PHYSICAL REVIEW LETTERS 122, 158701 (2019)

Editors' Suggestion Featured in Physics

Transitions across Melancholia States in a Climate Model: Reconciling the Deterministic and Stochastic Points of View

Observing the Climate System

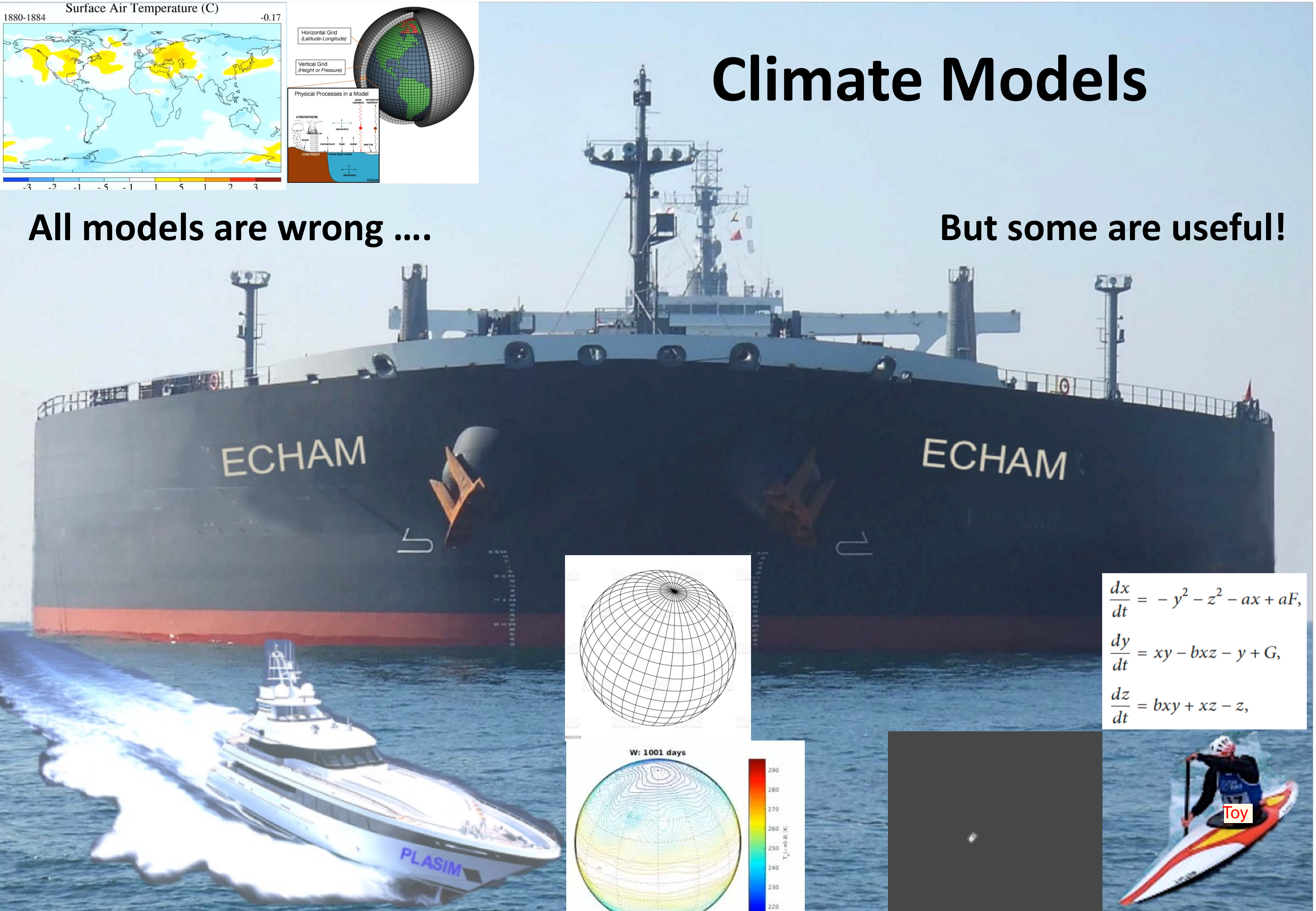


- Lack of synchronic and diachronic coherence in data collection
- Have to rely on proxy data for the past
- Uneven spatial coverage

Climate Models

All models are wrong ....

But some are useful!



$$\frac{dx}{dt} = -y^2 - z^2 - ax + aF,$$

$$\frac{dy}{dt} = xy - bxz - y + G,$$

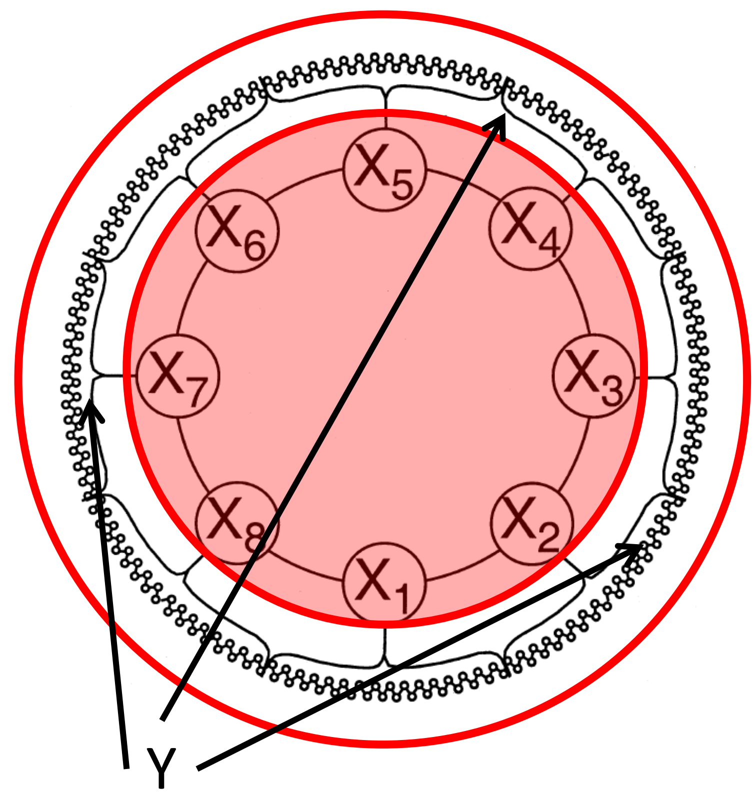
$$\frac{dz}{dt} = bxy + xz - z,$$

What is a parametrization?

- ✓ Full system: $\dot{z} = F(z) \quad z = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\dot{x} = F_x(x) + \varepsilon\Psi_x(x,y)$$
$$\dot{y} = F_y(y) + \varepsilon\Psi_y(x,y)$$

- ✓ We are interested only in the x variables
- ✓ "Optimal" reduction to ? $\quad \dot{x} = F_x(x) + \varepsilon\Gamma\{x\}$
- ✓ Usually: x are large scale, slow; y are small scale, fast variables.
- ✓ *coarse graining* -> constructing mesoscopic dynamics
- ✓ Reaction coordinates: optimal reduction (thermodynamics meets variational autoencoders)



Some comments

STOCHASTIC PARAMETERIZATION

Toward a New View of Weather and Climate Models

Judith Berner, Ulrich Achatz, Lauriane Batté, Lisa Bengtsson, Alvaro de la Cámara, Hannah M. Christensen, Matteo Colangeli, Danielle R. B. Coleman, Daan Crommelin, Stamen I. Dolaptchiev, Christian L. E. Franzke, Petra Friederichs, Peter Imkeller, Heikki Järvinen, Stephan Juricke, Vassili Kitsios, François Lott, Valerio Lucarini, Salil Mahajan, Timothy N. Palmer, Cécile Penland, Mirjana Sakradzija, Jin-Song von Storch, Antje Weisheimer, Michael Weniger, Paul D. Williams, and Jun-Ichi Yano

Stochastic parameterizations—empirically derived or based on rigorous mathematical and statistical concepts—have great potential to increase the predictive capability of next-generation weather and climate models.

- Parametrizations via empirical closures
  - Vertical transport of momentum

  at the boundary of the boundary layer as a function of ….
  - Evaporation over ocean given ….
- These formulas are deterministic functions of the larger scale variables:

$$\Gamma\{x\} = M(x)$$

- … or tables are used
- Recent push towards stochastic parametrizations (Palmer and Williams 2009, Franzke et al. 2015; Berner et al. 2017)
- Infinite spectral gap (homogeneization theory), parametrization is deterministic plus white noise (Pavliotis and Stewart 2008)

$$\Gamma\{x\} = M(x) + \sigma(t)$$

Mori-Zwanzig Projection

- Full system:

$$\dot{x} = F_x(x) + \varepsilon \Psi_x(x,y)$$
$$\dot{y} = F_y(y) + \varepsilon \Psi_y(x,y)$$

- Explicit expression for the three terms a (deterministic), b (stochastic), c (memory)- 2nd order expansion

a b c

$$\frac{dX}{dt} = F_X(X) + \epsilon D(X) + \epsilon S\{X\} + \epsilon^2 M\{X\}$$

Deterministic ✓ a

Stochastic ✓ b

Memory ✓ c

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Disentangling multi-level systems: averaging, correlations and memory

Jeroen Wouters and Valerio Lucarini
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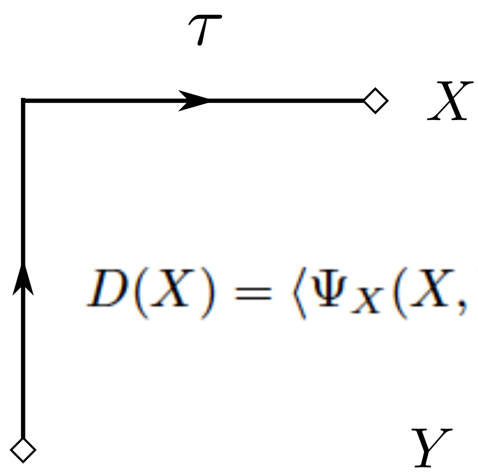
J Stat Phys (2013) 151:850–860
DOI 10.1007/s10955-013-0726-8

Multi-level Dynamical Systems: Connecting the Ruelle Response Theory and the Mori-Zwanzig Approach

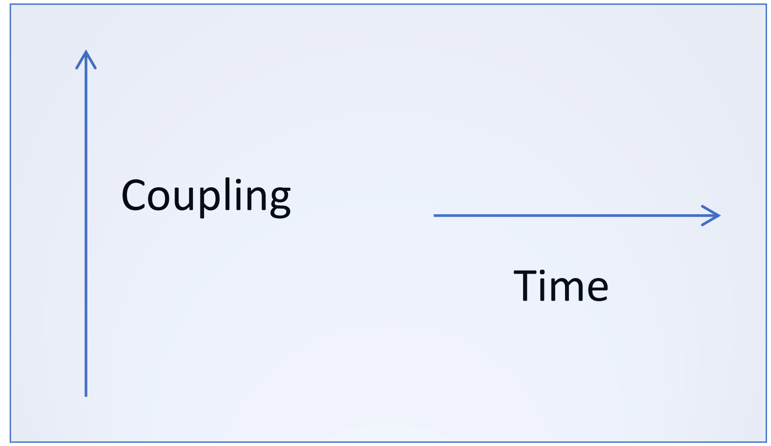
Jeroen Wouters · Valerio Lucarini

Diagrams: 1st & 2nd order "fastons"

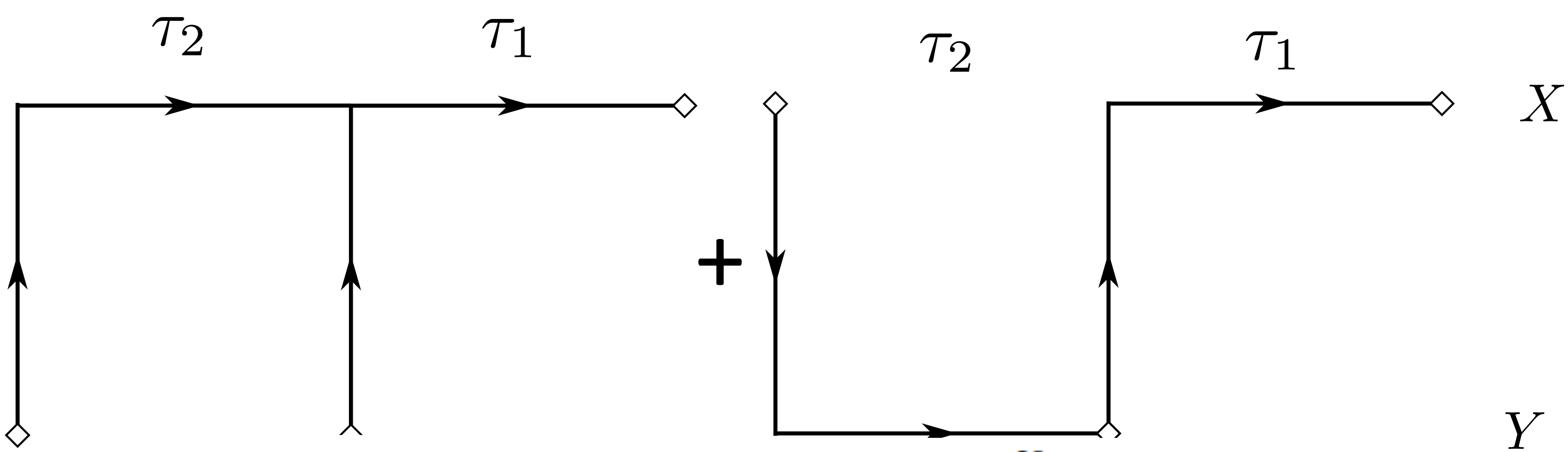
- 1



$$D(X) = \langle \Psi_X(X,Y) \rangle_{\rho_{0,Y}}$$



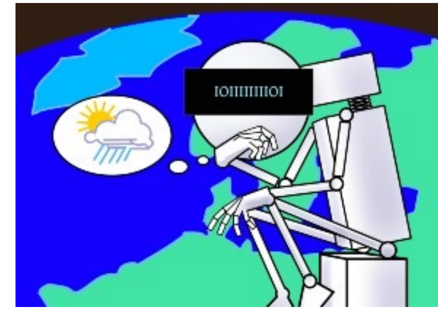
- 2

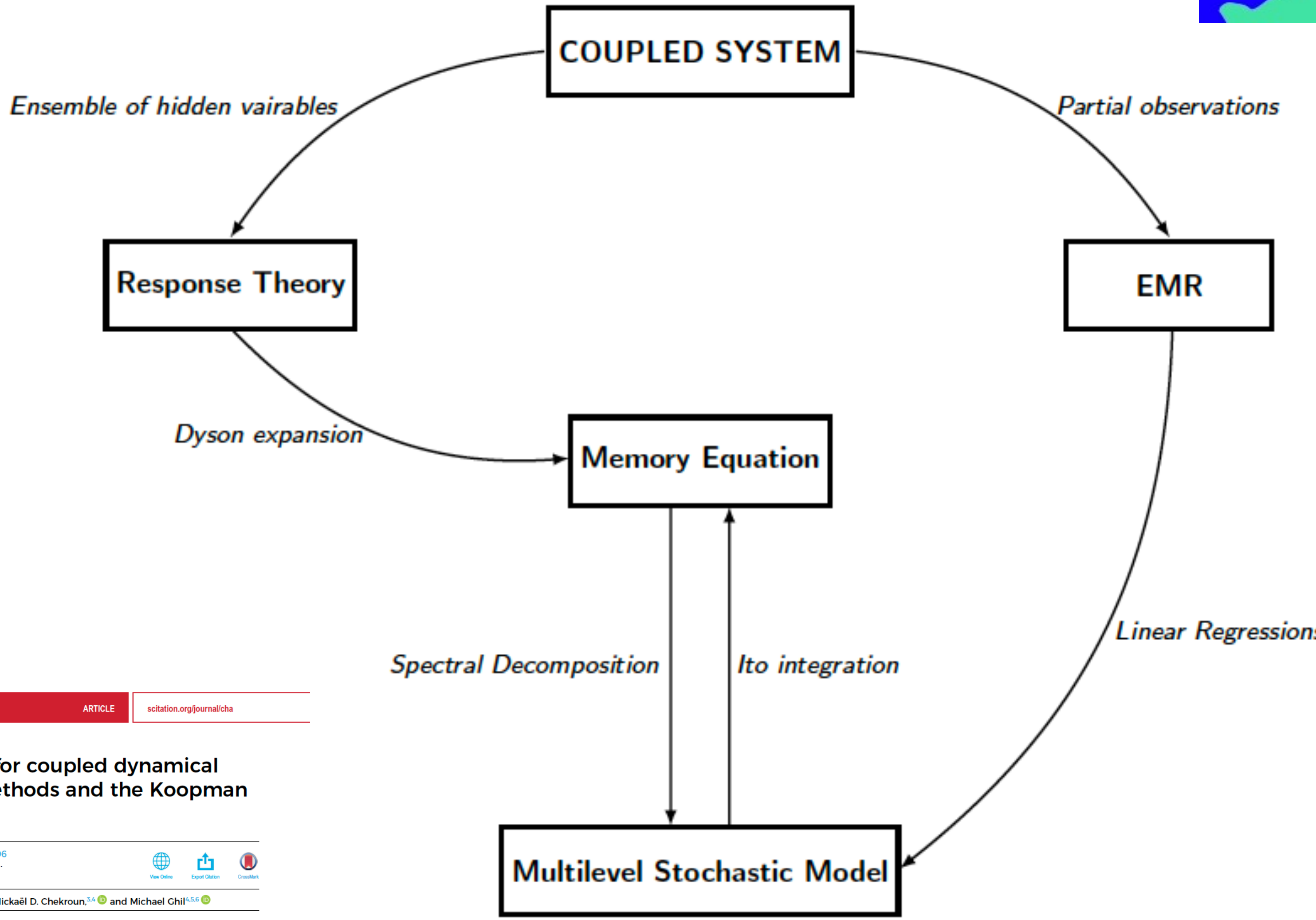


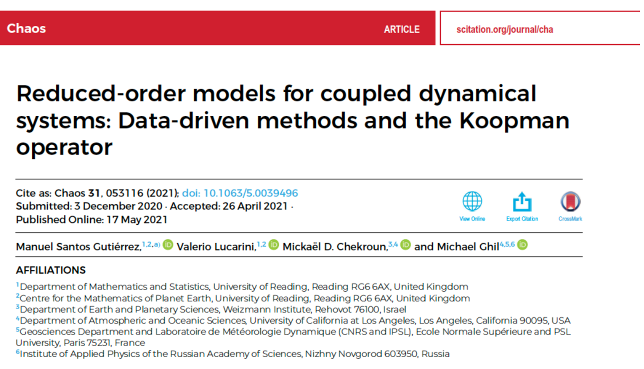
$$S\{X\} = \Psi'_{X,1}(X)\sigma(t)$$

$$M\{X\} = \int_0^\infty h(\tau, X(t-\tau_1))d\tau$$

Top-Down vs Bottom Up






Chaos ARTICLE scitation.org/journal/cha

Reduced-order models for coupled dynamical systems: Data-driven methods and the Koopman operator

Cite as: Chaos 31, 053116 (2021); doi: 10.1063/5.0039496
Submitted: 3 December 2020 · Accepted: 26 April 2021 ·
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View Online Export Citation CrossMark

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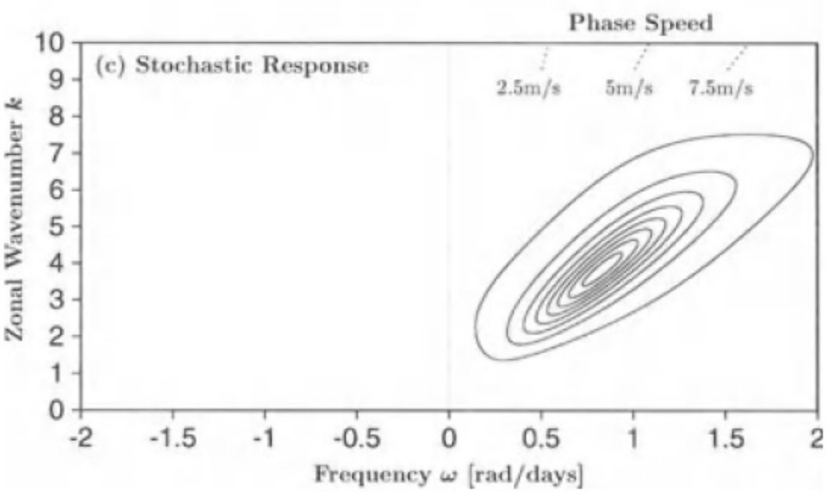
[a)]Author to whom correspondence should be addressed: m.santos@pgr.reading.ac.uk

Theories of Climate: Linking Variability and Response

Hasselmann programme (1976)

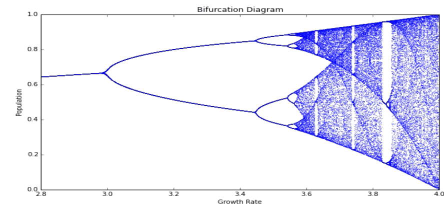
$$\lim_{\varepsilon \to 0} \begin{aligned} \dot{x}_t &= f(x_t, y_t), \quad x_0 = x \in \mathbb{R}^d \\ \dot{y}_t &= \frac{1}{\varepsilon} g(x_t, y_t), \quad y_0 = y \in \mathbb{R}^m \end{aligned}$$

Stochastic:
slow variables
integrate noise

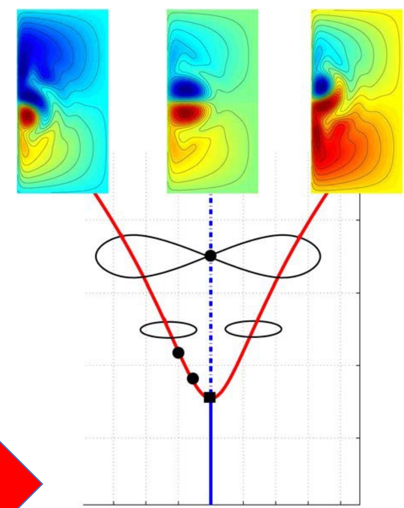


Deterministic Chaos (after Lorenz)

$$\dot{X} = f(X; \mu)$$



Deterministic:
Bifurcations & Route to Chaos



Regimes of Climate Response

A. Smooth response

Response Theory (Ruelle 1998, 2009; Majda and Hairer 2010)

Defining the sensitivity of the climate to perturbations; constructing the time-dependent measure of the pullback attractor.

B. High sensitivity

Ruelle-Pollicott Resonances (Ruelle 1986; Pollicott 1985)

Radius of expansion controlled by spectral gap; rough dependence of statistics on parameters when gap shrinks (Chekroun et al. 2014)

C. Critical Transitions

Freidlin & Wentzell (1984) meet Grebogi, Ott & York (1983)

Noise-induced transitions; Boundary crisis of the high-dimensional attractor (Ott 2002)

General Linear Response Formulas - Measures

$$dx = F(x)dt + \Sigma(x)dW + \varepsilon G(x)g(t)dt + \gamma\Psi(x)p(t)dW, \qquad x \in \mathbb{R}^D$$

Ruelle '98
Majda & Hairer 2010

Fokker-Planck Equation

$$\mathcal{L}_0\rho = -\nabla\cdot(F\rho) + \frac{1}{2}\nabla^2:(\Sigma\Sigma^T\rho)$$

$$\partial_t\rho = \mathcal{L}_0\rho + \varepsilon g(t)\mathcal{L}_1\rho + \gamma p(t)\mathcal{L}'_1\rho + h.o.t.$$

$$\mathcal{L}_1\rho = -\nabla\cdot(G\rho)$$

$$\mathcal{L}'_1\rho = \frac{1}{2}\nabla^2:\left((\Sigma\Psi^T + \Psi\Sigma^T)\rho\right)$$

Perturbative Expansion

$$\rho = \rho_0 + \varepsilon\rho_1(t) + \gamma\rho'_1(t) + h.o.t. \qquad \mathcal{L}_0\rho_0 = 0$$

$$\rho_1(t) = \int_0^t ds\, e^{\mathcal{L}_0(t-s)}g(s)\mathcal{L}_1\rho_0 \qquad \rho'_1(t) = \int_0^t ds\, e^{\mathcal{L}_0(t-s)}p(s)\mathcal{L}'_1\rho_0$$

Santos Gutierrez and L., arXiv:2205.08896 (2022)

General Linear Response Formulas - Observables

$$dx(t) = F(x)dt + \Sigma(x)dW + \varepsilon G(x)g(t)dt + \gamma\Psi(x)p(t)dW$$

$$\langle\Phi\rangle = \langle\Phi\rangle_0 + \varepsilon\langle\Phi\rangle_1(t) + \gamma\langle\Phi\rangle'_1(t) + h.o.t.$$

Fluctuation-Dissipation Theorem

$$\langle\Phi\rangle_1(t) = [\mathcal{G}^1 * g](t) \qquad \mathcal{G}^1(s) = \Theta(s)\left\langle\mathcal{L}_1^T e^{s\mathcal{L}_0^T}\Phi\right\rangle_0$$

$$\langle\Phi\rangle'_1(t) = [\mathcal{G}'^1 * p](t) \qquad \mathcal{G}'^1(s) = \Theta(s)\left\langle\mathcal{L}'^T_1 e^{s\mathcal{L}_0^T}\Phi\right\rangle_0$$

Free-Forced Fuctuations Dictionary works only for smooth invariant measures

Nonequilibrium deterministic systems require a different formulation: Climatic Surprises

Santos Gutierrez and L., arXiv:2205.08896 (2022)

Response Theory Meets Koopmanism

Fluctuation-Dissipation Theorem

$$\langle\Phi\rangle_1(t) = [\mathcal{G}^1 * g](t) \quad \mathcal{G}^1(s) = \Theta(s)\left\langle \mathcal{L}_1^T e^{s\mathcal{L}_0^T}\Phi \right\rangle_0 = -\Theta(s)\left\langle \frac{\mathcal{L}_1\rho_0}{\rho_0} e^{s\mathcal{L}_0^T}\Phi \right\rangle_0$$

Point Spectrum
(Ruelle-Pollicott Resonances)

Koopman operator
Spectral decomposition

$$\exp(\mathcal{L}_0^T\tau)\,\Phi = \Phi(\tau) \qquad \exp(\mathcal{L}_0^T\tau) = \sum_{j=1}^{M}\exp(\lambda_j\tau)\Pi_j + \mathcal{R}(\tau)$$

Essential
Spectrum

$$\mathcal{G}^1(s) = \Theta(s)\sum_{j=2}^{M}\exp(\lambda_j\tau)\langle\mathcal{L}_1^T(\Pi_j\Phi)\rangle_0 = \Theta(s)\sum_{j=2}^{M}\exp(\lambda_j\tau)\alpha_j$$

Neglecting
Degeneracies

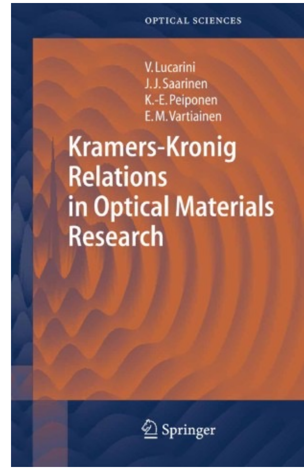
$$\chi^1(\omega) = \mathcal{F}[\mathcal{G}^1(s)] = \sum_{j=2}^{M}\frac{\alpha_j}{i\omega-\lambda_j}$$

Depends on the observable

Does not depend on the observable

Santos Gutierrez and L., arXiv:2205.08896 (2022)

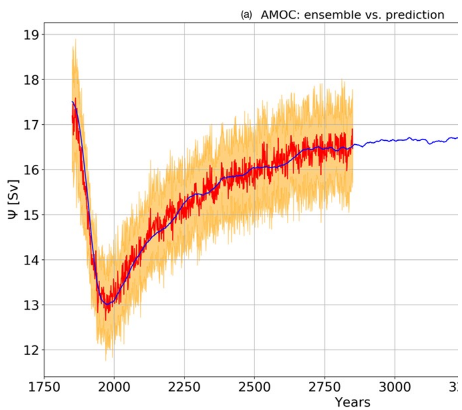
Equivalence Principle
Equilibrium ↔ Nonequilibrium

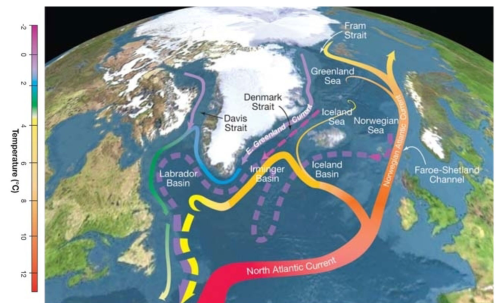


$$\chi^1(\omega) = \mathcal{F}[\mathcal{G}^1(s)] = \sum_{j=2}^{M} \frac{\alpha_j}{i\omega - \lambda_j}$$

- The problem is reduced to a (possibly infinite) set of harmonic oscillators
- … like a collection of Drude-Lorentz oscillator models
- $\alpha_j \backsim$ oscillator strength for the transition j in quantum mechanics (Heisenberg '26)

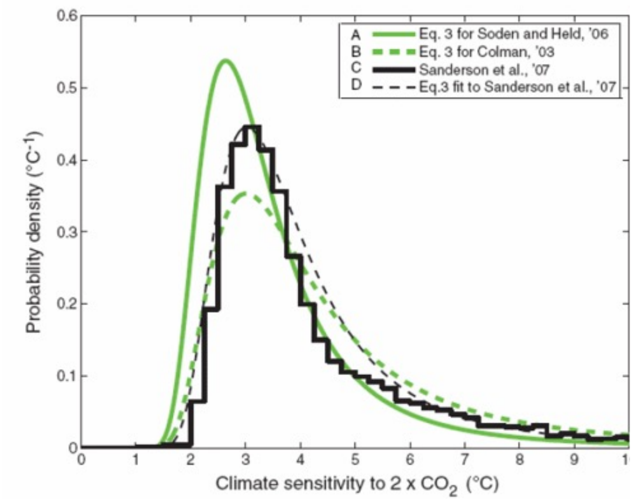
- All the classical results (Kramers-Kronig relations, sum rules, etc.) can be used to treat and interpret data.
- ….

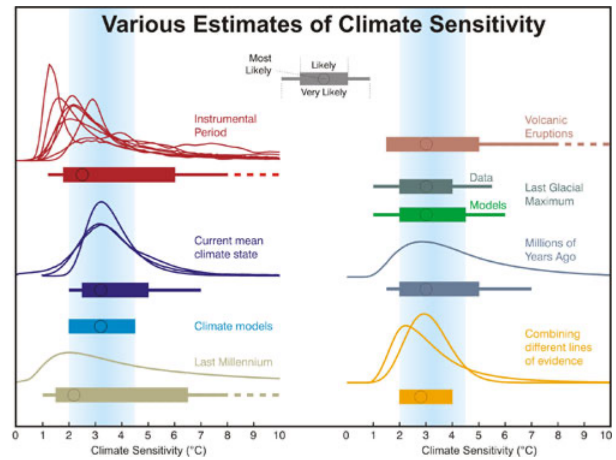




L. 2018; Tantet et al. 2018; Santos Gutierrez and L. 2022

Equilibrium Climate Sensitivity

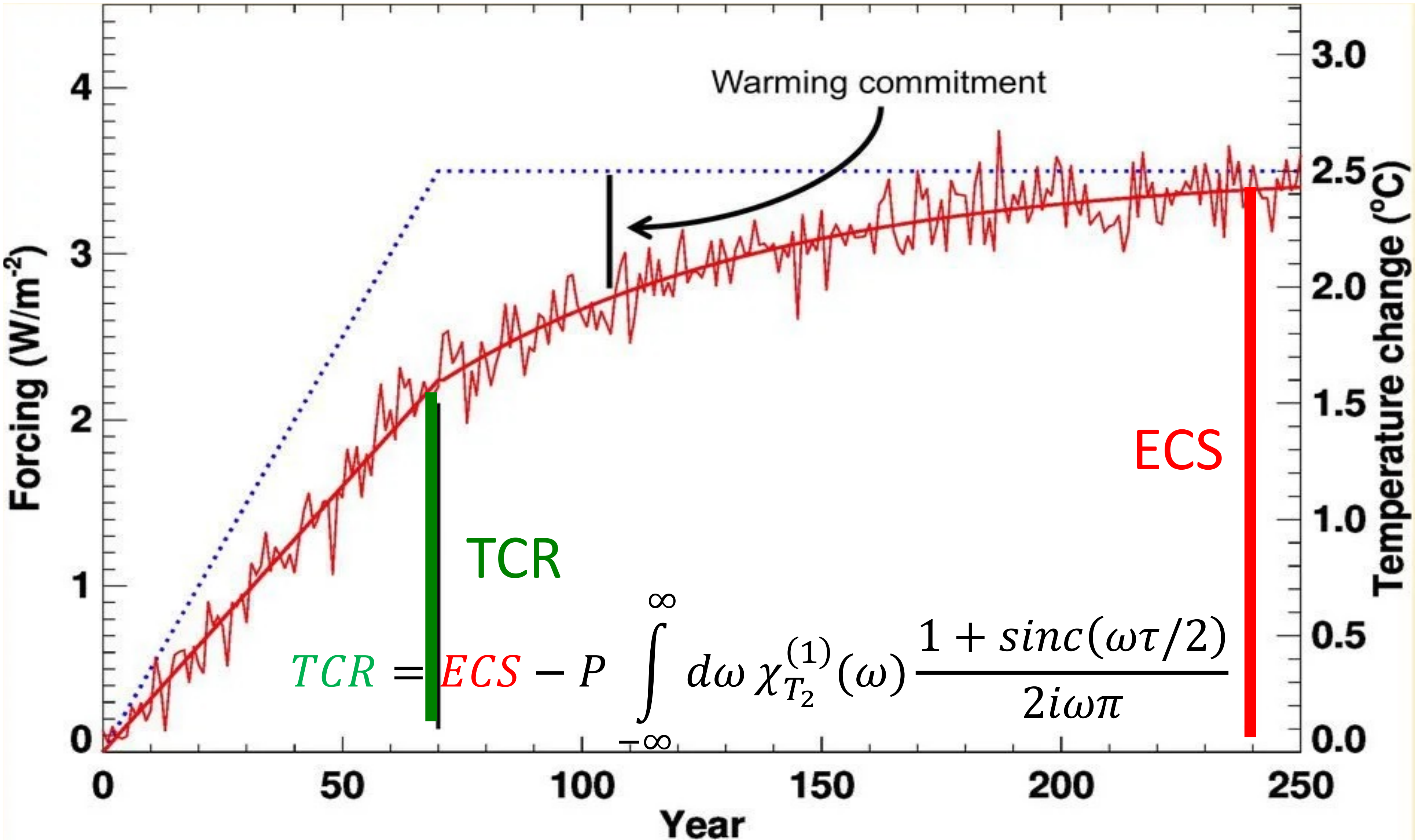




$$ECS = \Re\left\{\chi_{T_S}^{(1)}(0)\right\} = \frac{2}{\pi}\int d\omega\,\mathrm{Re}[\langle T_s\rangle^{(1)}(\omega)]$$

- Long Term Surface Temperature response to CO_2 doubling

Equilibrium Climate Sensitivity vs Transient Climate Response



$$TCR = ECS - P\int_{-\infty}^{\infty} d\omega\, \chi_{T_2}^{(1)}(\omega)\, \frac{1 + sinc(\omega\tau/2)}{2i\omega\pi}$$

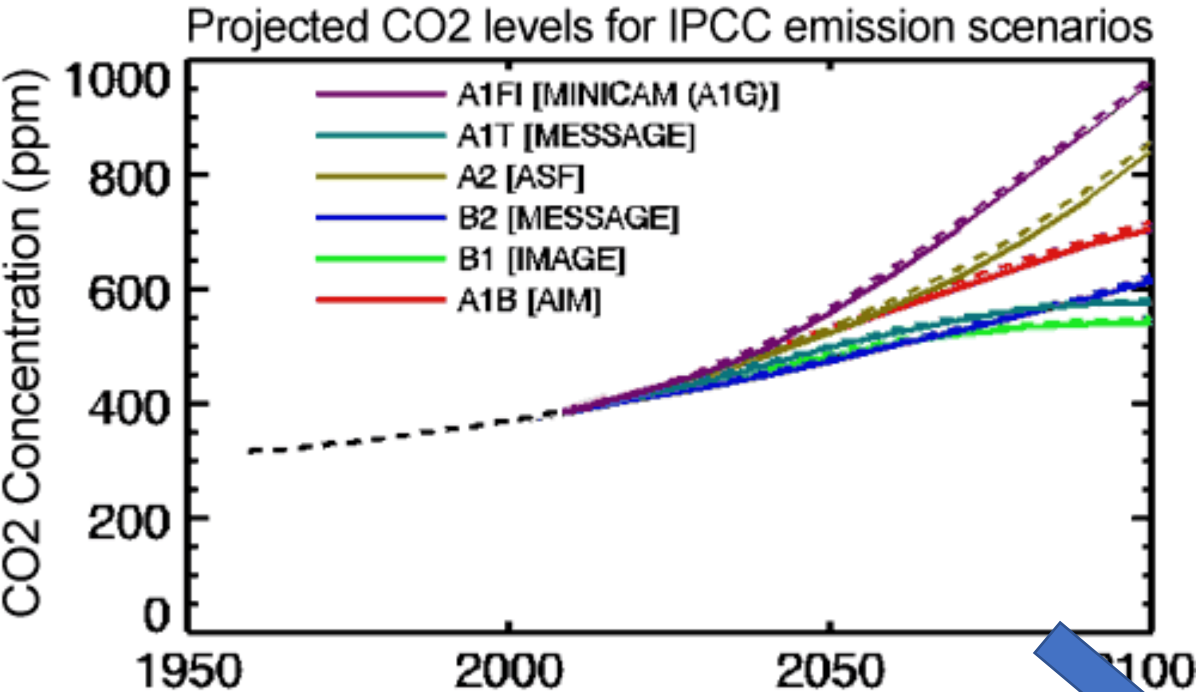
Computing the Green Function

- Observable: globally averaged T_S
- Forcing: increase of CO_2 concentration $f(t)$
- Linear response: $\langle T_S \rangle_f^{(1)}(t) = \int d\sigma G_{T_S}^{(1)}(\sigma) f(t-\sigma)$
- We perform ensemble experiments
  - Concentration ↑ at t=0 $f(t) = \varepsilon \Theta(t)$
- Fantastic, we estimate

$$\frac{d}{dt}\langle T_S \rangle_f^{(1)}(t) = \varepsilon G_{T_S}^{(1)}(t)$$

- ...and we predict: $\langle T_S \rangle_g^{(1)}(t) = \int d\sigma G_{T_S}^{(1)}(\sigma) g(t-\sigma)$

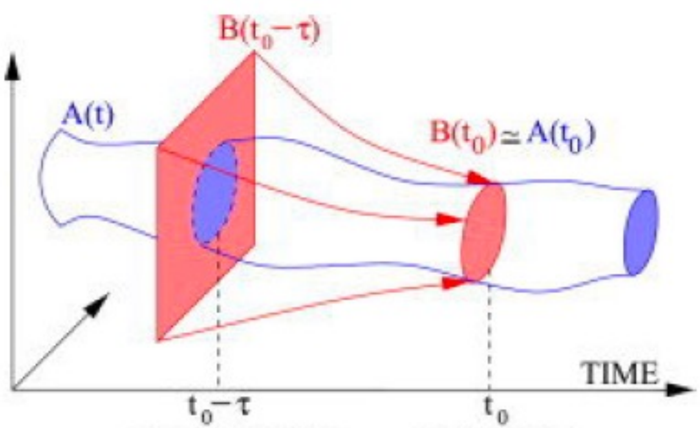
- <u>Note: we can use any test forcing pattern $f(t)$!</u>

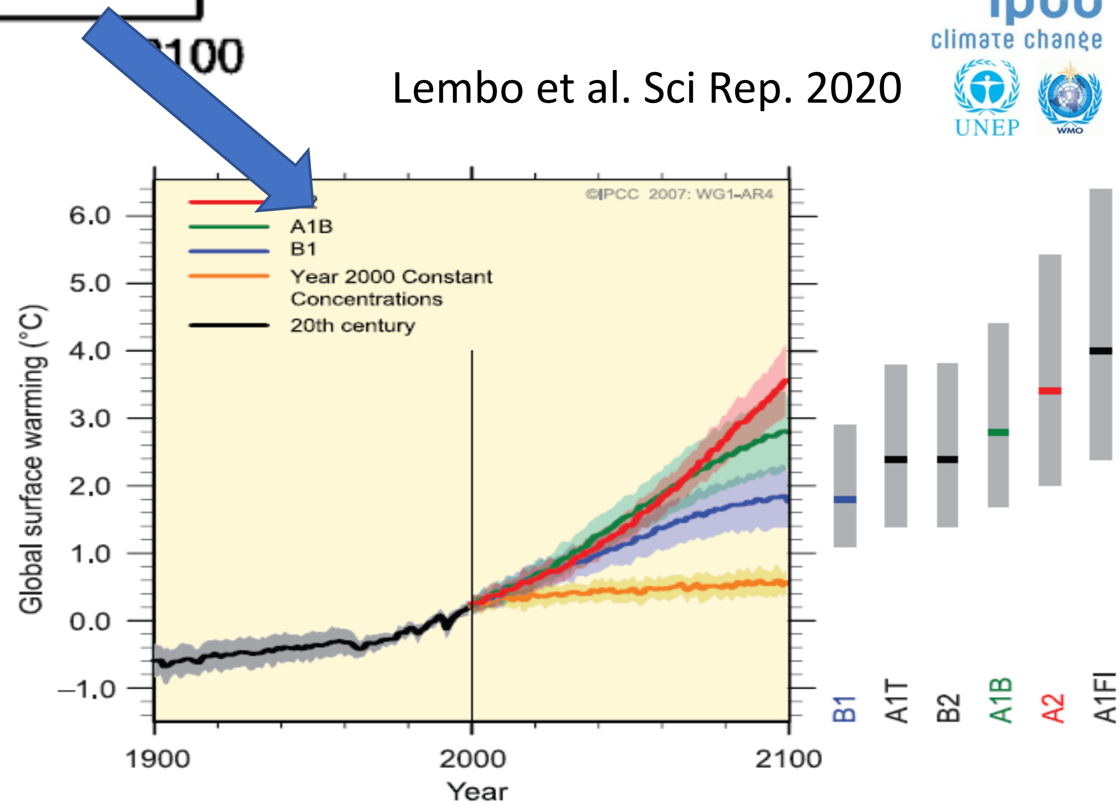


From forcing to response

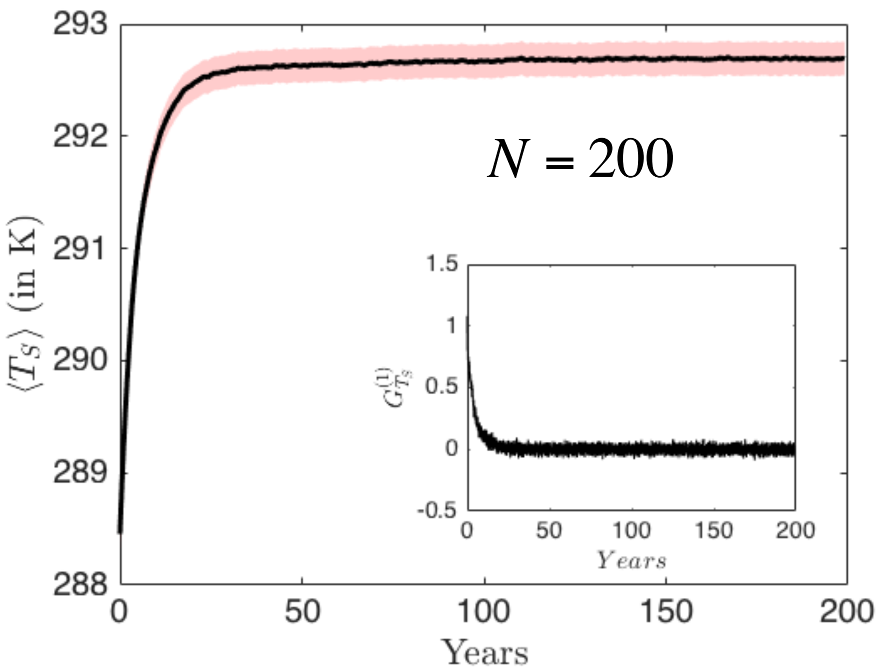
Lembo et al. Sci Rep. 2020

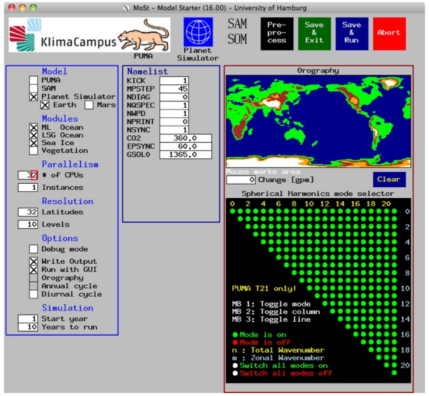
Pullback Attractor Flandoli, Tel, Ghil, …





Step 1: $[CO_2]$ Doubling 360 ppm ➔ 720 ppm

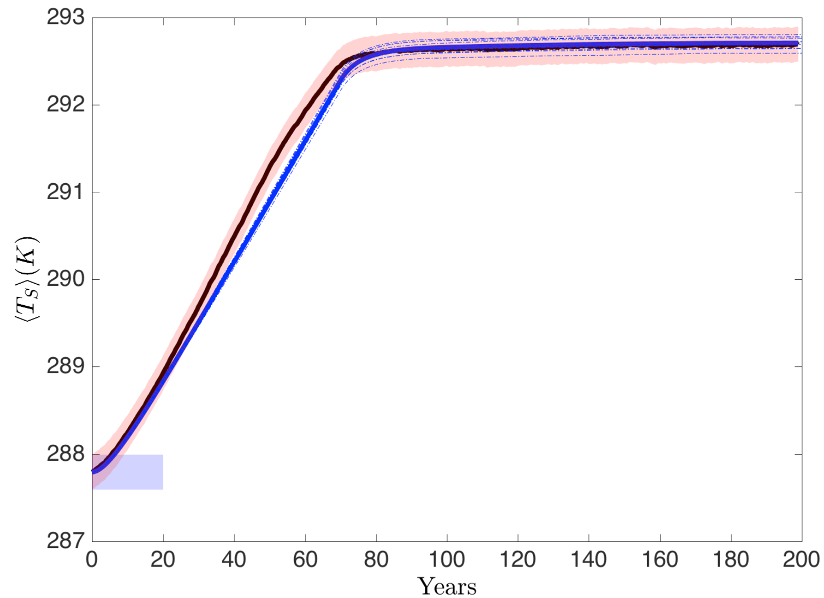




PLASIM
Intermediate
Complexity Model
$O(10^5)$ d.o.f.

Step 2: Climate Change Prediction - T_S

$[CO_2]$ 360 ppm → 720 ppm at 1% py
$[CO_2]$ is doubled after τ≈ 70 years
We keep $[CO_2]$ constant after that



$$\langle T_S \rangle^{(1)}_{g_\tau}(t) = \int d\sigma G^{(1)}_{T_S}(\sigma) g_\tau(t-\sigma)$$

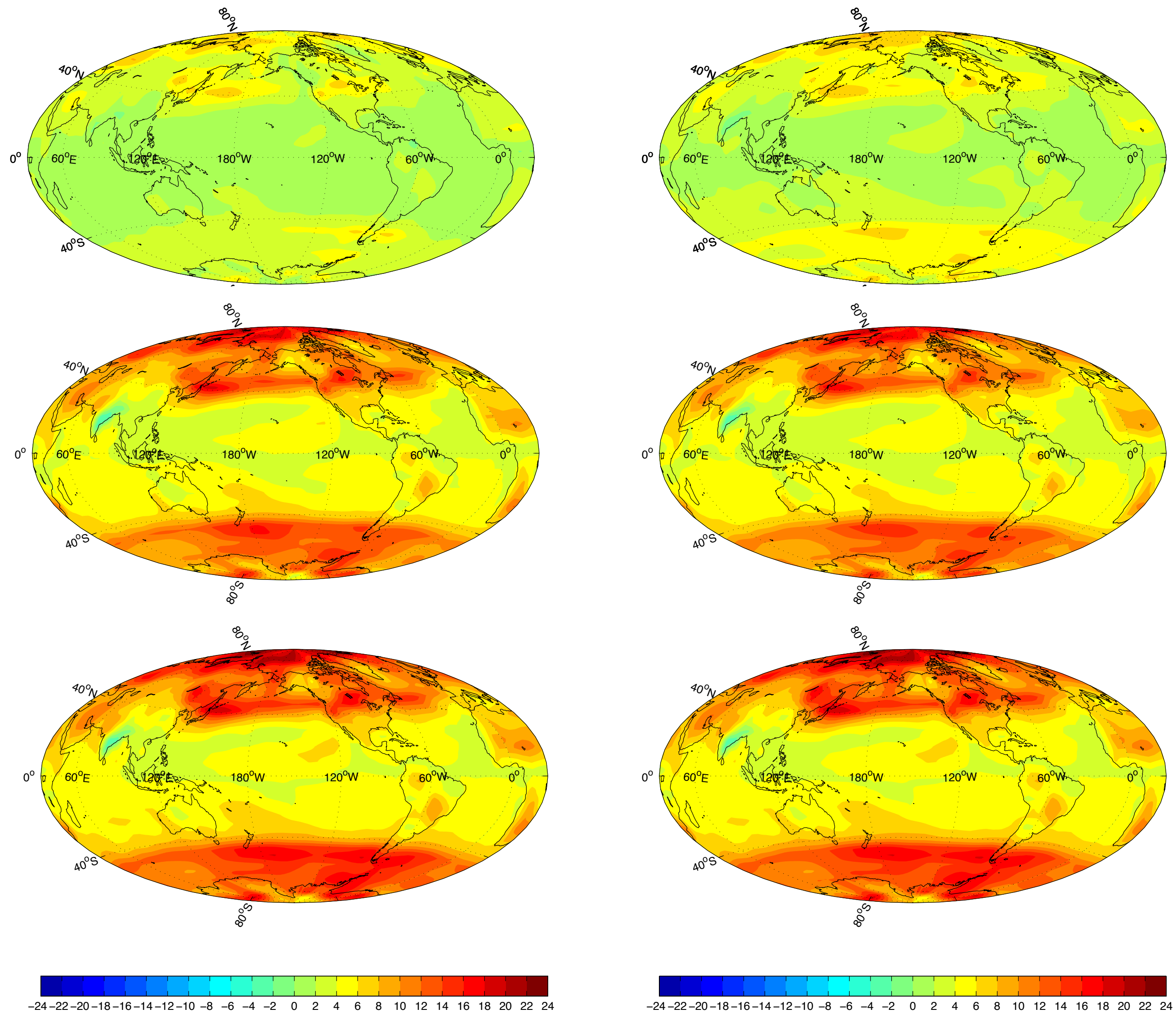
Not only global quantities!

Pattern T increase | **Forward run** | **Response Theory**

35 years

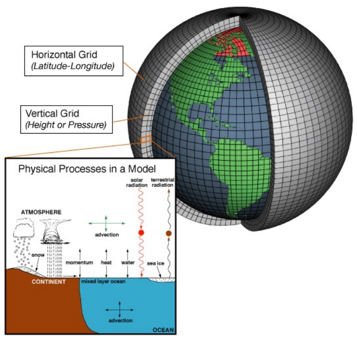
70 years

long term

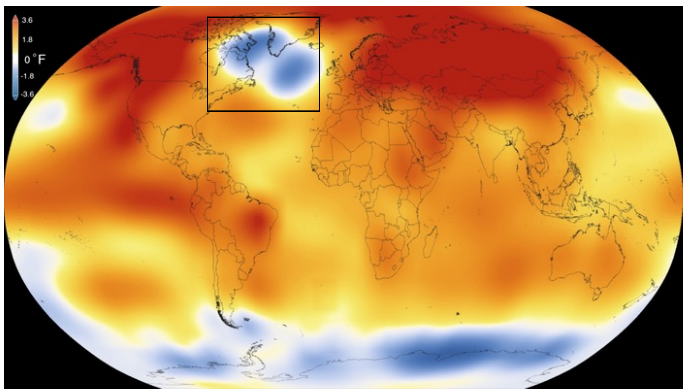


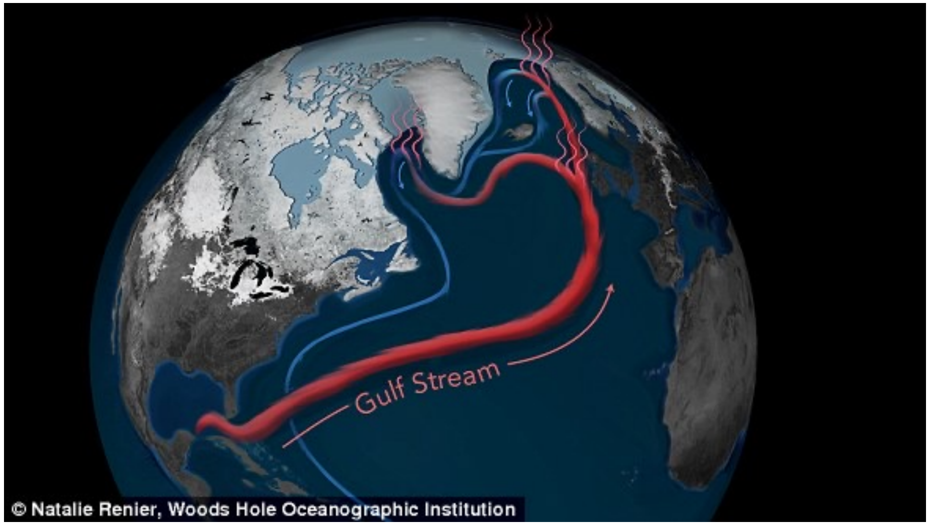
North Atlantic

Effect of AMOC on local climate

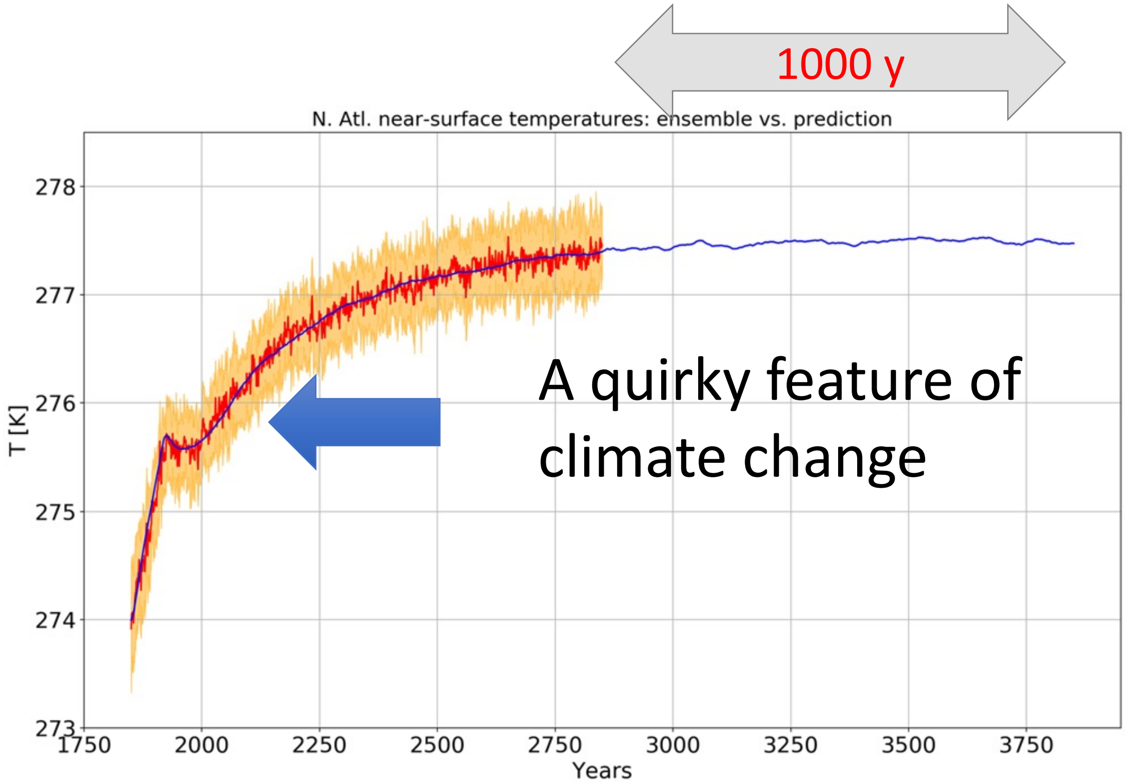


Temperature in the N. Atlantic (cold blob)



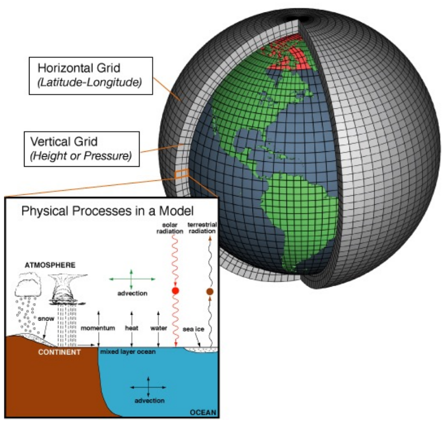


Lembo et al. Sci Rep. 2020



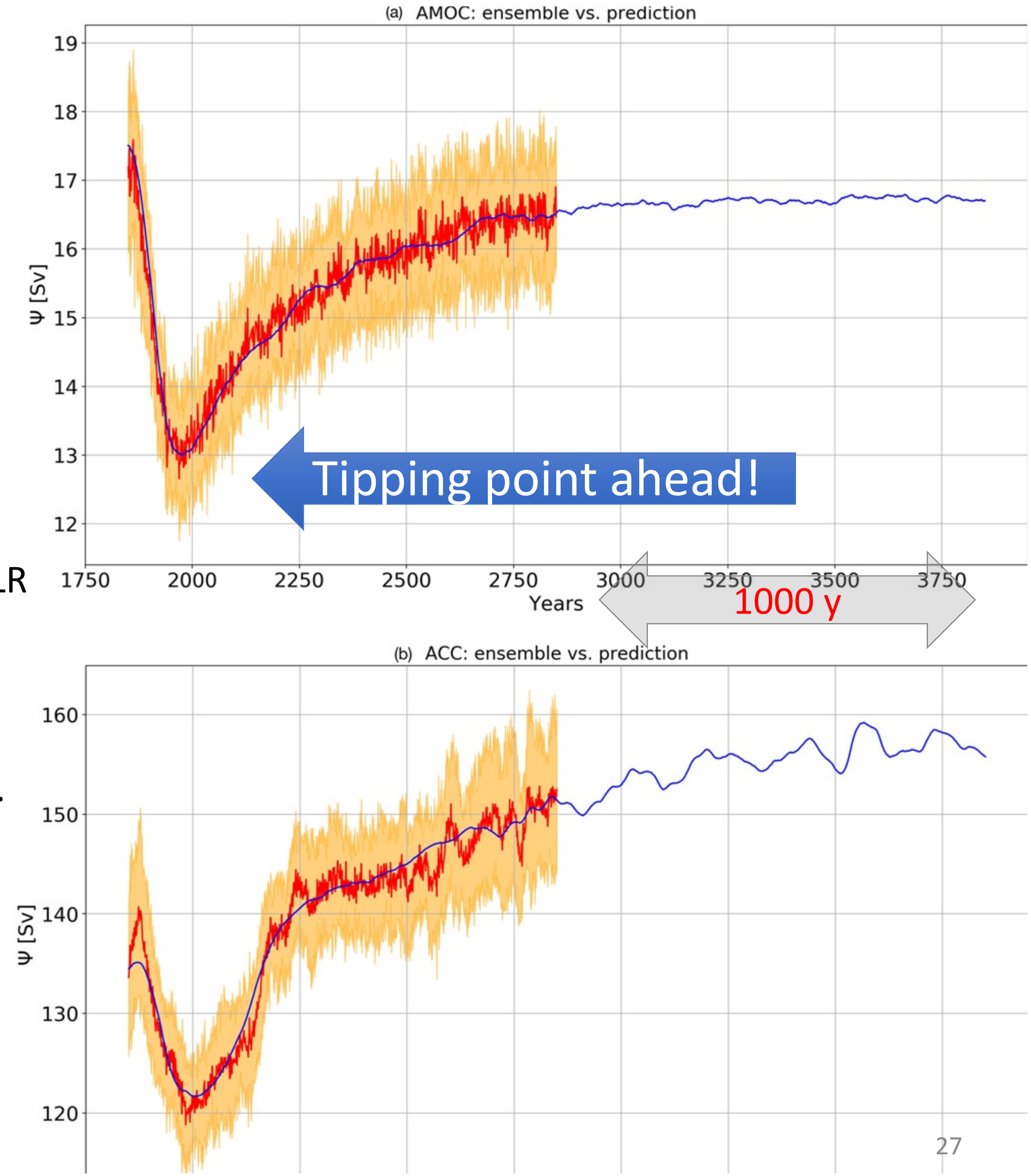
Ocean

Atlantic Meridional Overturning Circulation



MPI-ESM-LR

PLASIM
IPCC Class
O(10^7) d.o.f.



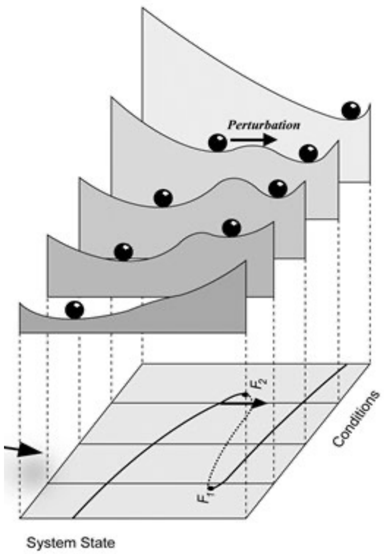
Antarctic Circumpolar Current

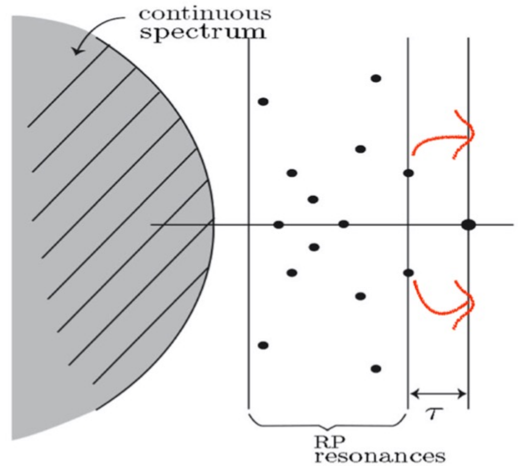
Lembo et al. Sci Rep. 2020

Emergence of Tipping Points

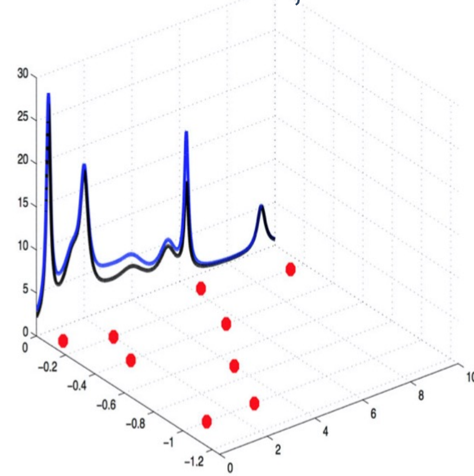
$$Susceptibility:\ \chi^{(1)}_{G,\Phi}(\omega) = \sum_{j=2}^{M} \frac{\alpha_j}{i\omega - \lambda_j} + R_{G,\Phi}(\omega)$$

$$Cospectrum::\ P_{\Psi,\Phi}(\omega) = \sum_{j=2}^{M} \frac{\beta_j}{i\omega - \lambda_j} + R_{\Psi,\Phi}(\omega)$$





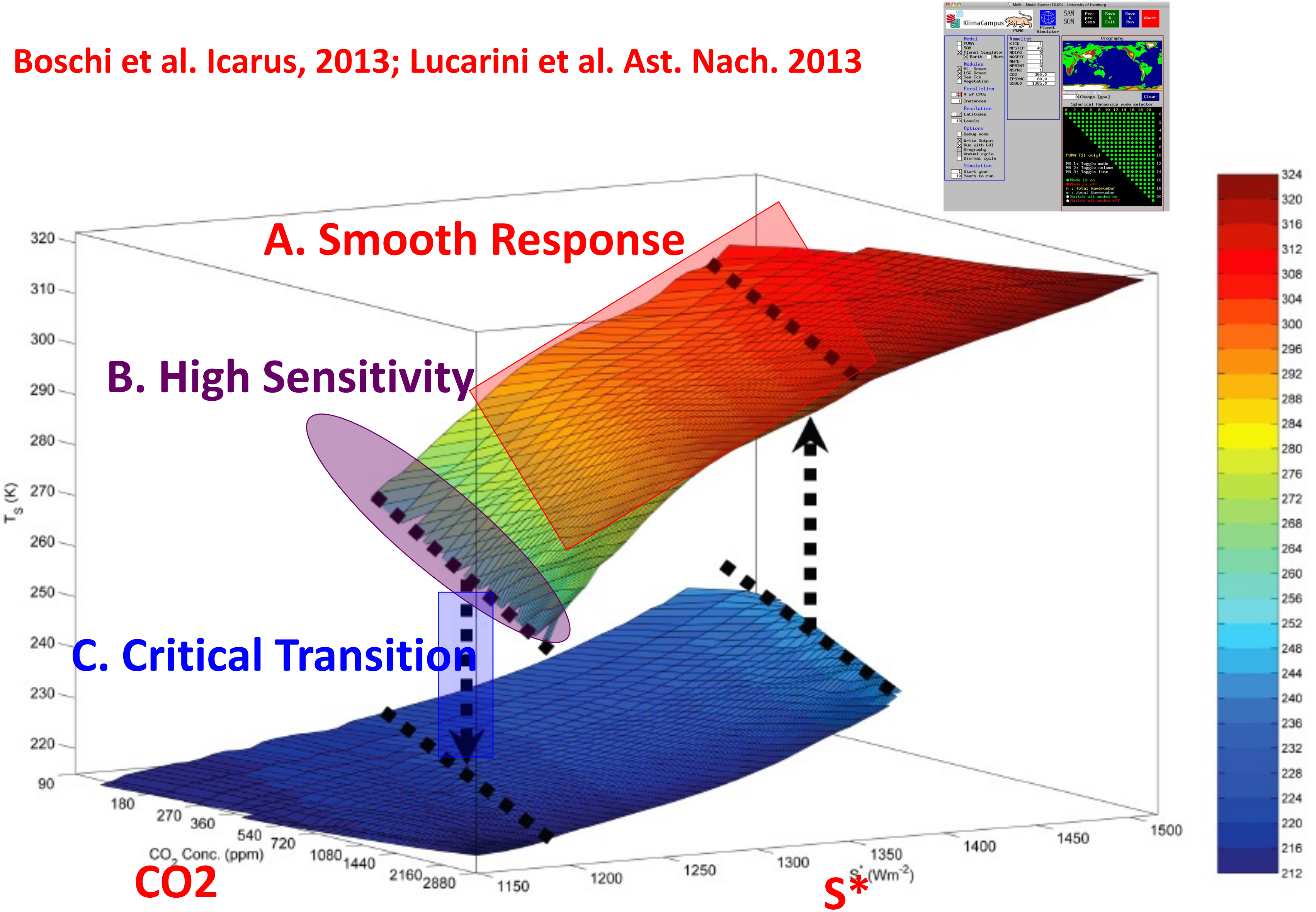
Tantet et al. 2018; Chekroun et al. 2020

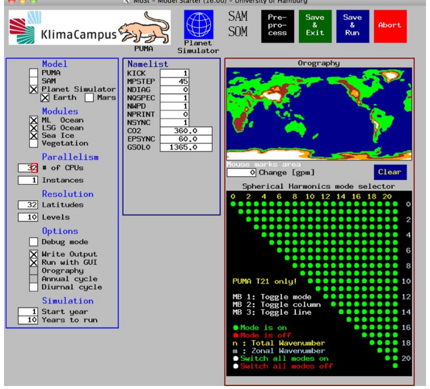


- Small spectral gap - $Re(\lambda_2) < 0 \rightarrow$ small radius of expansion
- Small spectral gap $\rightarrow$ amplified response
- Small spectral gap $\rightarrow$ slow decay correlations $\propto \exp[Re(\lambda_2)\, t]$

Climate Response: the General Picture

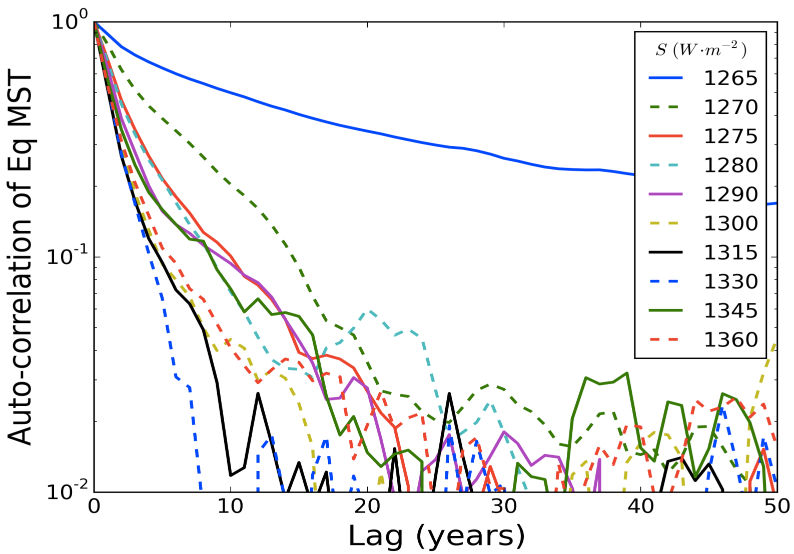
Boschi et al. Icarus, 2013; Lucarini et al. Ast. Nach. 2013

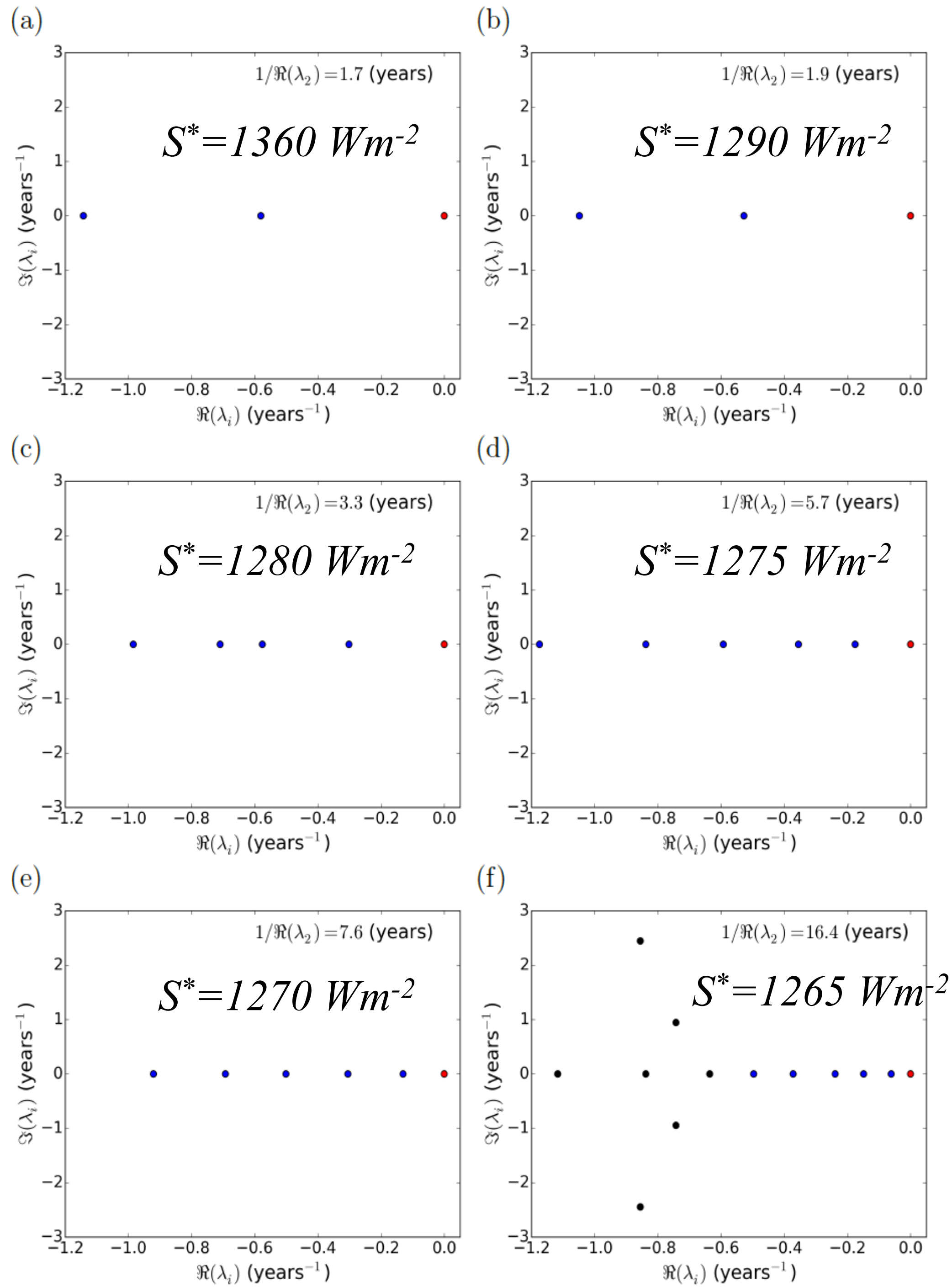




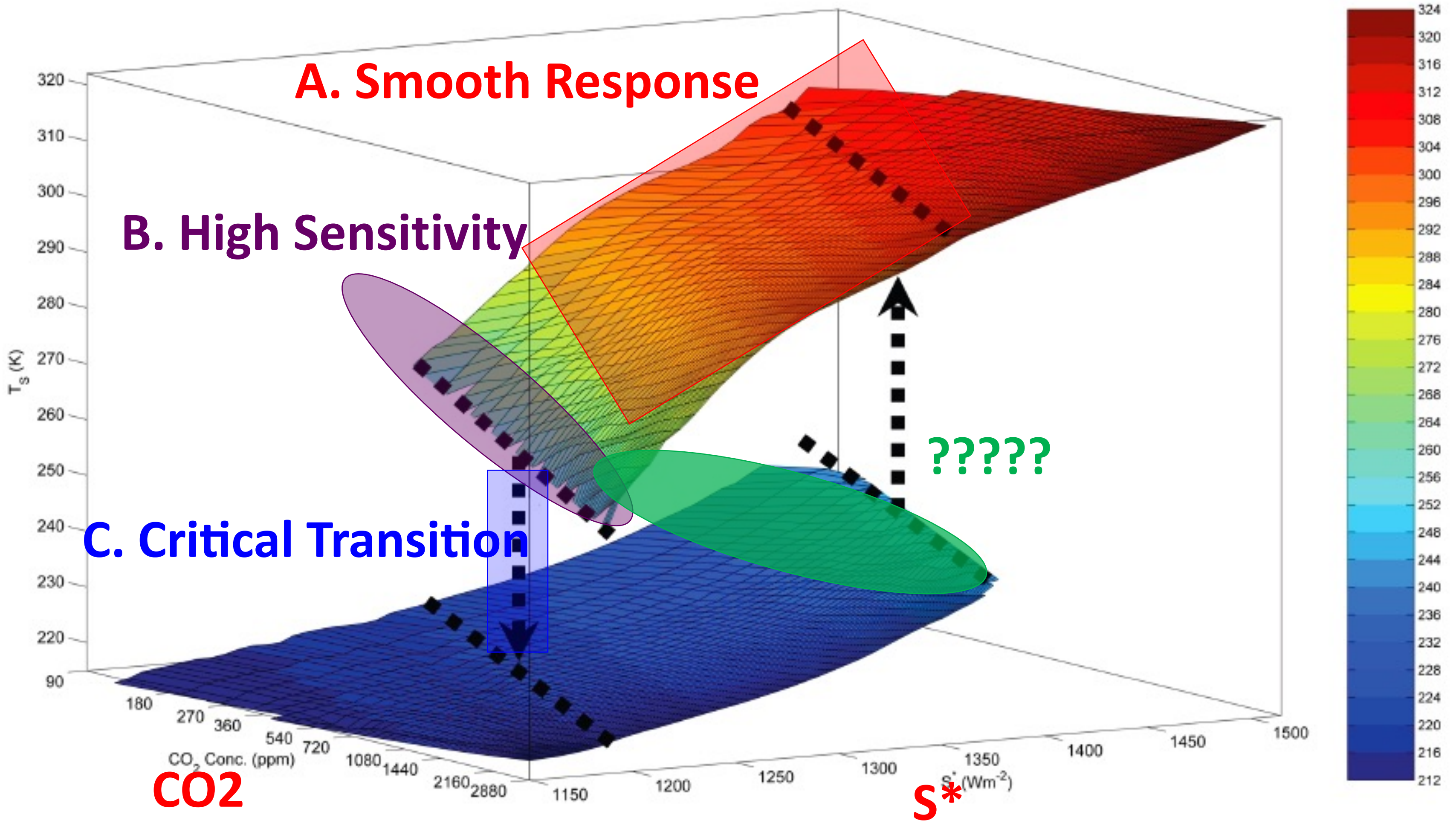
- As the estimate of Re(λ_2) gets closer to 0
- The decorrelation time goes to infinity …

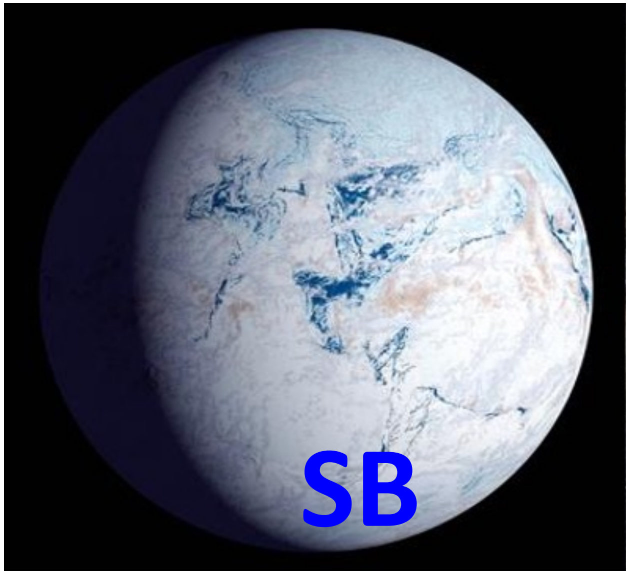
Tantet et al. 2018

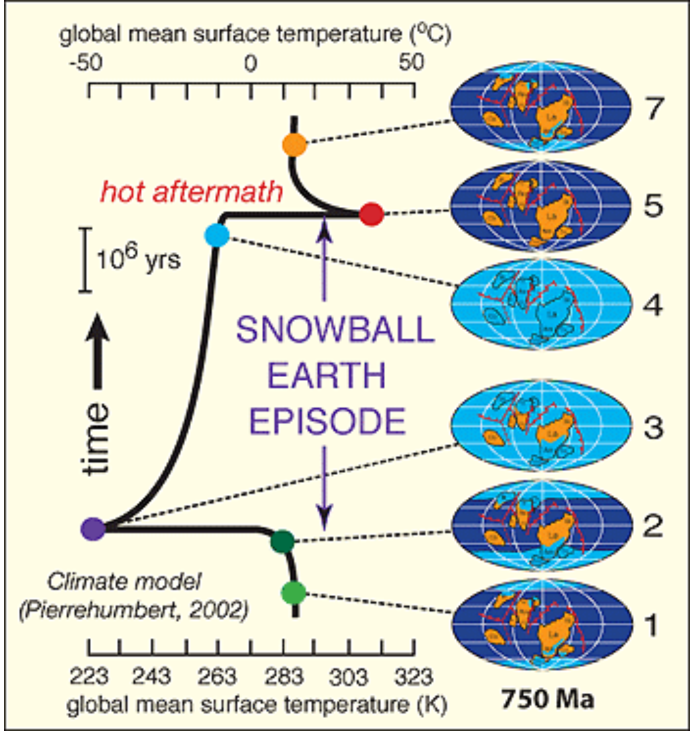




What can be found between the two climate states?

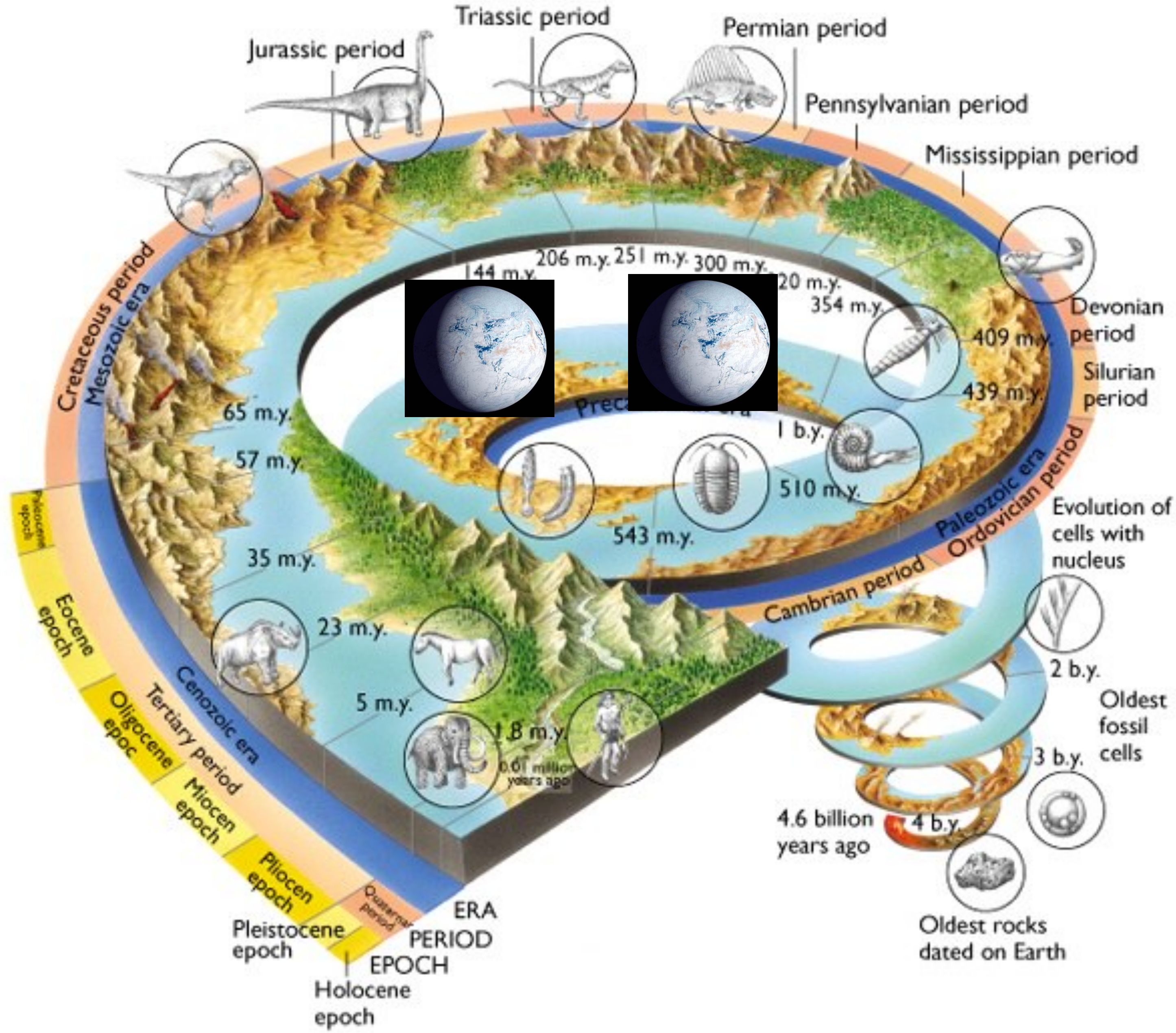






- The multistability of the Earth's climate was discovered when studying the possible effects of the nuclear winter
- Budyko, Sellers in the late '60 realized that a prolonged nuclear winter might lead to a global glaciation
- Ghil (1976): bifurcation and potential theory
- The community was very skeptical of this… but in early '90s paleo evidences emerged for SB state in the Neoproterozoic (650 mya)
  - Beware critical transitions!

Evolution

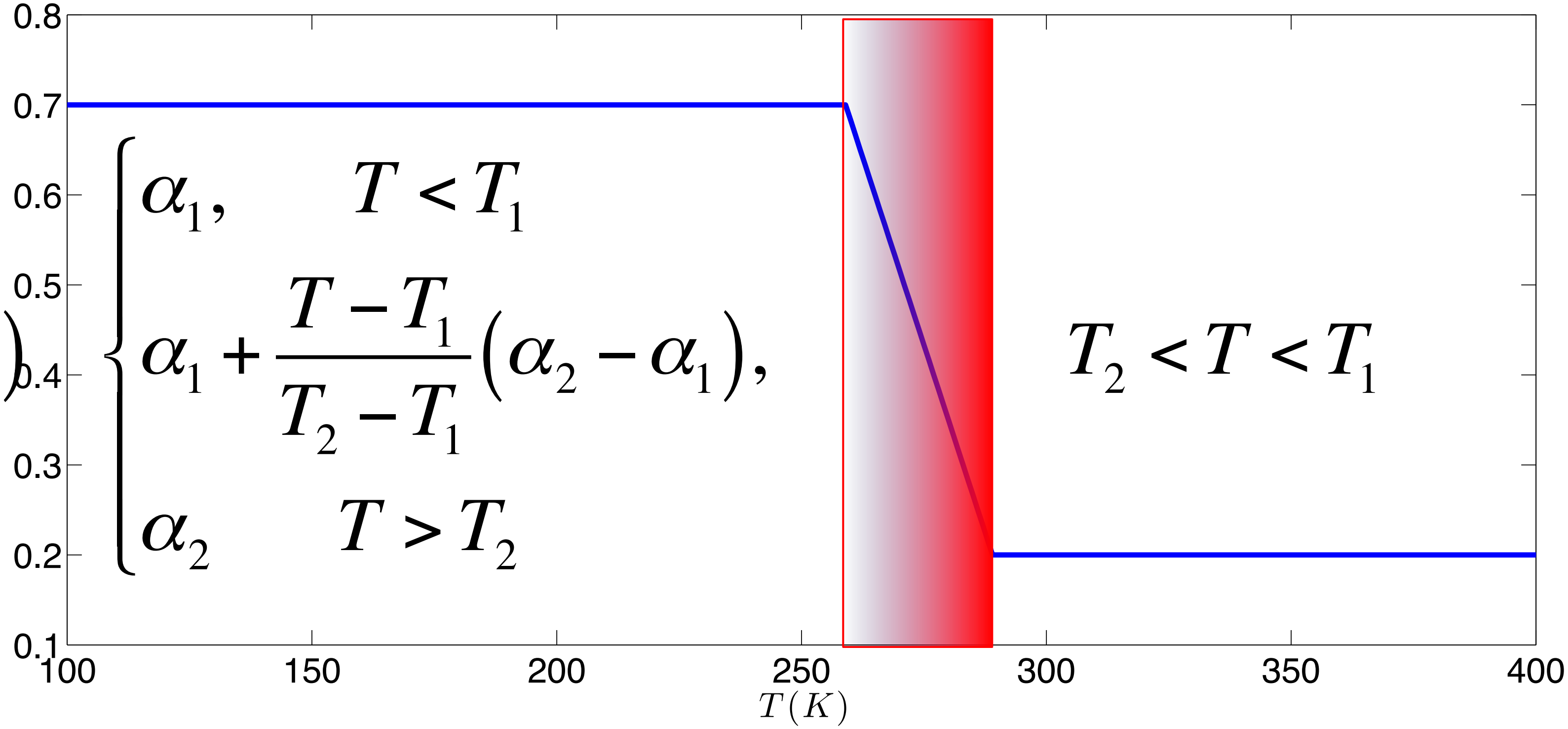


Triassic period
Permian period
Jurassic period
Pennsylvanian period
Mississippian period
Cretaceous period
Mesozoic era
206 m.y.
251 m.y.
300 m.y.
354 m.y.
Devonian period
409 m.y.
Silurian period
439 m.y.
65 m.y.
1 b.y.
57 m.y.
510 m.y.
Paleozoic era
Ordovician period
Evolution of cells with nucleus
543 m.y.
35 m.y.
Cambrian period
Eocene epoch
23 m.y.
2 b.y.
5 m.y.
Oldest fossil cells
Cenozoic era
Tertiary period
Oligocene epoc
1.8 m.y.
3 b.y.
Miocen epoch
4.6 billion years ago
4 b.y.
Pliocen epoch
ERA
PERIOD
EPOCH
Pleistocene epoch
Holocene epoch
Oldest rocks dated on Earth

Energy balance model

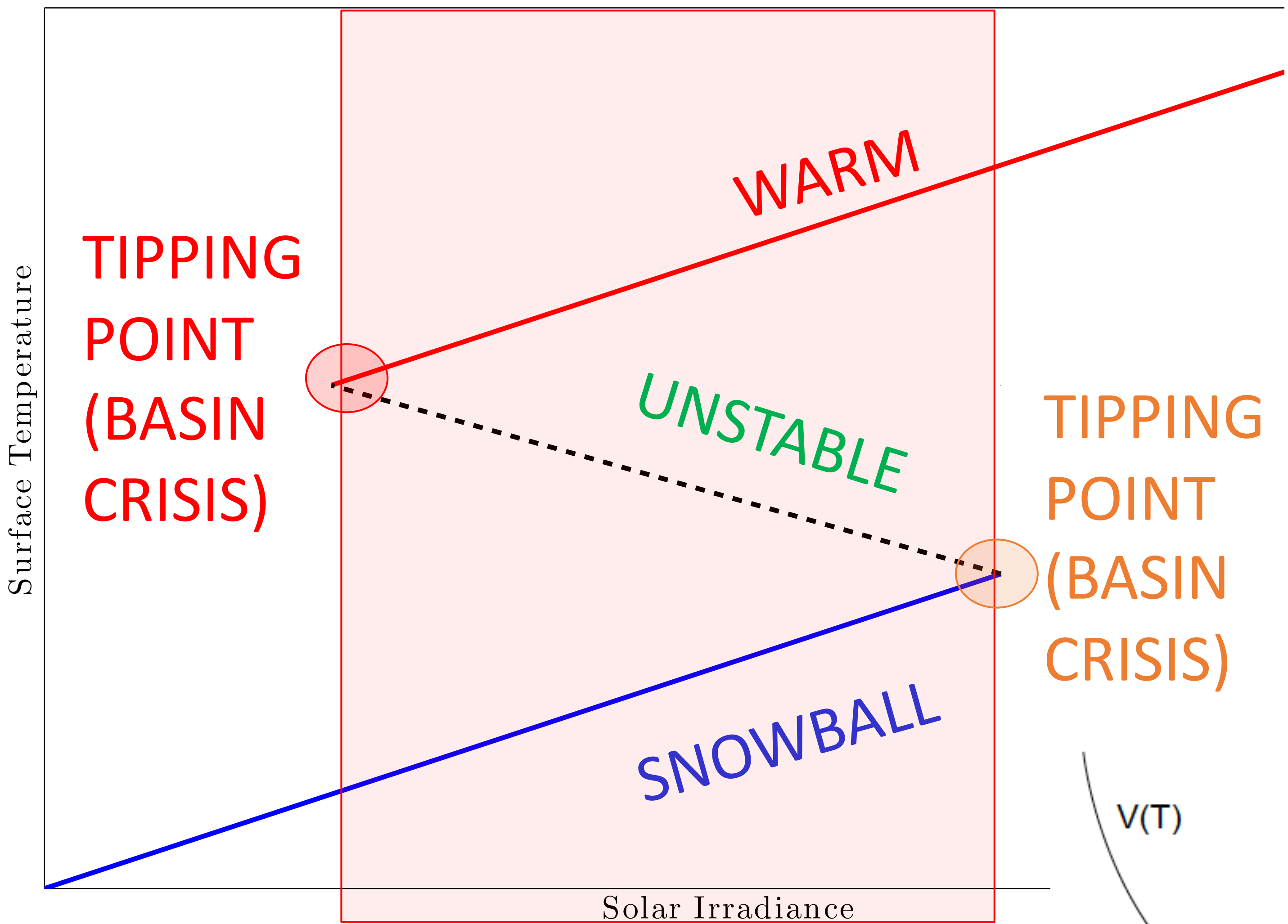
$$C\frac{dT}{dt} = I - O$$

$$C\frac{dT}{dt} = \underbrace{\left(1-\alpha_p(T)\right)\frac{S}{4}}_{I} - \underbrace{\left(A+BT\right)}_{O}$$

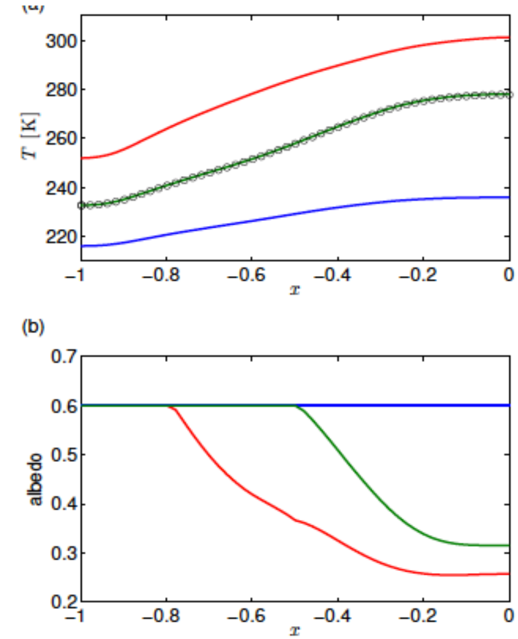


$$\alpha_p(T) \begin{cases} \alpha_1, & T < T_1 \\ \alpha_1 + \frac{T-T_1}{T_2-T_1}\left(\alpha_2-\alpha_1\right), & T_2 < T < T_1 \\ \alpha_2 & T > T_2 \end{cases}$$

Bifurcations

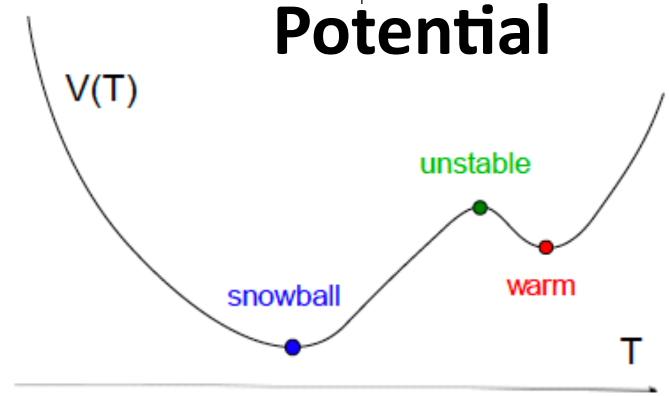


Bistability



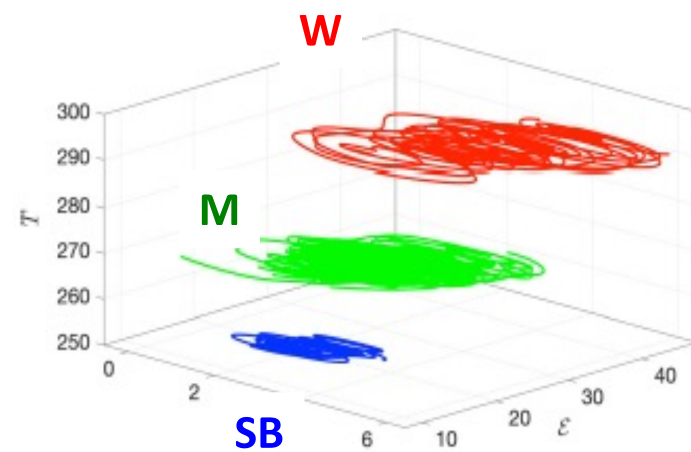
T, ice profile

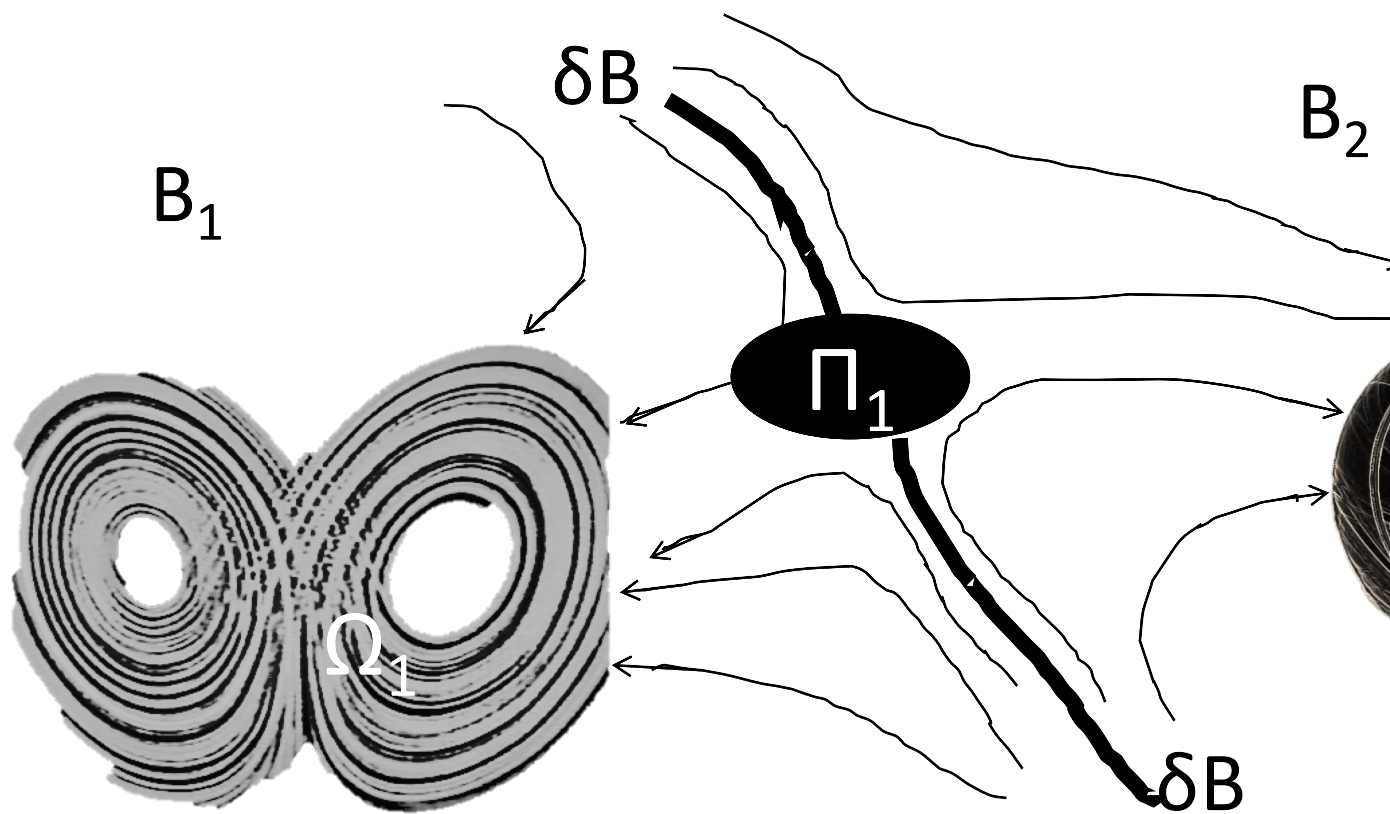
Potential



Competing Attractors and Melancholia State

- Dissipative system, co-existing attr. Ω_1, $\Omega_{2;}$ basins B_1, B_2
- M State Π_1 attracts orbits initialised on basin boundary δB between basins B_1 and B_2
- If near Π_1 state, uncertainty on final Ω_1 or Ω_2 state
    - Loss of Predictability of the second kind à la Lorenz





Melancholia (15) + presentation by Valerio Lucarini
Science on Screen

13 Nov 2018, 18:30, Barbican Cinema 2

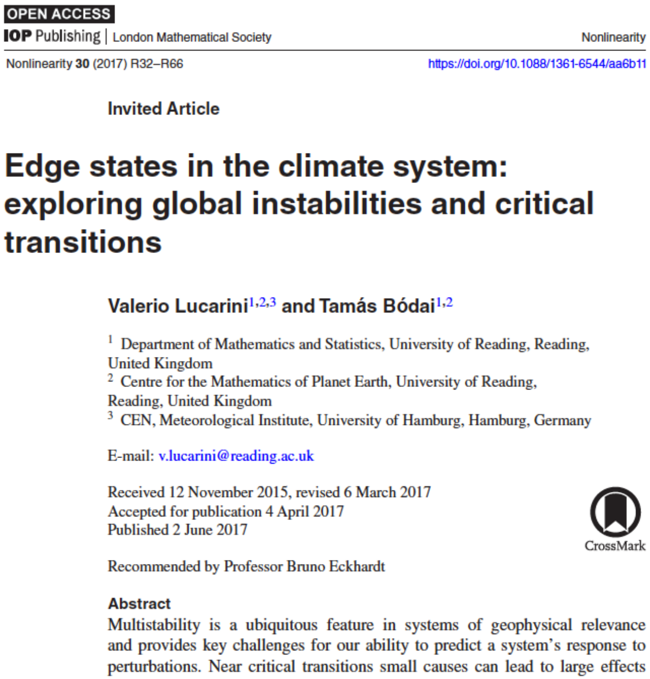
This is a past event

On-sale date and times ?

This is a past event. Sign up to our newsletters to hear about upcoming events

Albrecht Dührer, 1514

A somewhat simple climate model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Nonlinearity 30 (2017) R32–R66 https://doi.org/10.1088/1361-6544/aa6b11

Invited Article

Edge states in the climate system: exploring global instabilities and critical transitions

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Published 2 June 2017

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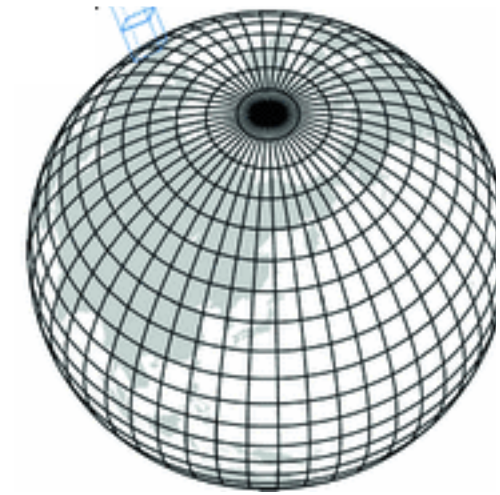
Recommended by Professor Bruno Eckhardt

Abstract
Multistability is a ubiquitous feature in systems of geophysical relevance and provides key challenges for our ability to predict a system's response to perturbations. Near critical transitions small causes can lead to large effects

Spectral Atmosphere
moist primitive equations
on σ levels

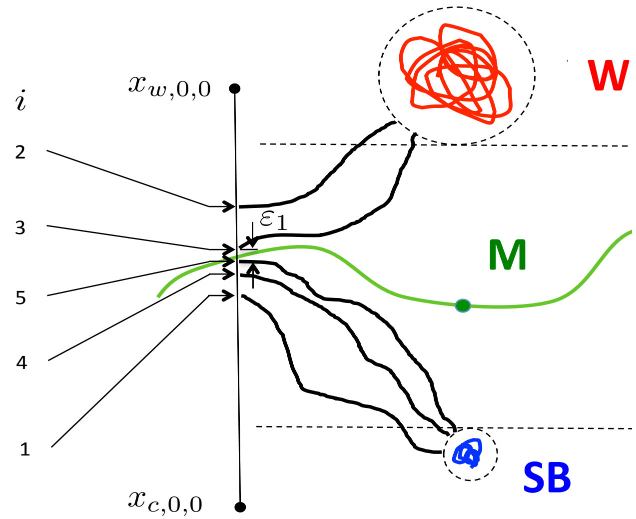
↕

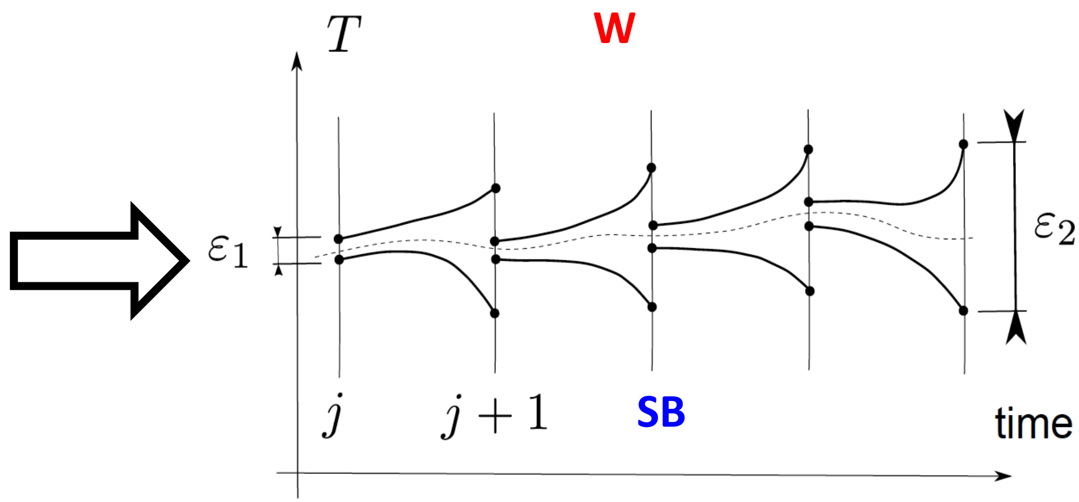
Diffusive Ghil-Sellers ocean
Ice-albedo feedback



- Turbulent atmosphere coupled to diffusive ocean
- Ice appears as surface temperature is below 0º C
- Ice-albedo effect

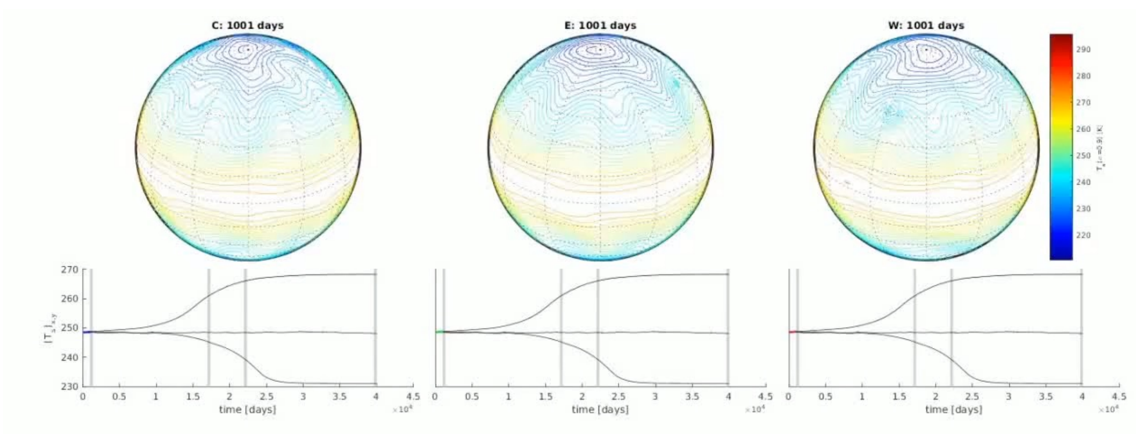
Tracking the M State (in 10^4 dimensions)

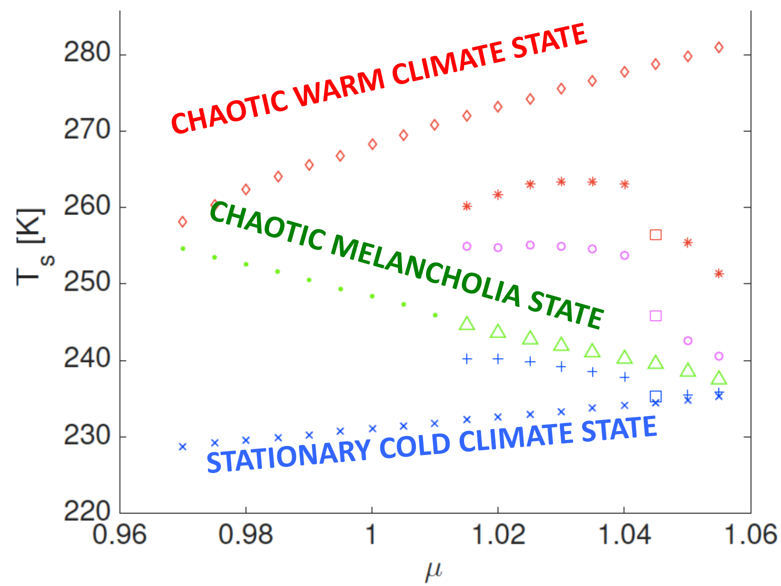




Bisection to shadow a trajectory on the basin boundary

Snowball Melancholia Warm State





A closer look at the boundary

- Is the basin boundary smooth?
  - It is folded, indeed fractal.
  - 1024 simulations between two trajectories near the boundary
  - Instability on the Melancholia states vs across it

Rough basin boundaries in high dimension: Can we classify them experimentally?

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Submitted: 26 January 2020 · Accepted: 16 September 2020 ·
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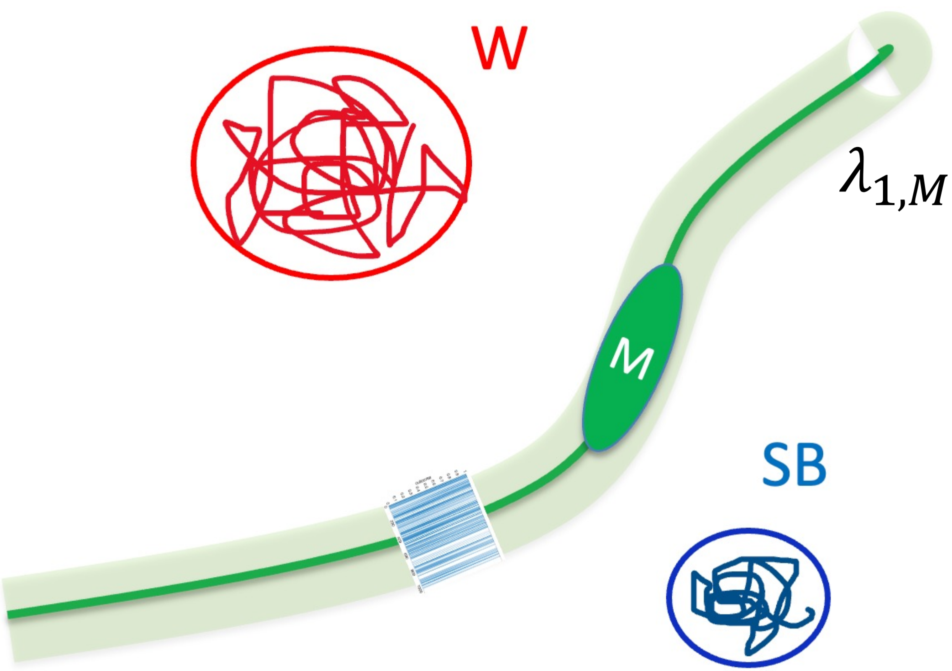
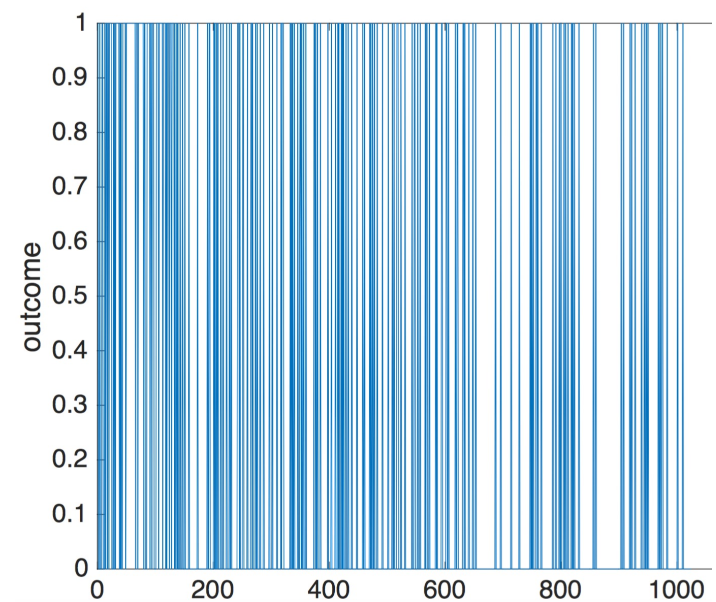
ABSTRACT
We show that a known condition for having rough basin boundaries in bistable 2D maps holds for high-dimensional bistable systems that possess a unique nonattracting chaotic set embedded in their basin boundaries. The condition for roughness is that the cross-boundary Lyapunov exponent λ_x on the nonattracting set is not the maximal one. Furthermore, we provide a formula for the generally noninteger co-dimension of the rough basin boundary, which can be viewed as a generalization of the Kantz–Grassberger formula. This co-dimension that can be at most unity can be thought of as a partial co-dimension, and, so, it can be matched with a Lyapunov exponent. We show in



$$\lambda_{1,M} > 1/\tau$$

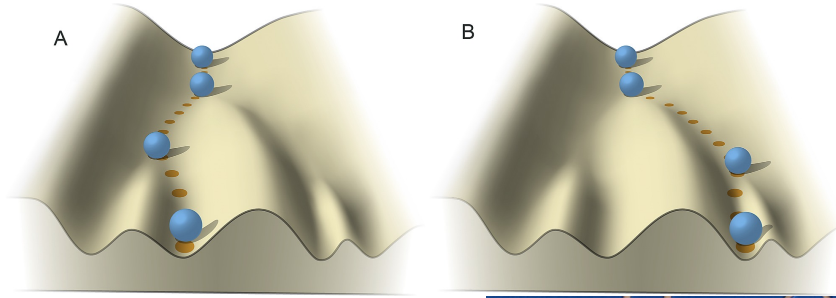


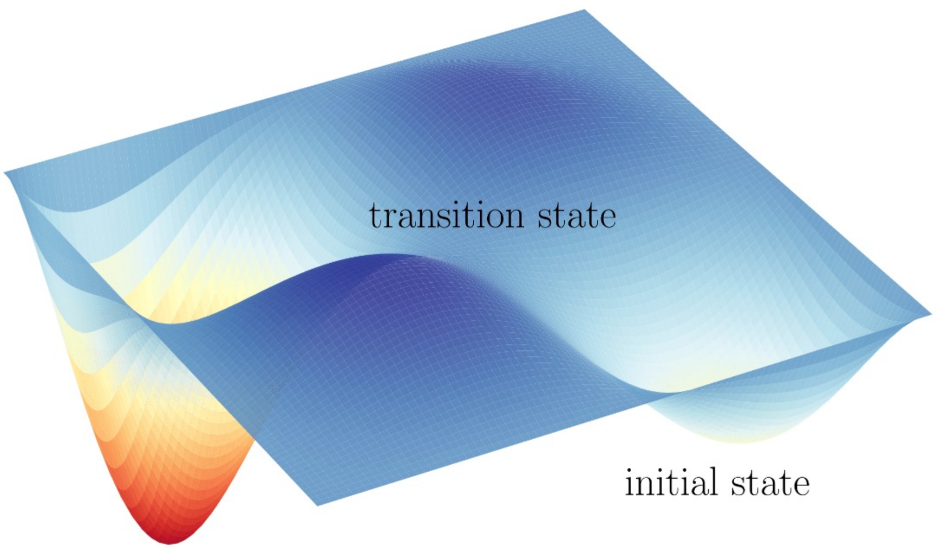
From Multistability to Metastability

- No noise: initial condition -> asymptotic state
- Classificatory; no jumps; sort of boring
- "Dynamical Landscape"

Biology

Epigenetic landscape punctuated equilibrium





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Nonlinearity 33 (2020) R59–R92 https://doi.org/10.1088/1361-6544/ab86cc

Global stability properties of the climate: Melancholia states, invariant measures, and phase transitions

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PHYSICAL REVIEW LETTERS **122,** 158701 (2019)

Editors' Suggestion Featured in Physics

Transitions across Melancholia States in a Climate Model: Reconciling the Deterministic and Stochastic Points of View

Valerio Lucarini[1,2,3,*] and Tamás Bódai[1,2]
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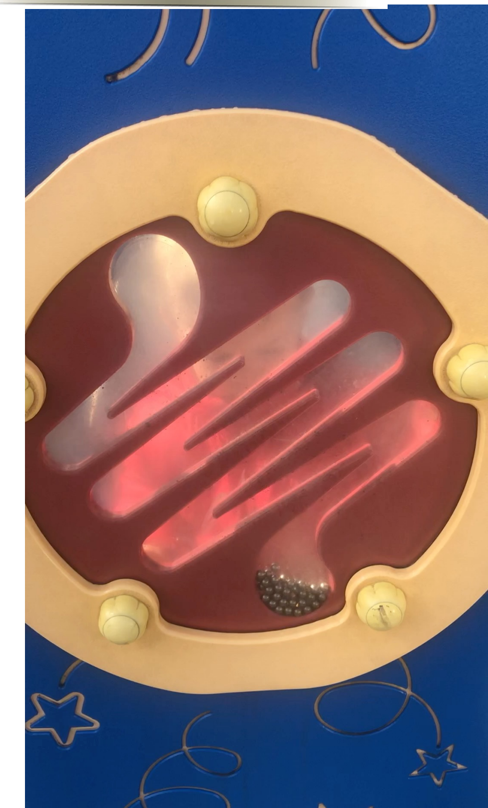
(Received 16 August 2018; published 16 April 2019)

The Earth is well known to be, in the current astronomical configuration, in a regime where two asymptotic states can be realized. The warm state we live in is in competition with the ice-covered snowball state. The bistability exists as a result of the positive ice-albedo feedback. In a previous investigation performed on a intermediate complexity climate model we identified the unstable climate states (melancholia states) separating the coexisting climates, and studied their dynamical and geometrical properties. The melancholia states are ice covered up to the midlatitudes and attract trajectories initialized on the basin boundary. In this Letter, we study how stochastically perturbing the parameter controlling the intensity of the incoming solar radiation impacts the stability of the climate. We detect transitions between the warm and the snowball state and analyze in detail the properties of the noise-induced escapes from the corresponding basins of attraction. We determine the most probable paths for the transitions and find evidence that the melancholia states act as gateways, similarly to saddle points in an energy landscape.

DOI: 10.1103/PhysRevLett.122.158701



Graham, Hamm, Tel … theory

- SDE: $\mathrm{d}x_i = F_i(\mathbf{x})\mathrm{d}t + \sigma s(\mathbf{x})_{ij}\mathrm{d}W_j,$
- Hasselmann's (1976) programme: stochastic climate modelling
- Hypoelliptic diffusion:
  - noise propagates via interaction with drift term (Hörmander)
- Ansatz: $W_\sigma(\boldsymbol{x}) \sim Z(\boldsymbol{x}) \exp(-2\Phi(\boldsymbol{x})/\sigma^2)$
- $\Phi(\boldsymbol{x})$ is the quasi-potential; depends on drift ***F*** and volatility ***s***
- Local minima of Φ are attractors; saddles are M states
- Orthogonal decomposition of drift: $F_i(\mathbf{x}) = R_i(\mathbf{x}) - C_{ij}(\mathbf{x})\partial_j\Phi(\mathbf{x})$
- One can frame stochastic resonance for this setting (L. PRE 2019)

Noise-induced Transitions

- In the weak-noise limit, escapes from attractors take place through the M state(s)
- Paths: instantons (variational formulation)
- Instantons obey a "strange equation"

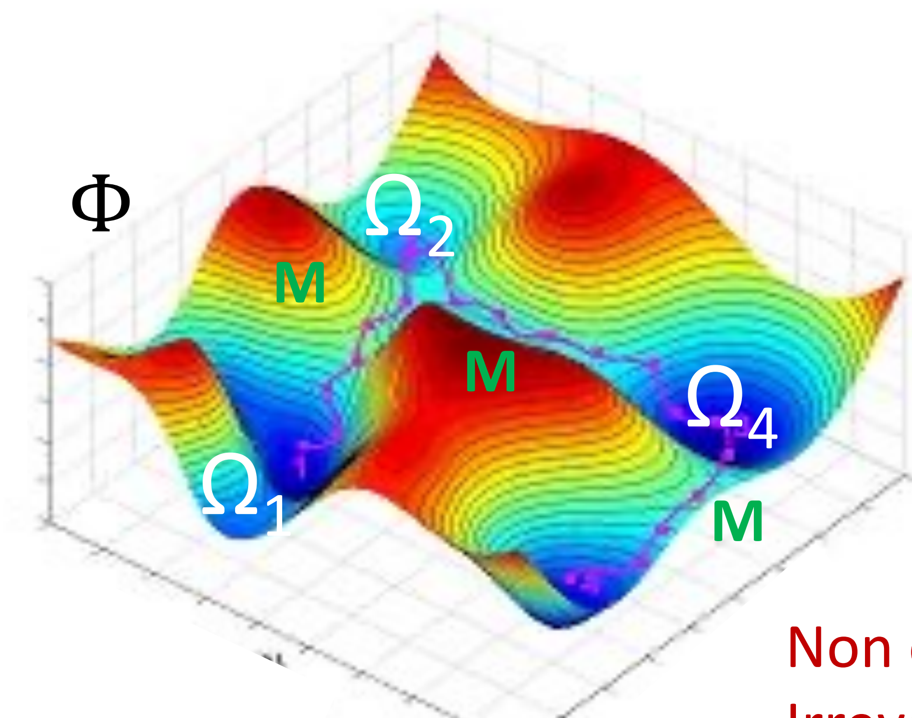
$$\frac{dx_i}{dt} = R_i(\vec{x}) + C_{ij}(\vec{x})\partial_j \Phi(\vec{x})$$

- Distribution of escape times: $P(p) = \frac{A(p)}{\bar{\tau}_\sigma} \exp\left(-\frac{p}{\bar{\tau}_\sigma}\right)$
- τ is controlled by local quasi-potential differences

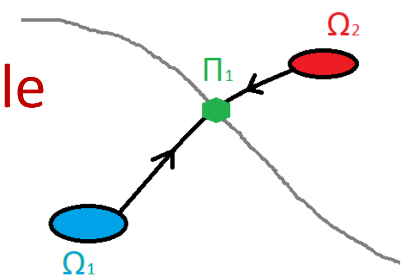
$$\bar{\tau}_\sigma \propto \exp\left(\frac{2(\Phi(\Pi_l) - \Phi(\Omega_j))}{\sigma^2}\right)$$

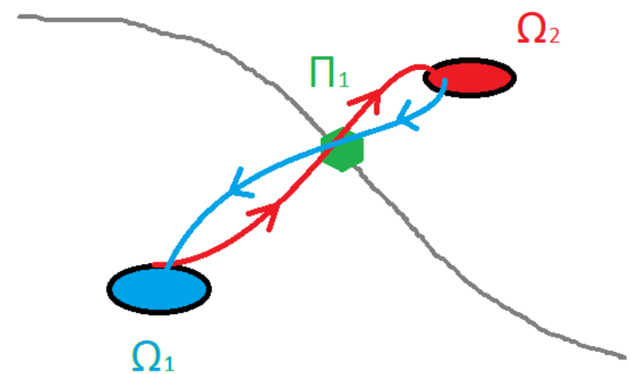
- $\Phi(\boldsymbol{x})$ is constant on attractors and M-states

Exploring the Dynamical Landscape

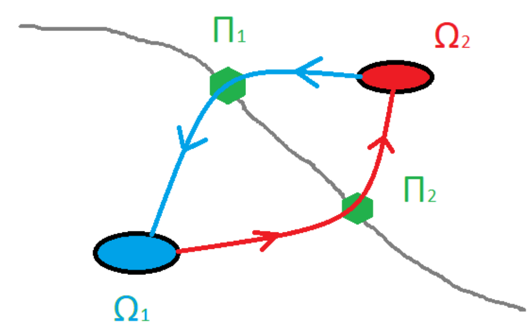


Equilibrium/Reversible



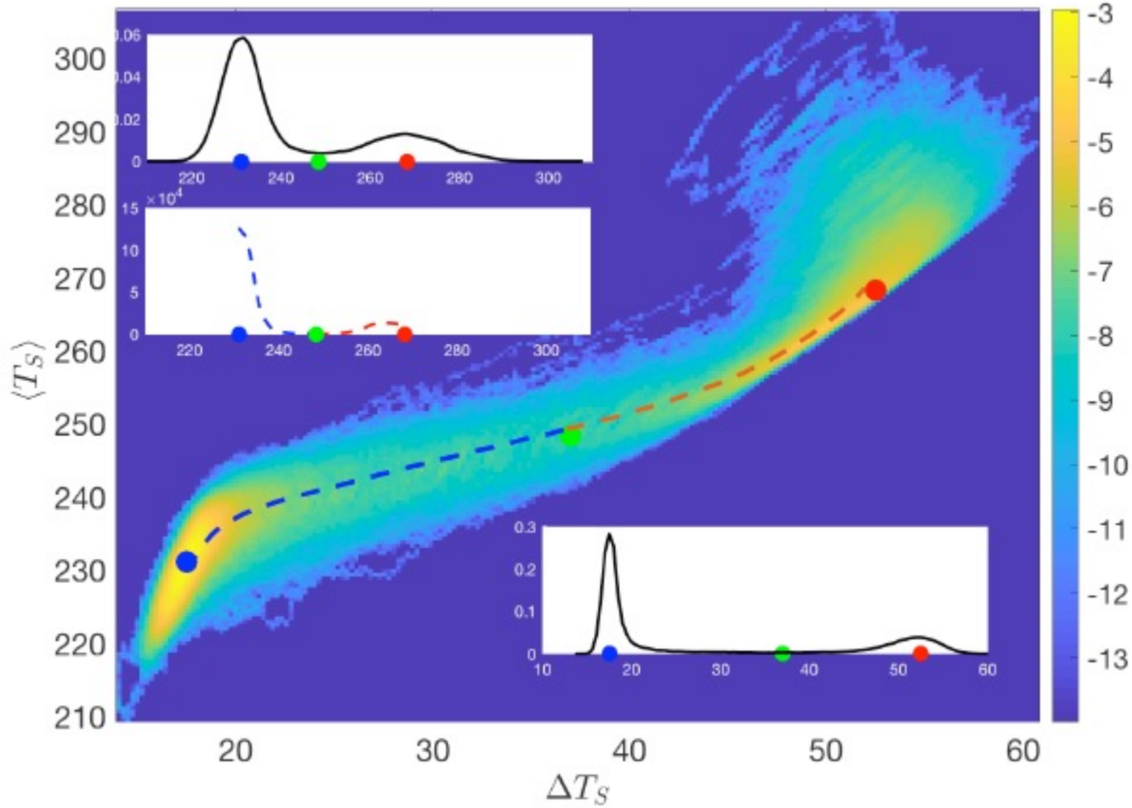


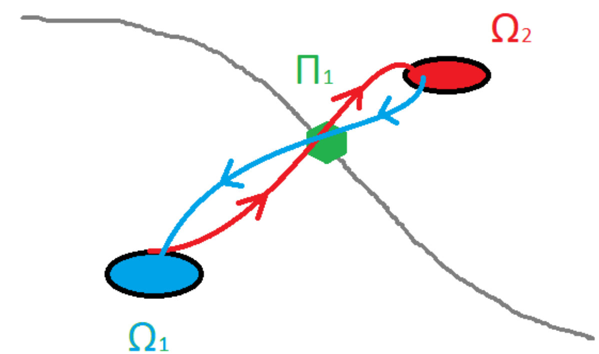
Non equilibrium/
Irreversible systems

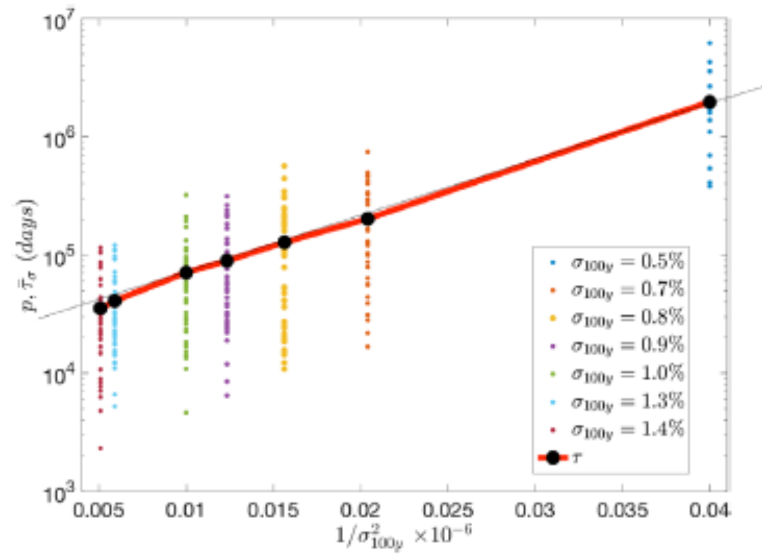


$$C(\phi)\frac{\partial T_S}{\partial t} = \mu(1+\sigma\frac{dW}{dt})I(\phi)\frac{S^*}{4}(1-\alpha(\phi,T_S)) - O(T_S) - D_\phi[T_S] + \chi[T_S,T_A],$$

Constructing the invariant measure







$$\Phi(s) \sim -\frac{\sigma^2}{2} \log \rho_\sigma(s) = -\frac{\sigma^2}{2} \log \int dx \delta(S(x) - s) \rho_\sigma(x)$$

- We have here the **W➔C** and the **C➔W** instantons
- The distributions peak at/near the attractors **W** and **C**
- **M** State is a saddle, instantons cross it

Outside the Comfort Zone

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Research

Check for updates

Cite this article: Margazoglou G, Grafke T, Laio A, Lucarini V. 2021 Dynamical landscape and multistability of a climate model. *Proc. R. Soc. A* **477**: 20210019.
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Subject Areas:
climatology, statistical physics, mathematical modelling

Keywords:
climate modelling, multistability, quasi-potential theory, non-equilibrium systems, data-driven methods, manifold learning

Dynamical landscape and multistability of a climate model

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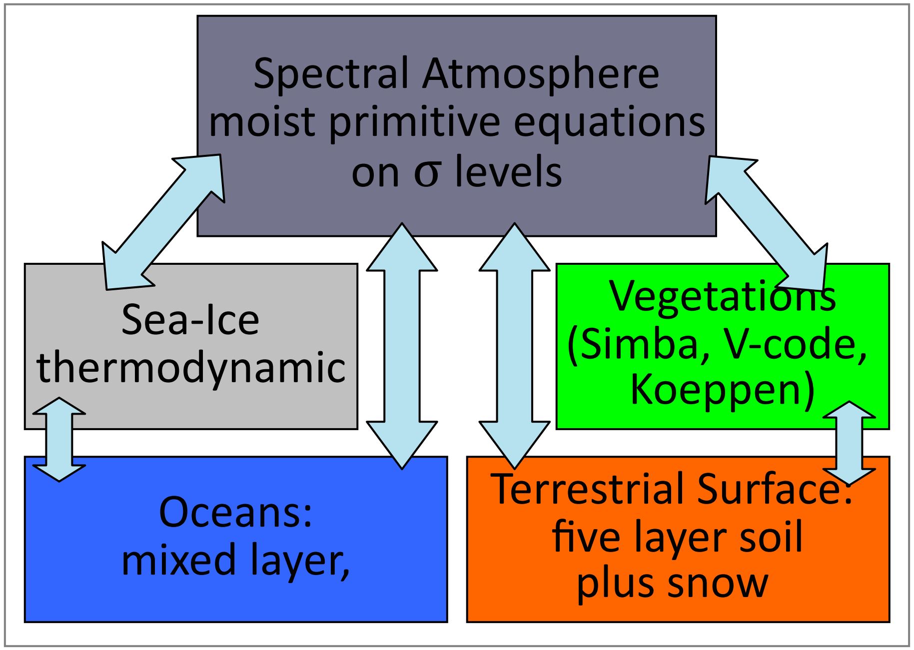
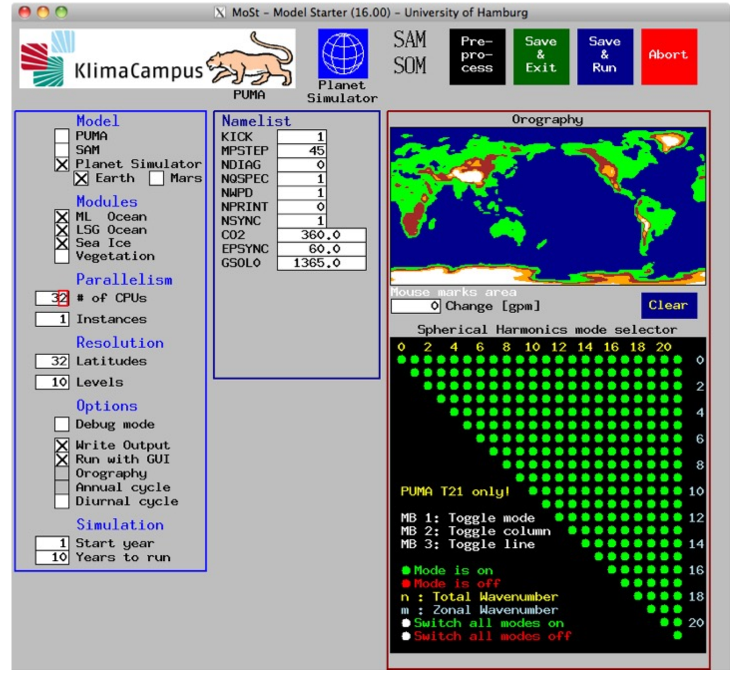
We apply two independent data analysis methodologies to locate stable climate states in an intermediate complexity climate model and analyse their interplay. First, drawing from the theory of quasi-potentials, and viewing the state space as an energy landscape with valleys and mountain ridges, we infer the relative likelihood of the identified multistable climate states and investigate the most likely transition trajectories as well as the expected transition times between them. Second, harnessing techniques from data science, and specifically manifold learning, we characterize the data landscape

Margazoglou et al. 2021

Using now a full climate model, same used for performing climate change projections with response theory
$O(10)^5$ degrees of freedom

PLASIM Model



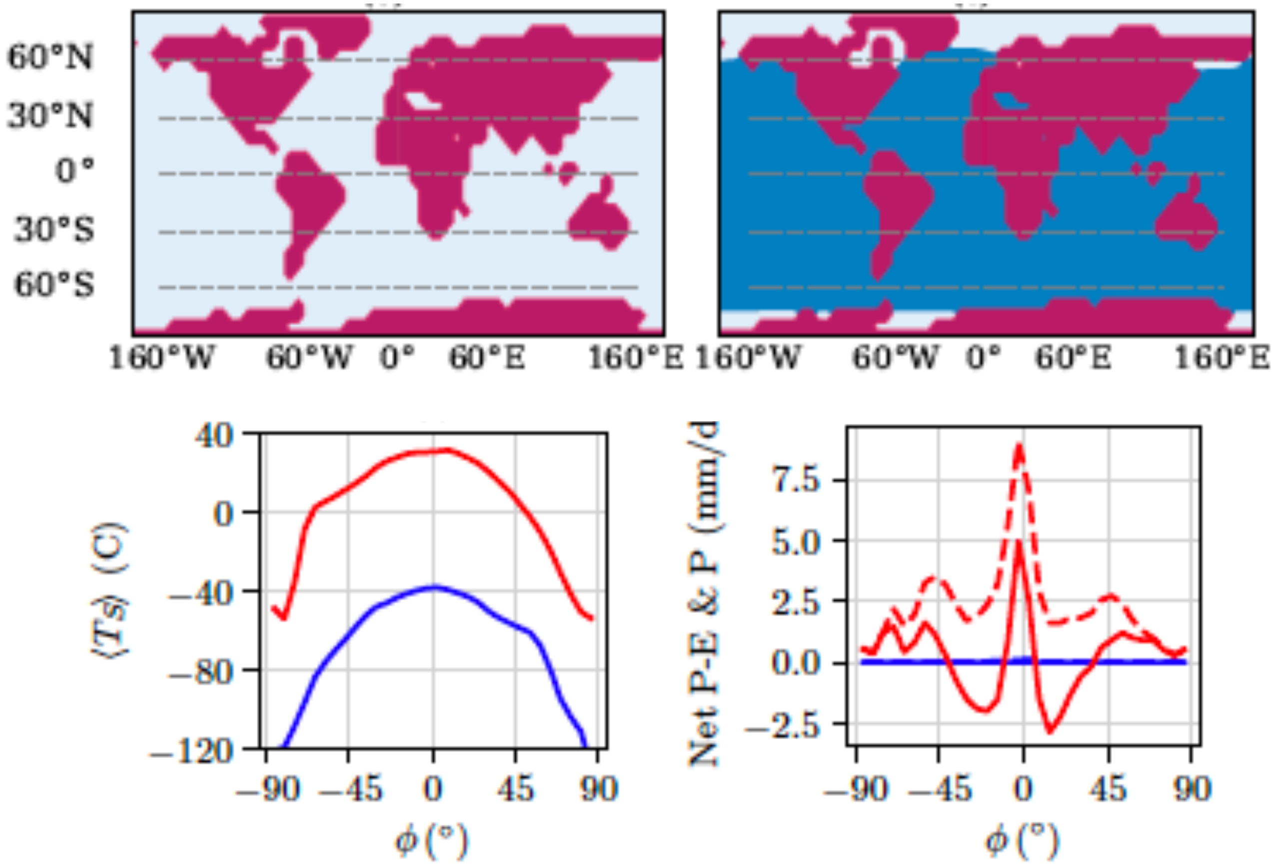


- Earth-like climate model
- Used for present climate, paleoclimate, exoplanets
- Most processes included
- Active hydrological cycle

Setup A

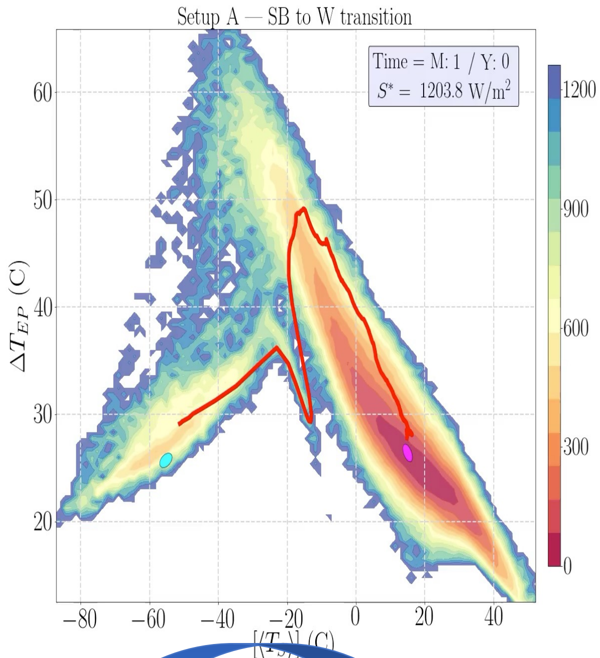
- Atmosphere and Ocean can transport heat
- We get W and SB competing climate states

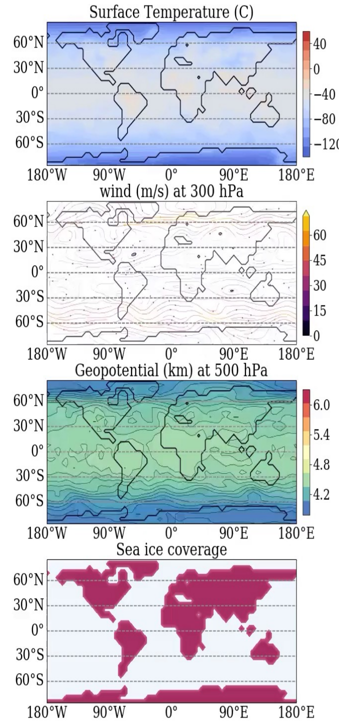
SB W

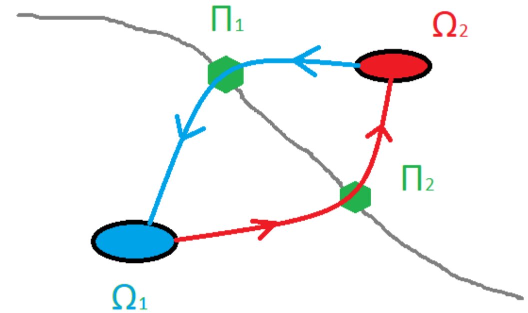


T_S P, P-E

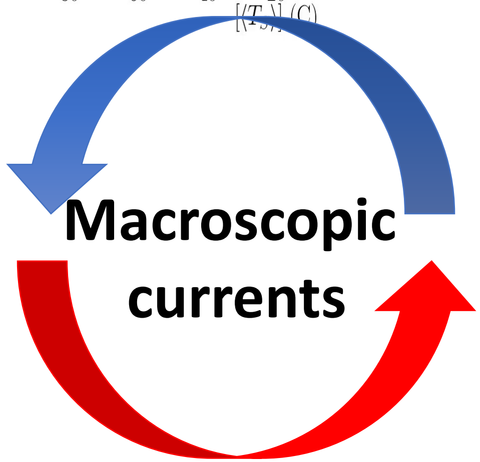
SB to W





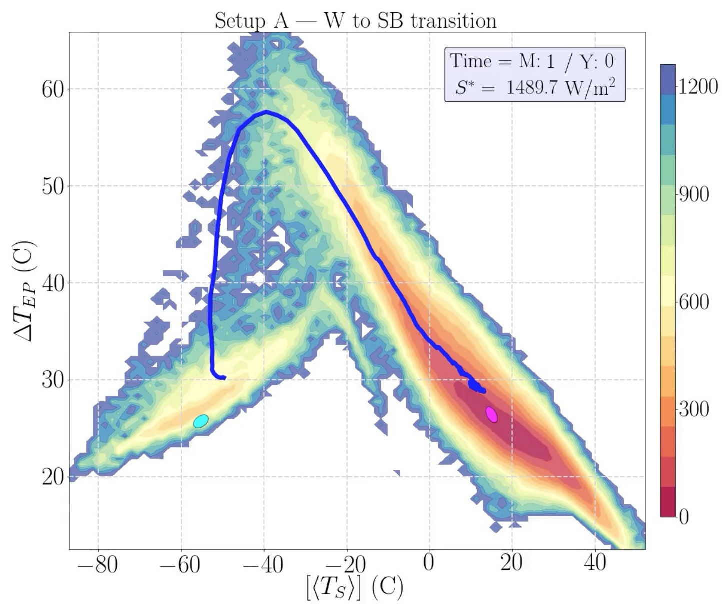


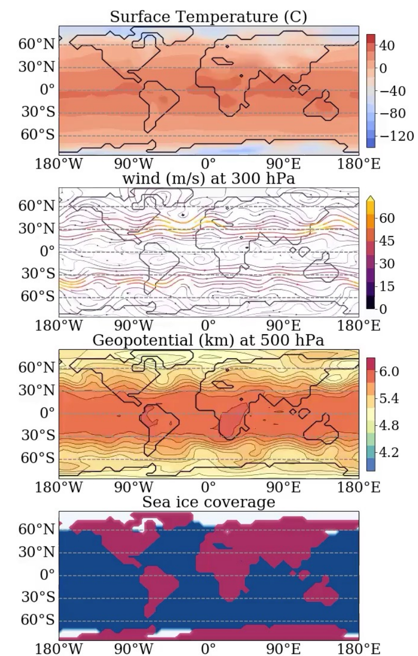
Hydrological Cycle
as “agent of irreversibility”



Stronger
Nonequilibrium

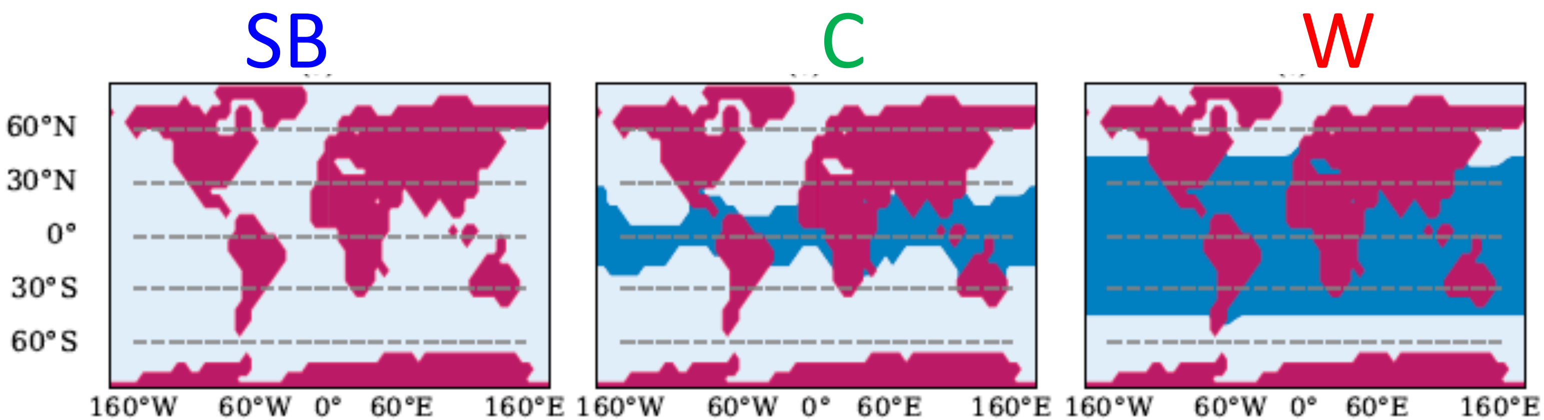
W to SB

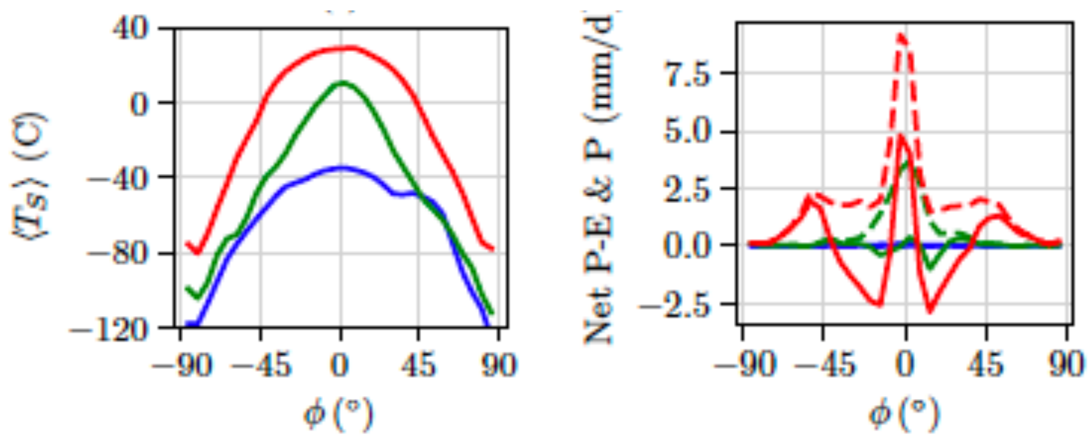




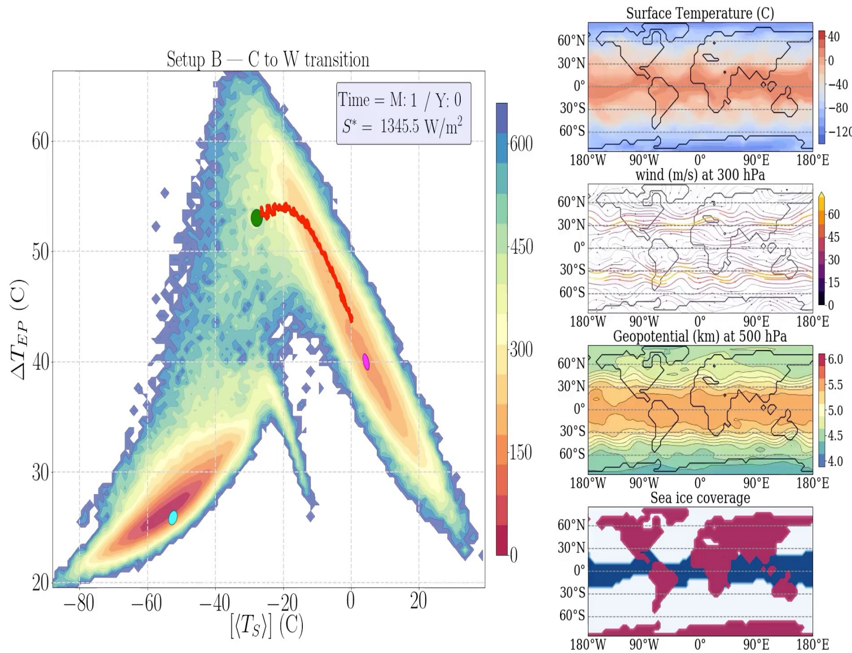
Setup B

- Only the Atmosphere can transport heat
- There competing climate states - serendipitous discovery
- In the 2D projection no sign of local minimum of Φ



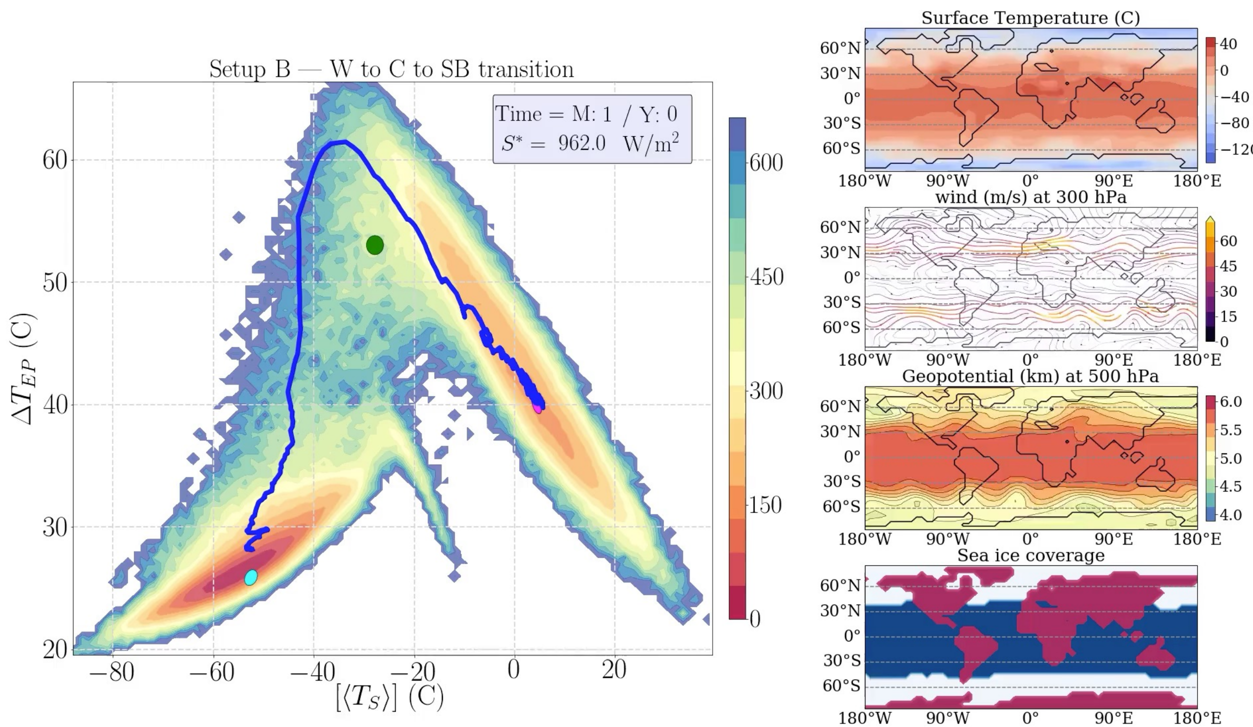


T_S P, P-E

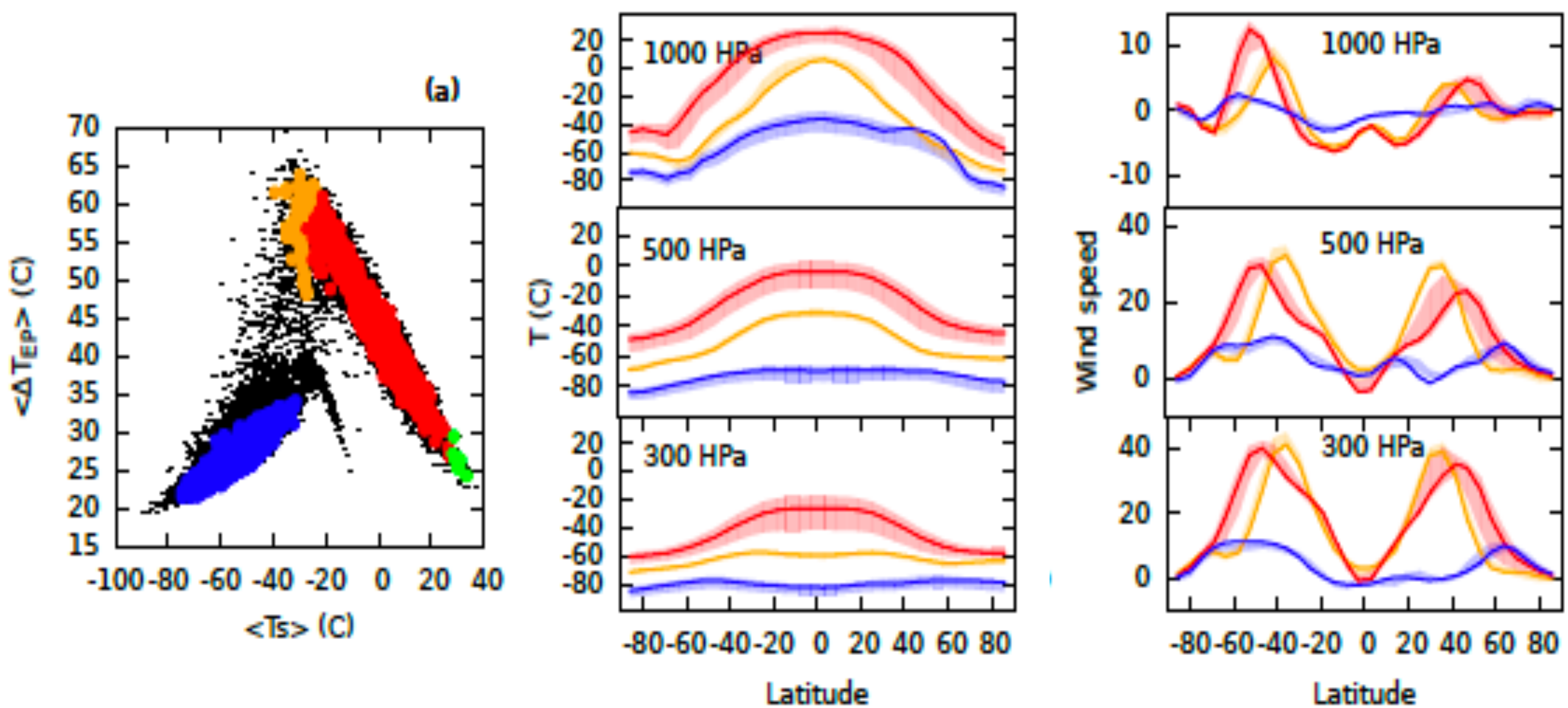


C to W

W to C to SB



Machine Learning Magic (1)

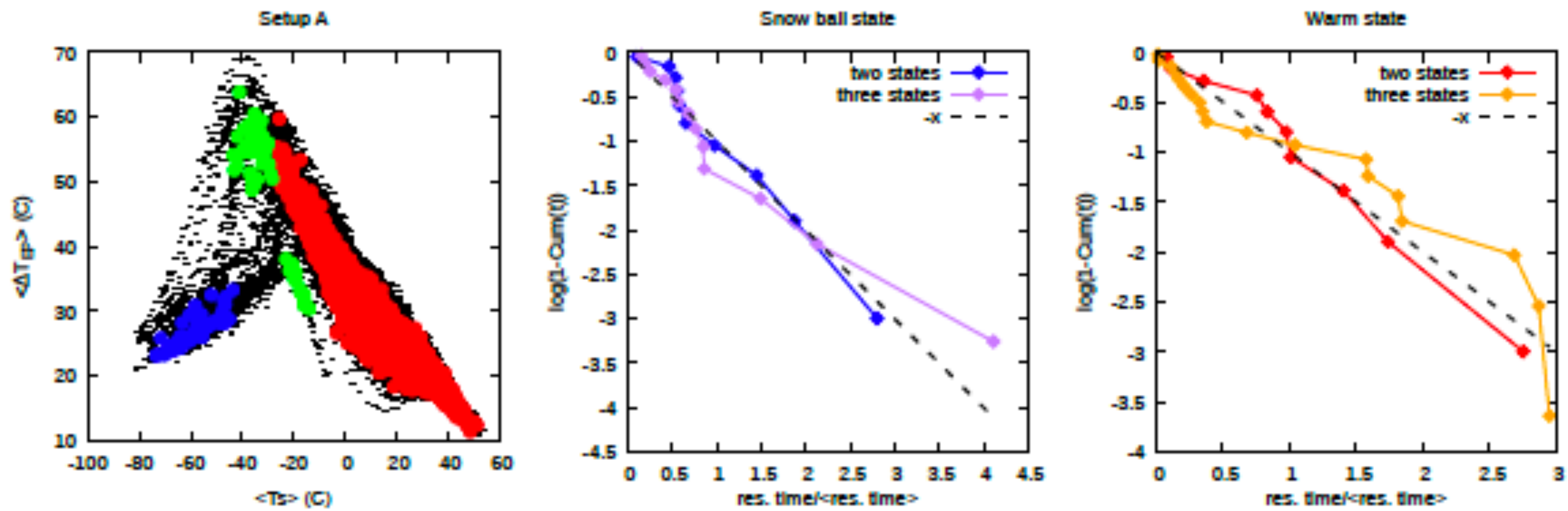


Local Maxima of $\Phi(s)$
k-Nearest Neighbour

$$\Phi(s) \sim -\frac{\sigma^2}{2} \log \rho_\sigma(s) = -\frac{\sigma^2}{2} \log \int dx \delta(S(x) - s) \rho_\sigma(x)$$

- Local dimension via statistics of first vs second neighbour to the reference point
- Topography of Φ (maxima, minima, saddles) via Density Peak Estimator
- We can estimate Φ *as a function of a large number of variables (no a-priori choice)*
- Probability density is estimated implicitly on the embedding manifold

Machine Learning Magic (2)



- In Setup B we find automatically the C state from a stochastic trajectory
- We can reconstruct the average properties of the competing climates
- Exclude spurious metastable states through statistics of permanence times
- Huge potential for analysing climate data
- Assess metastablity at different confidence levels

A complex dynamical landscape

Climate Dynamics
https://doi.org/10.1007/s00382-019-04926-7

Check for updates

Co-existing climate attractors in a coupled aquaplanet

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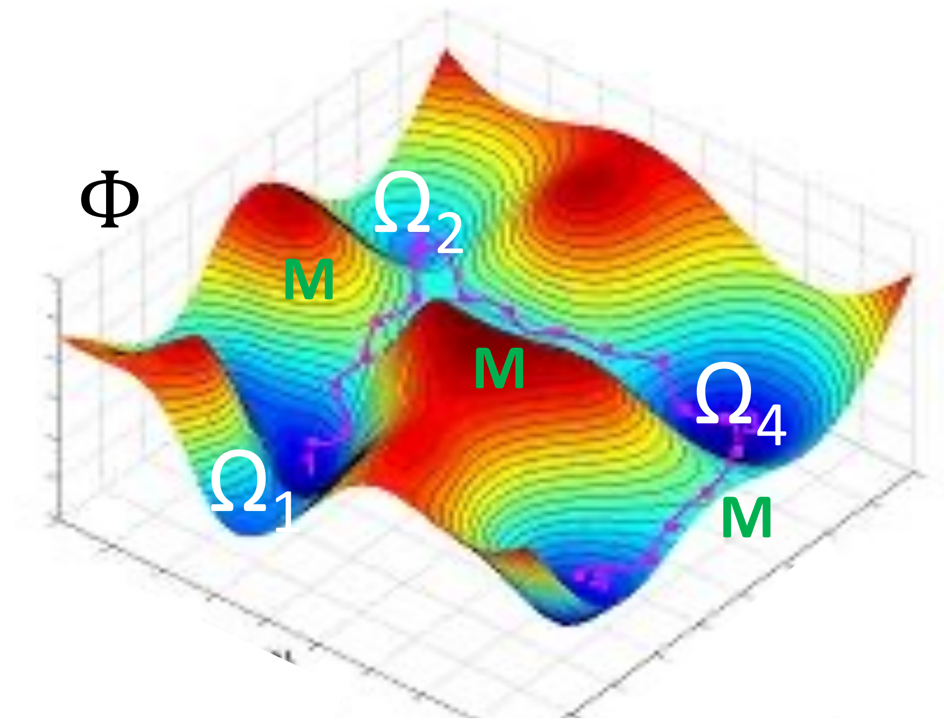
Abstract
The first step in exploring the properties of dynamical systems like the Earth climate is to identify the different phase space regions where the trajectories asymptotically evolve, called 'attractors'. In a given system, multiple attractors can co-exist under the effec http://orcid.org/0000-0001-8199-223X undaries of their basins of attraction, small changes produce large effects. Therefore, they are key regions for understanding the system response to perturbations. Here we prove the existence of up to five attractors in a simplified climate system where the planet is entirely covered by the ocean (aquaplanet). These attractors range from a snowball to a hot state without sea ice, and their exact number depends on the details of the coupled atmosphere–ocean–sea ice configuration. We characterise each attractor by describing the associated climate feedbacks, by using the principal component analysis, and by measuring quantities borrowed from the study of dynamical systems, namely instantaneous dimension and persistence.

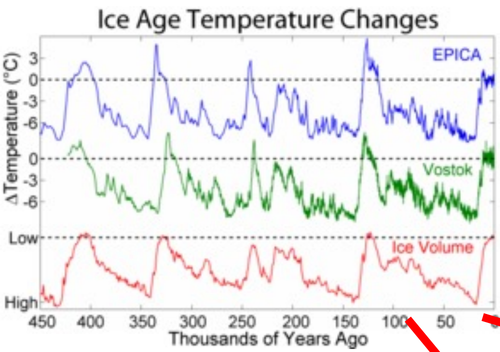


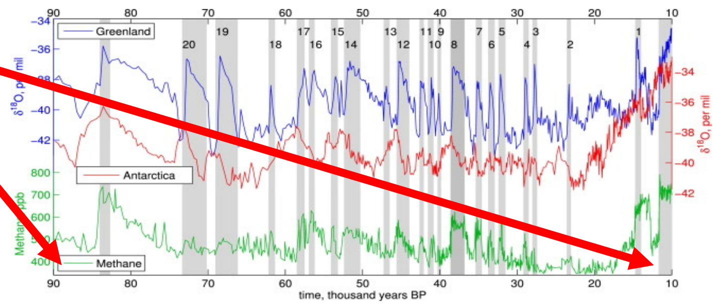
Brunetti et al. Clim. Dyn. 2020
Ragon et al. Clim. Dyn. 2022

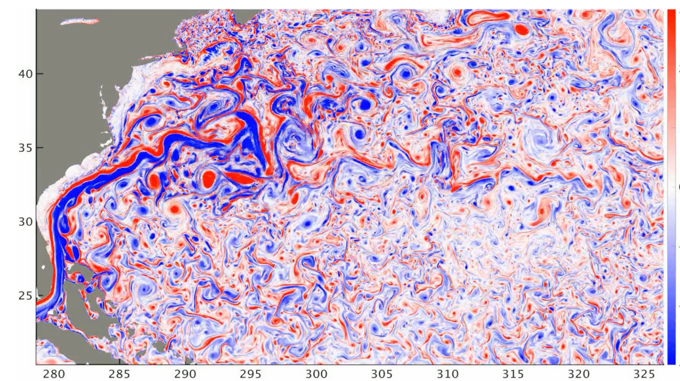
Five competing attractors!

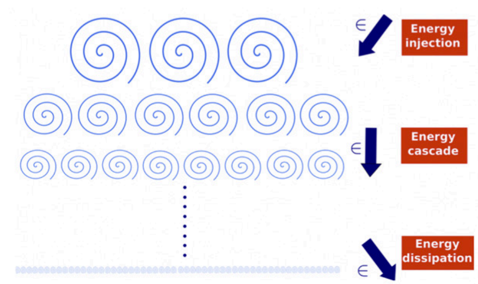
Conjecture: itinerancy between competing metastable states explains ultralong climate variability
Also: Lewis et al. 2007 JGR; Abbott et al. 2011 JGR

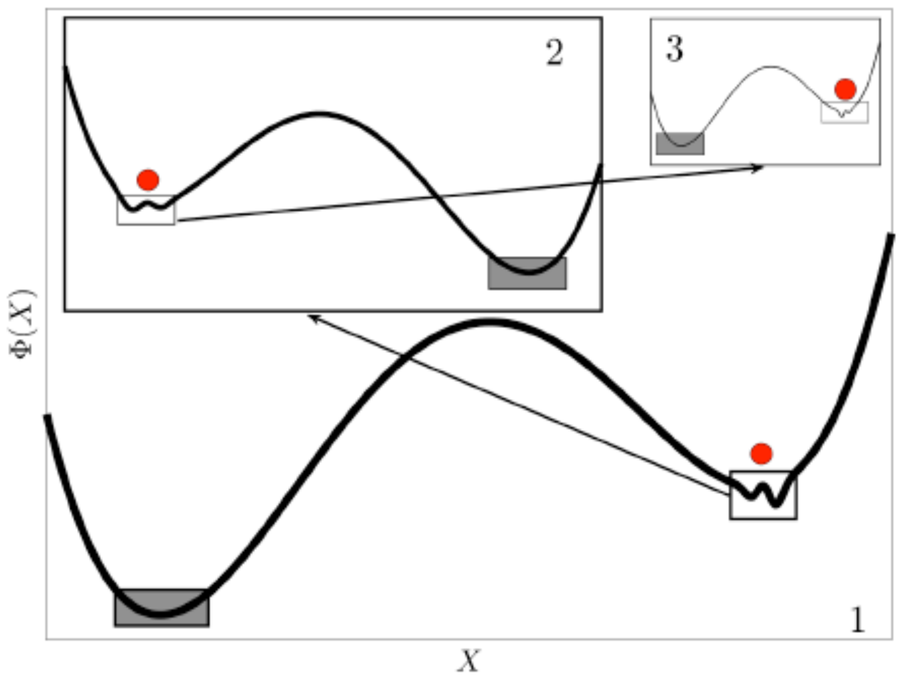
Multiple scales of Multistability

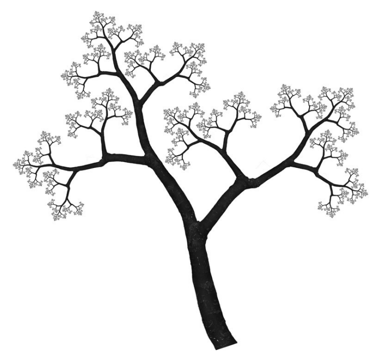








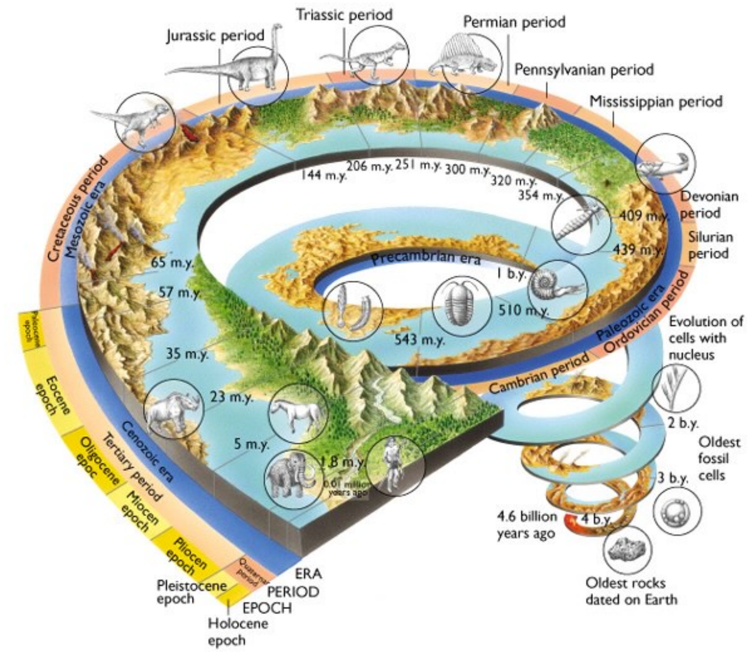




"Big whorls have little whorls
That feed on their velocity,
And little whorls have lesser
whorls, and so on to viscosity"

Lewis F. Richardson, 1920

Conclusions



- Climate as nonequilibrium statistical mechanical system
- Advancing Hasselmann’s programme
- Response theory for smooth response
  - Predict climate change
- High sensitivity and mixing rate
  - Nearing Tipping Points
- Multistability and Tipping points
  - Melancholia State, gate for the transitions

- Multiscale multistability in a hierarchy of models and tipping points
- Ultralong climate variability: itinerancy between metastable states?
- Linking natural history (realized trajectory) with the many “possibles”
- Running the movie again life might take different form (Gould 1989)

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Thanks for your attention!