# Operator Product Expansion

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based on joint work with M. Fröb, J. Holland and Ch. Kopper

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# QFT is hard

*"In those years a theoretical physicist of modest talent could harvest results of great interest, today persons of great talent produce results of modest interest"* [F. Hund]

The operator product expansion (OPE) expresses a decoupling of UV and IR dynamics in quantum/statistical field theory (QFT)

# OPE history

- 60s: OPE introduced by Wilson, later arguments by Zimmermann
- 70s: conformal bootstrap ideas [Polyakov, Mack, Gatto et al., Schroer et al., ...].
- 80s: CFTs in $d = 2$ [Belavin et al., ...].
- 90s: vertex operator algebras [Borcherds, Frenkel, Kac, ...] $\Rightarrow$ mathematics
- 00s-: numerical bootstrap in $d > 2$ [Dolan, Osborne, Penedones, Poland, Rychkov, Simmons-Duffin, ...]
- 10s-: lattice QFT [Monahan, ...]

Us: FLOW EQUATIONS, MATHEMATICAL BOUNDS [SH, Fröb, Holland, Kopper, ...]

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- OPE:

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- OPE coefficients are independent of $\Psi$ and generally covariant.
- OPE satisfies associativity law.

# When is OPE a good approximation?

If $\max|x_i - x_j| \lesssim$ **scale of state** $|\Psi\rangle$

- Thermal state at temperature $\beta^{-1}$: scale = $\beta$
- Bunch-Davies state in deSitter: scale $=$ Hubble radius
- Unruh state on black hole spacetime: scale $=$ Schwarzschild radius
- Particle state e.g. proton $|p\rangle$: scale $= |p|$
- ...

# Conformal field theory CFT

- operators form multiplets transforming under conformal group.
- each multiplet contains one primary field $\mathcal{O}_A$.
- OPE:

$$\mathcal{O}_A(x_1)\mathcal{O}_B(x_2) = \sum_{\text{primary } C} \frac{\lambda_{AB}^C}{|x_1 - x_2|^{\Delta_A+\Delta_B-\Delta_C}} \mathcal{P}(x_1 - x_2, \partial)\mathcal{O}_C(x_2)$$

where $\mathcal{P} = \mathcal{P}_{AB}^C$ is kinematical.

# CFT

In CFTs, the OPE is determined by representation theory of conformal group ("kinematical") $+$

## CFT data

- structure constants $\lambda_{AB}^{C}$
- dimensions $\Delta_A$.

Associativity + positivity put very stringent conditions on these data.

$\Longrightarrow$ conformal bootstrap

## This talk

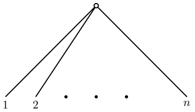
1. Do the expected properties of OPE hold?
2. What is the error term in the OPE?
3. How to get OPE coefficients?

# Action principle

To write down the action principle, use graphical notation. I draw an OPE coefficient

$$\mathcal{C}^{B}_{A_1 \dots A_n}(x_1, \dots, x_n)$$

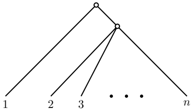
as




I draw a concatenation of OPE coefficients

$$\mathcal{C}^{B}_{A_1 C}(x_1, x_n)\mathcal{C}^{C}_{A_2 \dots A_n}(x_2, \dots, x_n)$$

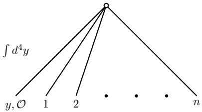
as




Attention: None of these diagrams is a “Feynman graph”!

# Action principle

I also write




where

- $\mathcal{O}$ denotes the “deformation”
- $\int d^4y =$ integral over $\{|y - x_n| < L\}$.
- $L =$ length scale that is part of the definition of the theory.

# Action principle

There is a kind of "action principle" for OPE coefficients if we "deform" $S \to S + g \int \mathcal{O}$:

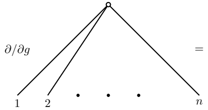



Figure: Left side: $\partial_g \mathcal{C}^B_{A_1 \dots A_n}(x_1, \dots, x_n)$

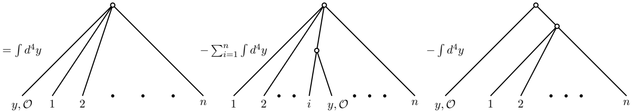



Figure: Right side: concatenations of OPE coefficients, e.g. the rightmost tree $\sum_C \mathcal{C}^C_{A_1 \dots A_n}(x_1, \dots, x_n) \mathcal{C}^B_{\mathcal{O}C}(y, x_n)$

# Action principle [SH, Holland 2017 $\phi^4$, Fröb, Holland 2018 QCD, Fröb 2020 curved space]

## Theorem

*Under* $S \to S + g \int \mathcal{O}$:

$$\partial_g \mathcal{C}_{A_1 \dots A_N}^B(x_1, \dots, x_N) = -\int_{|y-x_N|<L} \mathrm{d}^4 y \Big[ \mathcal{C}_{\mathcal{O}A_1\dots A_N}^B(y, x_1, \dots, x_N) - \sum_{i=1}^N \sum_{[C]\le[A_i]} \mathcal{C}_{\mathcal{O}A_i}^C(y, x_i)\, \mathcal{C}_{A_1\dots\widehat{A_i}\, C \dots A_N}^B(x_1, \dots, x_N) - \sum_{[C]<[B]} \mathcal{C}_{A_1\dots A_N}^C(x_1, \dots, x_N)\, \mathcal{C}_{\mathcal{O}C}^B(y, x_N) \Big] .$$

- Can compute OPE coefficients to any perturbation order by iteration.
- State independence obvious.
- $L \to L'$ equivalent to

$$\mathcal{O}_A \to \sum Z_A^B(g, \tau) \cdot \mathcal{O}_B \tag{0.1}$$

and $g \to g' = g(g, \tau)$. $\Rightarrow$ RG equations! ($\tau = \log L/L'$ = RG “time”).

# Proofs:

- The proof of this theorem as well as most other results in this talk uses a variant of the Wilson-Wegner-Polchinski-Wetterich RG flow equations. However, I will mostly only indicate the final results/bounds after removal of cutoffs, which no longer refer to this equation.
- Proof is perturbative but equation should hold non-perturbatively.
- Proofs exist for $\phi^4$, QCD
- For gauge theories suitable modifications involving BRST have to be taken into account.
- All statements refer to Euclidean signature and flat metric but some can be generalized to curved space.

# Exactly marginal flows of CFT [Hollands 2018, Behan 2018]

- For flows of CFTs, my action principle gives a closed system of ODEs for CFT data $\lambda = \{\lambda_{AB}^{C}(g)\}$ and $\Delta = \{\Delta_A(g)\}$.
- Dynamical system for CFT data:

$$\frac{d}{dg}\lambda = f(\Delta)\lambda\lambda$$
$$\frac{d}{dg}\Delta = g(\Delta)\lambda$$

where $f, g$ are explicit kinematical functions that only depend on $6j$-symbols of the conformal group

- We need initial condition.

# Outline

- OPE factorisation

# Associativity [SH, Holland 2016]

## Theorem

*In $\phi^4$-theory, any arbitrary but fixed loop order:*

$$\mathcal{C}_{A_1\ldots A_N}^{B}(x_1,\ldots,x_N) = \sum_C \mathcal{C}_{A_1\ldots A_M}^{C}(x_1,\ldots,x_M)\mathcal{C}_{C A_{M+1}\ldots A_N}^{B}(x_M,\ldots,x_N)$$

*holds on the domain* $\xi \equiv \frac{\max_{1\leq i\leq M} |x_i - x_M|}{\min_{M<j\leq N} |x_j - x_M|} < 1$. *(Sum over $C$ abs. convergent !)*

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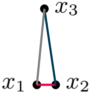
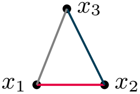
For $N = 3$: $\xi = \frac{|x_1 - x_2|}{|x_2 - x_3|} < 1$




for $\xi \ll 1$

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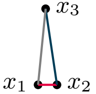


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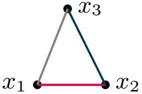


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This shows associativity really holds!

- Bound on remainder
- Justification of "action principle"

# Quantitative bound

## Theorem

*Up to any perturbation order* $r \in \mathbb{N}$ *the bound*

$$\left| \textit{Remainder in associativity} \right|$$

$$\leq \frac{K_r \xi^{D+1} \max\limits_{N \leq v < N} |x_i - x_n|^{[B]}}{\prod_{v=1}^{M} \min\limits_{1 \leq w \leq M, w \neq v} |x_v - x_w|^{[A_v]+\delta} \prod_{i=M+1}^{N} \min\limits_{M < j \leq N, i \neq j} |x_i - x_j|^{[A_i]+\delta}}$$

*holds for some* $\delta > 0$ *and where*

$$\xi := \frac{\max_{1 \leq i \leq M} |x_i - x_M|}{\min_{M < j \leq N} |x_j - x_M|}$$

*and where* $K_r$ *is a constant which does not depend on* $D$*. (Here* $[A] =$ *dim. of op. in free theory).*

# Bound on OPE remainder I [SH, Kopper 2015, ...]

## Theorem

*At any perturbation order $r$ and for any $D \in \mathbb{N}$,*

$$\Big|\overbrace{\Big\langle \Big( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\dim[B] \le D} \mathcal{C}^{B}_{A_1 \ldots A_N}(x_1, \ldots, x_N)\, \mathcal{O}_B(x_N) \Big) \underbrace{\hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n)}_{\text{Spectator fields}} \Big\rangle}^{\text{OPE-Remainder}}\Big|$$

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$$\le \frac{M^{n-1}}{\sqrt{D!}} \frac{\Big(K M \max_{1 \le i \le N} |x_i - x_N|\Big)^{D+1}}{\min_{1 \le i < j \le N} |x_i - x_j|^{\sum_i \dim[A_i] + 1}} \cdot \sup\left(1, \frac{|P|}{\sup(m, \kappa)}\right)^{(D+2)(r+5)}$$

- $M = \begin{cases} m & \text{for } m > 0 \\ \mu & \text{for } m = 0 \end{cases}$ mass or renormalization scale

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- $|P| = \sup_i |p_i|$: maximal momentum of spectators
- $\kappa := \inf(\mu, \varepsilon)$, where $\varepsilon = \min_{I\subset\{1,\ldots,n\}} |\sum_I p_i|$
  $\varepsilon$: distance of $(p_1,\ldots,p_n)$ to "exceptional" configurations

# Conclusions from bound on OPE remainder

$$\text{“OPE remainder”} \leq \frac{M^{n-1}}{\sqrt{D!}} \; \frac{\left(KM \max\limits_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min\limits_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

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   - maximal distance of points $x_i$ from reference point $x_N$ is large
   - ratio of max. and min. distances is large, e.g. for $N = 3$

# Bound on OPE remainder II SH, Kopper, Holland 2017

Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x)\varphi(x)\,\mathrm{d}^4x$.

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$M$: mass for $m > 0$ or renormalization scale $\mu$ for massless fields

$\|\hat{f}\|_s := \sup_{p\in\mathbb{R}^4}\left|(p^2 + M^2)^s\hat{f}(p)\right|$ (Schwartz norm)

# Bound on OPE remainder II SH, Kopper, Holland 2017

Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x)\varphi(x)\,\mathrm{d}^4x$.

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2. Let $\hat{f}_i(p) = 0$ for $|p| > |P|$: Bound vanishes as $D \to \infty$
   $\Rightarrow$ OPE converges at any finite distances!

# Conclusions & Outlook

1. QFT in CST is best formulated in terms of algebraic relations + states
2. The OPE converges at finite distances in perturbation theory.
3. The OPE factorizes (associativity) in perturbation theory.
4. The OPE satisfies an action principle which is also useful for calculations

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## Possible Generalizations

- Curved manifolds?
- Lorentzian signature?
- Light cone expansion?