FUNCTIONAL RENORMALIZATION GROUP AND THE 2PI EFFECTIVE ACTION

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It can be fruitful to establish connections between the FRG and other non-perturbative approaches (DS equations, NPI effective actions, ...)
[Dupuis (2005, 2014); Blaizot, Pawlowski, UR (2011, 2021);
Carrington et al (2015, 2018, 2019); Katanin (2019); Alexander et al (2019)]

This can help:

– unveiling new truncation schemes;
– clarifying the question of renormalization.

In this talk, we use the FRG to shed light on how renormalization works within the two-particle-irreducible (2PI) framework.

OUTLINE

1. 2PI Framework
2. 1PI Flows
3. Flow Reformulation of the 2PI Framework
4. 2PI Renormalization using Flows

For definiteness, we consider a scalar theory:

$$S[\varphi] = \int_{1/\Lambda_{\rm uv}} d^4x \left\{ \frac{1}{2}(\partial\varphi)^2 + \frac{m_{\rm b}^2}{2}\varphi^2 + \frac{\lambda_{\rm b}}{4!}\varphi^4 \right\}$$

1. 2PI Framework

2PI EFFECTIVE ACTION

The 1PI effective action is a functional $\Gamma[\phi]$ that gives access to the one-point function $\bar{\phi} \equiv \langle \varphi \rangle$ from a variational principle:

$$0 = \left.\frac{\delta \Gamma[\phi]}{\delta \phi}\right|_{\phi=\bar{\phi}}$$

Similarly, the 2PI effective action is a functional $\Gamma[G]$ that gives access to the two-point function $\bar{G}(x,y) \equiv \langle \varphi(x)\varphi(y)\rangle_c$:

$$0 = \left.\frac{\delta \Gamma[G]}{\delta G(x,y)}\right|_{G=\bar{G}}$$

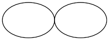
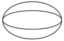
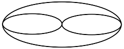
2PI EFFECTIVE ACTION

It admits the following loop expansion:

$$\Gamma[G] = \frac{1}{2}\int_{p<\Lambda_{\rm uv}} \log G^{-1}(p) + \frac{1}{2}\int_{p<\Lambda_{\rm uv}} (p^2 + m_{\rm b}^2)\, G(p) + \Phi[G,\lambda_{\rm b}]$$

with $\Phi[G,\lambda_{\rm b}]$ the sum of two-particle-irreducible (2PI) diagrams:

$$\Phi[G,\lambda_{\rm b}] \;=\; \lambda_{\rm b}\Phi^{(2)}[G] + \lambda_{\rm b}^2\Phi^{(3)}[G] + \cdots + \lambda_{\rm b}^{\ell-1}\Phi^{(\ell)}[G] + \cdots$$

$\Phi^{(2)}[G] \;=$ , $\Phi^{(3)}[G] =$ , $\Phi^{(4)}[G] =$ , ...

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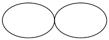
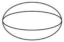
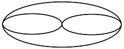
$\Phi^{(2)}[G] \;=$ , $\Phi^{(3)}[G] =$ , $\Phi^{(4)}[G] =$ , $\ldots$

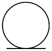
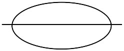
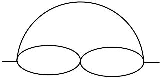
In practice, one retains in $\Phi[G]$ the diagrams up to a certain loop order:

$$\Phi[G,\lambda_{\rm b}] \to \Phi_L[G,\lambda_{\rm b}] \equiv \lambda_{\rm b}\Phi^{(2)}[G] + \lambda_{\rm b}^2\Phi^{(3)}[G] + \cdots + \lambda_{\rm b}^{L-1}\Phi^{(L)}[G]$$

GAP EQUATION

The variational principle then gives $\bar{G}(p)$ as the solution to an implicit "gap equation":

$$\bar{G}^{-1}(p) = p^2 + m_{\rm b}^2 + \left.\frac{2\delta\Phi[G]}{\delta G(p)}\right|_{\bar{G}}$$

$= p^2 + m_{\rm b}^2 +$  $+$  $+$  $+ \dots$

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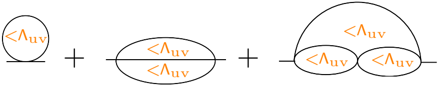
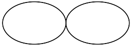
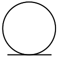


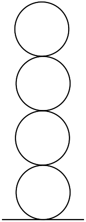
$+ \ldots$

This diagrammatic formulation comes inevitably with the problem of UV divergences.

UV DIVERGENCES

$\Phi[G] =$  $\Rightarrow \quad \bar{G}^{-1} = p^2 + m_{\rm b}^2 +$ 

Iterations:



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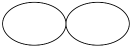
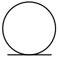
$\Phi[G] =$  $\Rightarrow$ $\bar{G}^{-1} = p^2 + m_{\rm b}^2 +$ 

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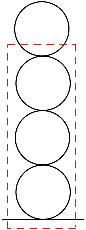

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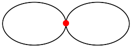
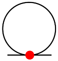
$\Phi[G] =$  $\Rightarrow \quad \bar{G}^{-1} = p^2 + m_\mathrm{b}^2 +$ 

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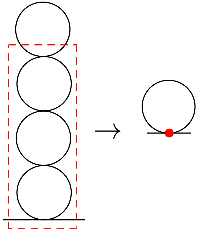
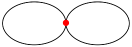
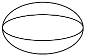
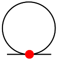
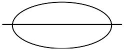


$$\Gamma^{(4)} = \mathcal{I}[G] - \frac{1}{2}\int_r \mathcal{I}[G]\,\bar{G}^2(r)\,\Gamma^{(4)} \quad \text{[Bethe-Salpeter]}$$

$$\mathcal{I}[G] = \frac{4\delta^2\Phi[G]}{\delta G^2} =$$  [Kernel]

UV DIVERGENCES

$\Phi[G] =$  $+$  $\Rightarrow$ $\bar{G}^{-1} = p^2 + m_{\rm b}^2 +$  $+$ 

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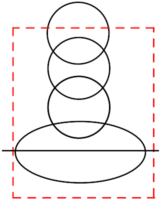
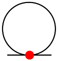
 $\rightarrow$ 

$$\Gamma^{(4)}(p,q) = \mathcal{I}(p,q) - \frac{1}{2}\int_r \mathcal{I}(p,r)\bar{G}^2(r)\Gamma^{(4)}(r,q)$$

$\mathcal{I}(p,q) = \frac{4\delta^2\Phi[G]}{\delta G(p)\delta G(q)} =$  $+$ 

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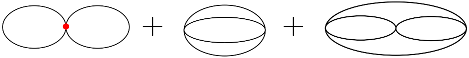
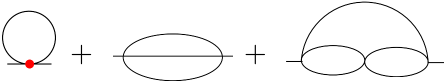
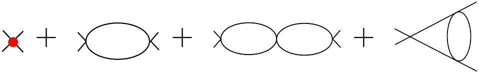
$\Phi[G] =$ 

$\bar{G}^{-1} = p^2 + m_{\rm b}^2 +$ 

$\mathcal{I}(p,q) = \frac{4\delta^2\Phi[G]}{\delta G(p)\delta G(q)} =$ 

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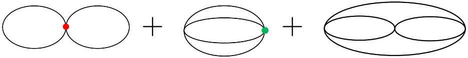
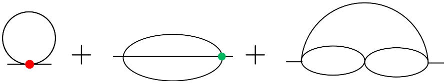
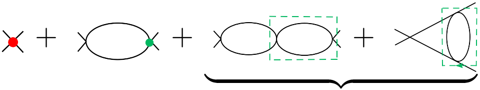
$\Phi[G] =$ 

$\bar{G}^{-1} = p^2 + m_{\rm b}^2 +$ 

$\mathcal{I}(p,q) = \frac{4\delta^2\Phi[G]}{\delta G(p)\delta G(q)} =$ 

$\Rightarrow \Gamma^{(4)}_{L=1} \neq \Gamma^{(4)}_{\rm BS}$

- New divergences that have to do with a loop-expanded four-point function $\Gamma^{(4)}_L$ rather than with the BS equation.
- Asymmetrical treatment of the bare vertices.

2. 1PI Flows

PROPERTIES OF 1PI FLOWS

Infinite hierarchy of equations for the vertex functions:

$$\partial_\kappa \Gamma^{(2)}_\kappa(p) = -\frac{1}{2}\int_{q<\Lambda_{\rm uv}} \partial_\kappa R_\kappa(q)\, G^2_\kappa(q)\, \Gamma^{(4)}_\kappa(q,p)$$

$$\dots$$

Two crucial properties:

- no reference to the parameters of the model;
- the flow is UV finite (thanks to $\partial_\kappa R_\kappa$).

$\Rightarrow$ access to the renormalized vertex functions from the start!

In what follows we investigate the possibility to reformulate the 2PI framework in terms of flow equations.

Our two guidelines will be:

- removing any reference to the parameters of the model.
- making the flow UV finite thanks to $\partial_\kappa R_\kappa$ or power counting.

This will provide a renormalized 2PI framework from the start.

This should also clarify the need for various approximations to the four-point function.

3. Flow Reformulation of the 2PI Framework

DEFORMED 2PI EFFECTIVE ACTION

As usual $S + \int_q \varphi(-q) R_\kappa(q) \varphi(q)$. The modified 2PI effective action is:

$$\Gamma[G] = \frac{1}{2}\int_{p<\Lambda_{\rm uv}} \log G^{-1}(p) + \frac{1}{2}\int_{p<\Lambda_{\rm uv}} (p^2 + m_{\rm b}^2 + R_\kappa(q))\, G(p) + \Phi[G]$$

The corresponding gap equation reads

$$G_\kappa^{-1}(p) - R_\kappa(p) = p^2 + m_{\rm b}^2 + \left.\frac{2\delta\Phi[G]}{\delta G(p)}\right|_{G_\kappa}$$

It still refers to $m_{\rm b}^2$.

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The corresponding gap equation reads

$$\Gamma^{(2)}_\kappa(p)\equiv G_\kappa^{-1}(p)-R_\kappa(p)=p^2+m_{\rm b}^2+\left.\frac{2\delta\Phi[G]}{\delta G(p)}\right|_{G_\kappa}$$

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The corresponding gap equation reads

$$\partial_\kappa \Gamma^{(2)}_\kappa(p) = \underbrace{\partial_\kappa(p^2 + m_{\rm b}^2)}_{=0} + \partial_\kappa \left.\frac{2\delta\Phi[G]}{\delta G(p)}\right|_{G_\kappa}$$

To remove the reference to $m_{\rm b}^2$, we promote $\Gamma^{(2)}_\kappa$ to a flowing quantity.

FLOW OF THE TWO-POINT FUNCTION

With the help of the chain rule, this becomes:

$$\partial_\kappa \Gamma^{(2)}_\kappa(p) = \frac{1}{2}\int_{q<\Lambda_{\rm uv}} \underbrace{\left.\frac{4\delta^2\Phi[G]}{\delta G(p)\delta G(q)}\right|_{G_\kappa}}_{\mathcal{I}[G_\kappa](p,q)\equiv\mathcal{I}_\kappa(p,q)} \partial_\kappa G_\kappa(q)$$

$$= -\frac{1}{2}\int_{q<\Lambda_{\rm uv}} \mathcal{I}_\kappa(p,q)\, G^2_\kappa(q)\, (\partial_\kappa \Gamma^{(2)}_\kappa(p) + \partial_\kappa R_\kappa(q))$$

This equation is not finite however:

	$\int_q$	$\mathcal{I}_\kappa$	G^2_κ	$\partial_\kappa \Gamma^{(2)}_\kappa$
δ	4	0	-4	0

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FLOW OF THE TWO-POINT FUNCTION

We solve formally for $\partial_\kappa \Gamma_\kappa^{(2)}$: [Blaizot, Pawlowski, UR, 2011]

$$\partial_\kappa \Gamma_\kappa^{(2)}(p) = -\frac{1}{2}\int_{q<\Lambda_{\rm uv}} \Gamma_\kappa^{(4)}(p,q)\, G_\kappa^2(q)\, \partial_\kappa R_\kappa(q)$$

with (Bethe-Salpeter equation)

$$\Gamma_\kappa^{(4)}(p,q) = \mathcal{I}_\kappa(p,q) - \frac{1}{2}\int_{r<\Lambda_{\rm uv}} \mathcal{I}_\kappa(p,r)\, G_\kappa^2(r)\, \Gamma_\kappa^{(4)}(r,q)$$

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As anticipated, the BS relation between $\Gamma^{(4)}_\kappa$ and $\mathcal{I}_\kappa$ plays a key role.

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The flow equation is now finite thanks to $\partial_\kappa R_\kappa$.

As anticipated, the BS relation between $\Gamma^{(4)}_\kappa$ and $\mathcal{I}_\kappa$ plays a key role.

However, $\Gamma^{(4)}_\kappa$ is still given in terms of UV regulated diagrams contained in $\mathcal{I}[G] \propto \delta^2\Phi[G,\lambda_{\rm b}]/\delta G^2$ which depends on $\lambda_{\rm b}$.

FLOW OF THE FOUR-POINT FUNCTION

Once more, we promote $\Gamma^{(4)}_\kappa$ to a flowing quantity:

$$\partial_\kappa\Gamma^{(4)}_\kappa(q,p)=\partial_\kappa\mathcal{I}_\kappa(q,p)\;-\;\frac{1}{2}\int_{r<\Lambda_{\rm uv}}\Gamma^{(4)}_\kappa(q,r)\,G^2_\kappa(r)\,\partial_\kappa\mathcal{I}_\kappa(r,p)$$

$$-\;\frac{1}{2}\int_{r<\Lambda_{\rm uv}}\Gamma^{(4)}_\kappa(q,r)\,\partial_\kappa G^2_\kappa(r)\,\mathcal{I}_\kappa(r,p)$$

$$-\;\frac{1}{2}\int_{r<\Lambda_{\rm uv}}\partial_\kappa\Gamma^{(4)}_\kappa(q,r)\,G^2_\kappa(r)\,\mathcal{I}_\kappa(r,p)$$

This equation is not finite however:

	$\int_q$	$\mathcal{I}_\kappa$	G^2_κ	$\Gamma^{(4)}_\kappa$	$\partial_\kappa\mathcal{I}_\kappa$	$\partial_\kappa G^2_\kappa$	$\partial_\kappa\Gamma^{(4)}_\kappa$
δ	4	0	-4	0	-2	-6	0

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$$- \frac{1}{2}\underbrace{\int_{r<\Lambda_{\rm uv}} \Gamma^{(4)}_\kappa(q,r)\, \partial_\kappa G^2_\kappa(r)\, \mathcal{I}_\kappa(r,p)}_{\delta = 4+0-6+0=-2}$$

$$- \frac{1}{2}\underbrace{\int_{r<\Lambda_{\rm uv}} \partial_\kappa \Gamma^{(4)}_\kappa(q,r)\, G^2_\kappa(r)\, \mathcal{I}_\kappa(r,p)}_{\delta = 4+0-4+0=0}$$

This equation is not finite however:

	$\int_q$	$\mathcal{I}_\kappa$	G^2_κ	$\Gamma^{(4)}_\kappa$	$\partial_\kappa \mathcal{I}_\kappa$	$\partial_\kappa G^2_\kappa$	$\partial_\kappa \Gamma^{(4)}_\kappa$
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[Blaizot, Pawlowski, UR, (2021)]

$$
\begin{aligned}
\partial_\kappa \Gamma_\kappa^{(4)}(p,q) &= \partial_\kappa \mathcal{I}_\kappa(p,q) \\
&- \frac{1}{2}\int_{r<\Lambda_{\rm uv}} \Gamma_\kappa^{(4)}(p,r)\,\partial_\kappa G_\kappa^2(r)\,\Gamma_\kappa^{(4)}(r,q) \\
&- \frac{1}{2}\int_{r<\Lambda_{\rm uv}} \partial_\kappa \mathcal{I}_\kappa(p,r)\,G_\kappa^2(r)\,\Gamma_\kappa^{(4)}(r,q) \\
&- \frac{1}{2}\int_{r<\Lambda_{\rm uv}} \Gamma_\kappa^{(4)}(p,r)\,G_\kappa^2(r)\,\partial_\kappa \mathcal{I}_\kappa(r,q) \\
&+ \frac{1}{4}\iint_{r,s<\Lambda_{\rm uv}} \Gamma_\kappa^{(4)}(p,r)\,G_\kappa^2(r)\,\partial_\kappa \mathcal{I}_\kappa(r,s)\,G_\kappa^2(s)\,\Gamma_\kappa^{(4)}(s,q)
\end{aligned}
$$

FLOW OF THE FOUR-POINT FUNCTION

We solve formally for $\partial_\kappa \Gamma_\kappa^{(4)}$:

[Blaizot, Pawlowski, UR, (2021)]

	$\int_q$	G_κ^2	$\Gamma_\kappa^{(4)}$	$\partial_\kappa \mathcal{I}_\kappa$	$\partial_\kappa G_\kappa^2$
δ	4	-4	0	-2	-6

$$\begin{aligned}
\partial_\kappa \Gamma_\kappa^{(4)}(p,q) &= \partial_\kappa \mathcal{I}_\kappa(p,q) \\
&- \frac{1}{2}\int_{r<\Lambda_{\rm uv}} \Gamma_\kappa^{(4)}(p,r)\,\partial_\kappa G_\kappa^2(r)\,\Gamma_\kappa^{(4)}(r,q) \to \delta = 4+0-6+0=-2 \\
&- \frac{1}{2}\int_{r<\Lambda_{\rm uv}} \partial_\kappa \mathcal{I}_\kappa(p,r)\, G_\kappa^2(r)\,\Gamma_\kappa^{(4)}(r,q) \to \delta = 4-2-4+0=-2 \\
&- \frac{1}{2}\int_{r<\Lambda_{\rm uv}} \Gamma_\kappa^{(4)}(p,r)\, G_\kappa^2(r)\,\partial_\kappa \mathcal{I}_\kappa(r,q) \to \delta = 4+0-4-2=-2 \\
&+ \frac{1}{4}\iint_{r,s<\Lambda_{\rm uv}} \Gamma_\kappa^{(4)}(p,r)\, G_\kappa^2(r)\,\partial_\kappa \mathcal{I}_\kappa(r,s)\, G_\kappa^2(s)\,\Gamma_\kappa^{(4)}(s,q) \\
&\qquad \to \delta = 4+4+0-4-2-4+0=-2
\end{aligned}$$

This equation is now finite by power counting.

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$$
\begin{aligned}
\partial_\kappa \Gamma_\kappa^{(4)}(p,q) \; = \; & \partial_\kappa \mathcal{I}_\kappa(p,q) \\
& - \frac{1}{2}\int_{r<\Lambda_{\rm uv}} \Gamma_\kappa^{(4)}(p,r)\, \partial_\kappa G_\kappa^2(r)\, \Gamma_\kappa^{(4)}(r,q) \to \delta = 4+0-6+0=-2 \\
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& + \frac{1}{4}\iint_{r,s<\Lambda_{\rm uv}} \Gamma_\kappa^{(4)}(p,r)\, G_\kappa^2(r)\, \partial_\kappa \mathcal{I}_\kappa(r,s)\, G_\kappa^2(s)\, \Gamma_\kappa^{(4)}(s,q) \\
& \qquad \to \delta = 4+4+0-4-2-4+0=-2
\end{aligned}
$$

This equation is now finite by power counting.

However, $\partial_\kappa \mathcal{I}_\kappa$ still given in terms of diagrams and thus in terms of $\lambda_{\rm b}$.

FLOW OF THE KERNEL

$$\partial_\kappa \mathcal{I}_\kappa(q,p) = \int_{r<\Lambda_{\rm uv}} \partial_\kappa G_\kappa(r) \, \left.\frac{\delta \mathcal{I}[G](q,p)}{\delta G(r)}\right|_{G=G_\kappa}$$

From here, two different strategies are possible:

- descending equations for derivatives of the kernel [Carrington et al]
- four-skeleton expansion of the flow of the kernel [This work]

DESCENDING EQUATIONS

[Carrington et al]

$$\partial_\kappa \left.\frac{\delta\mathcal{I}(q,p)}{\delta G(r_1)}\right|_{G=G_\kappa} = \int_{r_2<\Lambda_{\rm uv}} \partial_\kappa G_\kappa(r_2)\, \left.\frac{\delta^2\mathcal{I}(q,p)}{\delta G(r_2)\delta G(r_1)}\right|_{G=G_\kappa}$$

DESCENDING EQUATIONS

[Carrington et al]

$$\partial_\kappa \left. \frac{\delta^k \mathcal{I}(q,p)}{\delta G(r_k)\cdots\delta G(r_1)} \right|_{G=G_\kappa} = \int_{r_{k+1}<\Lambda_{\rm uv}} \partial_\kappa G_\kappa(r_{k+1}) \left. \frac{\delta^{k+1}\mathcal{I}(q,p)}{\delta G(r_{k+1})\cdots\delta G(r_1)} \right|_{G=G_\kappa}$$

DESCENDING EQUATIONS

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If $\Phi[G]$ contains a finite number of loops, so does $\mathcal{I}[G]$ and then the procedure stops eventually.

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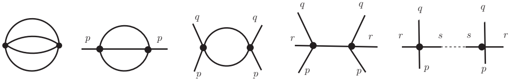
If $\Phi[G]$ contains a finite number of loops, so does $\mathcal{I}[G]$ and then the procedure stops eventually.

Problem: $\delta^k\mathcal{I}[G]/\delta G^k$ are not simple to initialize since not 1PI.



FOUR-SKELETON EXPANSION

[This work]

$$\partial_\kappa \mathcal{I}_\kappa(q,p) = \int_{r<\Lambda_{\rm uv}} \partial_\kappa G_\kappa(r)\; \frac{\delta \mathcal{I}[G](q,p)}{\delta G(r)}\bigg|_{G=G_\kappa}$$

FOUR-SKELETON EXPANSION

[This work]

$$\partial_\kappa \mathcal{I}_\kappa(q,p) = \sum_{\text{diagrams } \mathcal{D}} \int_{r<\Lambda_{\rm uv}} \partial_\kappa G_\kappa(r)\, \mathcal{D}[G_\kappa;\lambda_{\rm b}](q,p,r)$$

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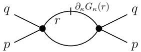
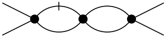
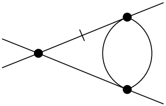
4-skeletons are defined as diagrams with no 4-point subgraphs.



4-skeleton



no 4-skeleton



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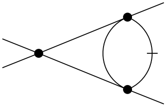

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FOUR-SKELETON EXPANSION

[This work]

$$\partial_\kappa \mathcal{I}_\kappa(q,p) = \sum_{4-\text{skeletons } \mathcal{S}} \int_{r<\Lambda_{\rm uv}} \partial_\kappa G_\kappa(r)\, \mathcal{S}[G_\kappa; \Gamma^{(4)}_\kappa](q,p,r)$$

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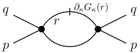

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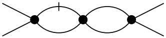

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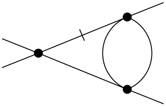

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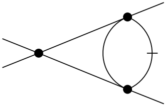

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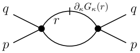
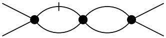
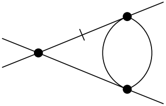


4-skeleton

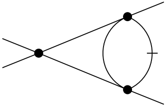
4-skeletons have no subdivergences and are then completely finite.



no 4-skeleton



no 4-skeleton



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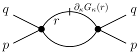

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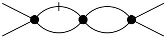

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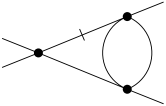

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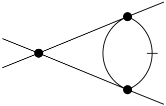

4-skeleton

We thus arrive at a flow equation for the kernel which is finite and makes no reference to the bare parameters.

2PI FLOW

Reformulation of the exact 2PI effective action in terms of finite flow equations that make no reference to the parameters:

$$\partial_\kappa \Gamma^{(2)}_\kappa(p) = -\frac{1}{2}\int_{q<\Lambda_{\rm uv}} \Gamma^{(4)}_\kappa(p,q)\, G^2_\kappa(q)\, \partial_\kappa R_\kappa(q)$$

$$\partial_\kappa \Gamma^{(4)}_\kappa(p,q) = \partial_\kappa \mathcal{I}_\kappa(p,q) - \frac{1}{2}\int_{r<\Lambda_{\rm uv}} \Gamma^{(4)}_\kappa(p,r)\, \partial_\kappa G^2_\kappa(r)\, \Gamma^{(4)}_\kappa(r,q) + \dots$$

$$\partial_\kappa \mathcal{I}_\kappa(q,p) = \sum_{\mathcal{S}} \int_{r<\Lambda_{\rm uv}} \partial_\kappa G_\kappa(r)\, \mathcal{S}[G_\kappa;\Gamma^{(4)}_\kappa](q,p,r)$$

LOOP TRUNCATED 2PI FLOW

Reformulation of the L-loop truncated 2PI effective action in terms of finite flow equations that make no reference to the parameters:

$$\partial_\kappa \Gamma^{(2)}_\kappa(p) = -\frac{1}{2}\int_{q<\Lambda_{\rm uv}} \Gamma^{(4)}_\kappa(p,q)\, G^2_\kappa(q)\, \partial_\kappa R_\kappa(q)$$

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$$\partial_\kappa \mathcal{I}_\kappa(q,p) = \sum_{\mathcal{S}}\int_{r<\Lambda_{\rm uv}} \partial_\kappa G_\kappa(r)\, \mathcal{S}[G_\kappa;\{\Gamma^{(4)}_{L',\kappa}; L'\leq L-3\}](r,q,p)\Big|_{L-2}$$

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$$\partial_\kappa \mathcal{I}_\kappa(q,p) = \sum_{\mathcal{S}} \int_{r<\Lambda_{\rm uv}} \partial_\kappa G_\kappa(r)\, \mathcal{S}[G_\kappa;\{\Gamma^{(4)}_{L',\kappa}; L'\leq L-3\}](r,q,p)\Big|_{L-2}$$

$$\partial_\kappa \Gamma^{(4)}_{L',\kappa}(p_i) = \sum_{\hat{\mathcal{S}}} \int_{r<\Lambda_{\rm uv}} \partial_\kappa G_\kappa(r)\, \hat{\mathcal{S}}[G_\kappa;\{\Gamma^{(4)}_{L'',\kappa}; L''\leq L'-1\}](r,p_i)\Big|_{L'}$$

4. 2PI Renormalization using Flows

RENORMALIZED 2PI FRAMEWORK

$\Lambda_{\rm uv} \gg \Lambda \gg \Lambda_{\rm phys}$: momenta are either large or have a large mass Λ.

$\Rightarrow$ Loop diagrams are expandable w.r.t. the external momenta:

$$\Gamma^{(2)}_\Lambda(p) \to m^2_\Lambda + Z_\Lambda p^2\,, \quad \Gamma^{(4)}_\Lambda \to \lambda_{{\rm BS},\Lambda}\,, \quad \Gamma^{(4)}_{L,\Lambda} \to \lambda_{L,\Lambda}$$

Moreover, since $\kappa < \Lambda \ll \Lambda_{\rm uv}$, the cut-off $\Lambda_{\rm uv}$ plays no role in the flow:

$$\Gamma^{(2)}_{\kappa=0}(p) = m^2_\Lambda + Z_\Lambda p^2 + \int_\Lambda^0 d\kappa \left[-\frac{1}{2}\int_{q<\cancel{\Lambda_{\rm uv}}} \partial_\kappa R_\kappa(q)\, G^2_\kappa(q)\, \Gamma^{(4)}_\kappa(q,p)\right]$$

$$\ldots$$

$\Rightarrow$ 2PI renormalized framework from the start!

BACK TO THE DIAGRAMS

$\Lambda \gg \Lambda_{\rm uv} \gg \Lambda_{\rm phys}$: loop momenta are all very massive.

$\Rightarrow$ Loop diagrams are completely suppressed as $\Lambda \to \infty$:

$$\Gamma^{(2)}_\Lambda(p) \to Zm_{\rm b}^2 + Zp^2\,, \quad \Gamma^{(4)}_\Lambda \to Z^2\lambda_{\rm b,BS}\,, \quad \Gamma^{(4)}_{L,\Lambda} \to Z^2\lambda_{\rm b,L}$$

Note that $\Lambda_{\rm uv}$ cannot be removed in the range $\Lambda_{\rm uv} < \kappa$:

$$\Gamma^{(2)}_{\kappa=0}(p) = Zm_{\rm b}^2 + Zp^2 + \int_{\Lambda\gg\Lambda_{\rm uv}}^0 d\kappa \left[-\frac{1}{2}\int_{q<\Lambda_{\rm uv}} \partial_\kappa R_\kappa(q)\, G^2_\kappa(q)\, \Gamma^{(4)}_\kappa(q,p)\right]$$
$$\ldots$$

$\Rightarrow$ Fancy rewriting of the diagrammatic 2PI framework. Useful?

MAPPING OUT DIVERGENCES

The flow separates the overall divergences from the subdivergences of a given vertex function and dictates how the later appear as overall divergences of other vertex functions:

$$\Gamma^{(2)}_\kappa(p) = Zp^2 + Zm_{\rm b}^2 + \int_{\Lambda\gg\Lambda_{\rm uv}}^0 d\kappa\, \mathcal{F}^{(2)}_\kappa\big[\Gamma^{(2)}_\kappa, \Gamma^{(4)}_\kappa\big](p)\Big|_{\Lambda_{\rm uv}}$$

$$\Gamma^{(4)}_\kappa(p,q) = Z^2\lambda_{\rm b,BS} + \int_{\Lambda\gg\Lambda_{\rm uv}}^0 d\kappa\, \mathcal{F}^{(4)}_\kappa\big[\Gamma^{(2)}_\kappa, \Gamma^{(4)}_\kappa, \Gamma^{(4)}_{L-3,\kappa}, \dots, \Gamma^{(4)}_{1,\kappa}\big](p,q)\Big|_{\Lambda_{\rm uv}}$$

$$\Gamma^{(4)}_{L',\kappa}(p_i) = Z^2\lambda_{\rm b,L'} + \int_{\Lambda\gg\Lambda_{\rm uv}}^0 d\kappa\, \mathcal{F}^{(4)}_{L',\kappa}\big[\Gamma^{(2)}_\kappa, \Gamma^{(4)}_{L'-1,\kappa}, \dots, \Gamma^{(4)}_{1,\kappa}\big](p_i)\Big|_{\Lambda_{\rm uv}}$$

$\Rightarrow$ complete map of UV divergences in the 2PI framework!

RENORMALIZED 2PI EFFECTIVE ACTION

With these intial conditions at $\Lambda \gg \Lambda_{\rm uv}$, the flow integrates into

$$\frac{1}{2}\int_{p<\Lambda_{\rm uv}} \log G^{-1}(p) + \frac{1}{2}\int_{p<\Lambda_{\rm uv}} (Z p^2 + Z m_{\rm b}^2)\, G(p) + Z^2 \lambda_{\rm b,BS} \Phi^{(2)}[G] + (Z^2\lambda_{\rm b,L})^2 \Phi^{(3)}[G] + \cdots + (Z^2\lambda_{\rm b,L})^{L-1}\Phi^{(L)}[G]\Big|_L$$

instead of the original

$$\frac{1}{2}\int_{p<\Lambda_{\rm uv}} \log G^{-1}(p) + \frac{1}{2}\int_{p<\Lambda_{\rm uv}} (p^2 + m_{\rm b}^2)\, G(p) + \lambda_{\rm b}\Phi^{(2)}[G] + \lambda_{\rm b}^2\Phi^{(3)}[G] + \cdots + \lambda_{\rm b}^{L-1}\Phi^{(L)}[G]$$

$\Rightarrow$ All order renormalized expression for the 2PI effective action!

CONCLUSIONS

- We have reformulated the 2PI framework in terms of finite flow equations that do not make reference to the parameters.

- This opens the way to new 2PI/FRG approximations schemes.

- In the case of the standard 2PI loop expansion, it provides:
  - a renormalized 2PI framework from the start;
  - a map of the UV divergences among the various vertex functions;
  - a renormalized expression for the 2PI effective action.

- Strategy applicable to other non-perturbative approaches.

THANK YOU!