ARNOLD SOMMERFELD CENTER FOR THEORETICAL PHYSICS

Phase transitions in Group field theory:

Towards the phase structure of the complete Lorentzian Barrett-Crane model

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Outline

- General motivation: Landau-Ginzburg mean-field method
- Group field theory
- LG theory applied to Group field theory
- Conclusions

What is LG theory? How does it relate to the RG?

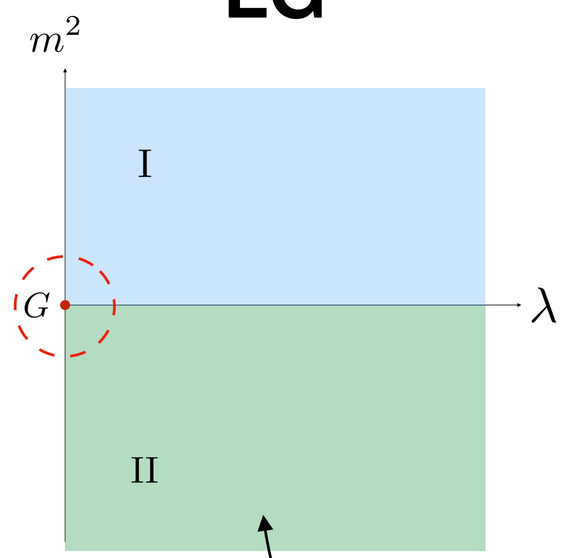
- statistical field theory method to describe **1st and 2nd order phase transitions at mean-field level**

- LG mean-field analysis **clarifies phase structure** of local field theories (coarse account)

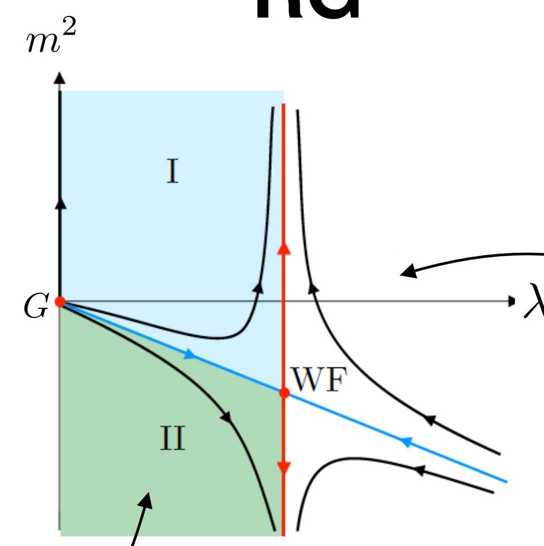
- transition to condensate phase with **nontrivial VEV** (non-perturbative vacuum) $\langle\varphi\rangle \neq 0$

e.g. $$S[\varphi] = \frac{1}{2}\int \mathrm{d}^3x\varphi(x)(-\Delta + m^2)\varphi(x) + \frac{\lambda}{4!}\int \mathrm{d}^3x\varphi(x)^4$$

LG



RG



LG theory studies fluctuations around the Gaussian fixed point (free theory)

quasi-Gaussian distribution

involved RG studies go beyond small fluctuations

generally non-Gaussian distribution

$\langle\varphi\rangle \neq 0$

→ coarse account of the phase diagram

(good enough to start with…)

→ detailed account of the phase diagram

What is LG theory? Setup (I)

- start with **free energy functional** as an expansion in terms of even + odd powers of the local field (order parameter) and its gradient
- consider **truncation** of this functional assumed to be valid from mesoscale to macroscale
- details on microphysics encoded in couplings and order parameter
- **order parameter** features only **universal properties** of the system (dimension of space, symmetries of order parameter)

dimension of underlying space

$$S[\varphi] = \frac{1}{2}\int \mathrm{d}^d x \varphi(x)(-\Delta + m^2)\varphi(x) + \frac{\lambda}{4!}\int \mathrm{d}^d x \varphi(x)^4$$

free energy functional

order parameter

here: global symmetry $\mathbb{Z}_2$

goal: evaluate $Z = \int \mathcal{D}\varphi \mathrm{e}^{-S[\varphi]}$

partition function (sums all configs)

- allows to **control thermodynamic phases** of the system by studying long-range correlations of order parameter fluctuations over the distance ξ (**correlation length**)
- beyond ξ correlations decay exponentially; it diverges at criticality

What is LG theory? Setup (II)

1) determine **uniform field configurations** which are minimizers of the free energy functional

$$\varphi_0 = 0 \text{ if } m^2 > 0 \text{ and } \varphi_0 = \pm\sqrt{-\frac{m^2}{\lambda/3!}} \text{ if } m^2 < 0$$

2) study **correlations of fluctuations** around this uniform background (aka Gaussian approximation)

2. a) **linearize classical equations of motion** using fluctuations over the background $\varphi(\vec{x}) \to \varphi_0 + \delta\varphi(\vec{x})$

$$\longrightarrow \quad \left(-\Delta + m^2\right)\delta\varphi(\vec{x}) + \frac{\lambda\varphi_0^2}{2}\delta\varphi(\vec{x}) = 0$$

2. b) solve for **correlation function** $\left(-\Delta + m^2 + \frac{\lambda\varphi_0^2}{2}\right)C(\vec{x}) = \delta(\vec{x})$ (go to Fourier representation)

2. c) correlator is exponentially decaying function $\longrightarrow$ determine **correlation length** $\xi^2 = \frac{1}{-2m^2}, \quad m^2 < 0$

3) determine **domain of validity**

$\longrightarrow$ fluctuations and coupling should remain small then mean-field theory self-consistent

Ginzburg parameter $Q = \frac{\int_\xi \mathrm{d}^d x C(\vec{x})}{\int_\xi \mathrm{d}^d x \varphi_0^2}$

(measures strength of fluctuations)

$$Q \sim \lambda\xi^{4-d} \longrightarrow d_c = 4$$

critical dimension (of *flat space*) below which MFT seizes to be accurate; accounts for coarse picture of phase diagram (good enough)

Why bother in Group field theory? Applicable?

What is GFT?

Motivation via Matrix models for 2d gravity

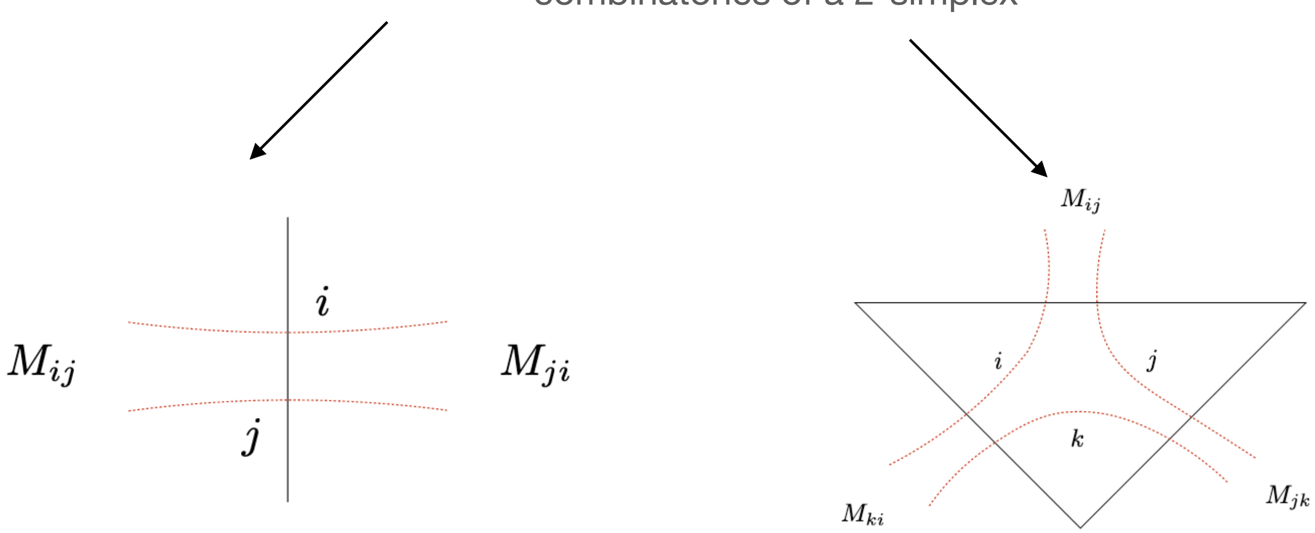
- Matrix models generate 2d random lattices
- at criticality they give rise to continuum geometries of dimension $d \leq 2$
- phase diagram of simple matrix models obtainable via diagonalization of matrices, computation of the partition function for large matrixes and then checking the (non-) analyticity of the free energy
- alternative route: phase structure via functional renormalization group
- continuum limit of simple matrix models agrees with that of 2d Liouville gravity

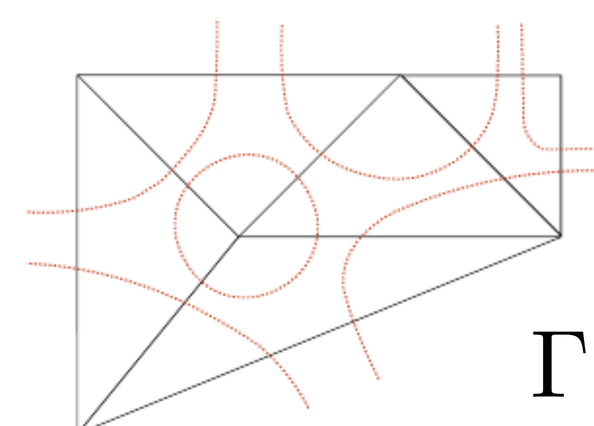
for example

$$S(M) = \frac{1}{2}\mathrm{tr}(M^2) + \frac{\lambda}{3!}\mathrm{tr}(M^3)$$

$$Z_{\mathrm{MM}} = \int \mathrm{d}M \mathrm{e}^{-S(M)} = \sum_{\Gamma} \frac{\lambda^{n(\Gamma)}}{\mathrm{sym}(\Gamma)} \mathcal{A}_{\Gamma}$$

combinatorics of a 2-simplex





Γ

graph

snapshot of a triangulation

Group Field Theory

[Oriti, Freidel, Rovelli, Livine, Gurau, Baratin,...]

classical theory (kinematics):

group field $\varphi(g_1, ..., g_r) : G^r \to \mathbb{R}, \mathbb{C}, \quad \varphi \in L^2(G^r)$ (rank r; Lie group G)

parallel transport $g_I = \mathcal{P}e^{\int_{e_I} A}$ for $I = 1, ..., r$, link e_I, connection A

typically work with $r = 4$ and $G = \mathrm{SL}(2, \mathbb{C})$

$\longrightarrow$ to model 4-dimensional Lorentzian quantum geometries

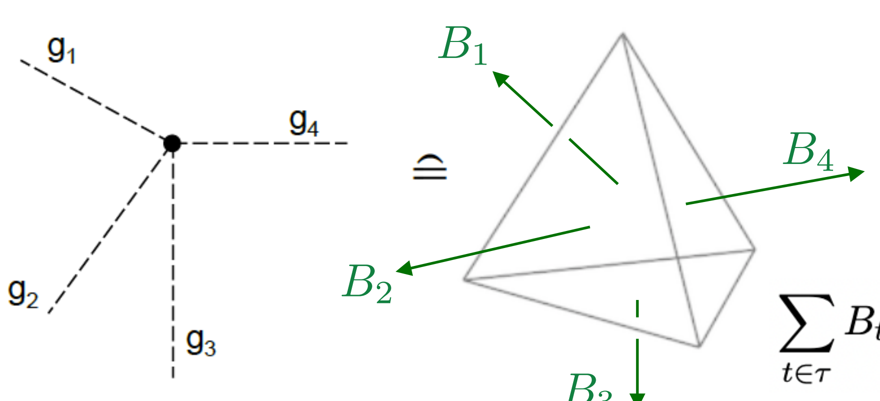
supplement field with invariance property: $\varphi(g_1, ..., g_r) = \varphi(g_1 h^{-1}, ..., g_r h^{-1}), \quad \forall h \in G$

(known as closure constraint)

$\longrightarrow$ phase space: $T^*G^d \cong G^d \times \mathfrak{g}^d$

$\longrightarrow$ dual formulation: $\tilde{\varphi}(B_1, ..., B_4) = \int (\mathrm{d}g)^4 \varphi(g_1, g_2, g_3, g_4) \prod_{I=1}^{4} e_{g_I}(B_I)$

e.g. for **r=4** invariant field corresponds to a 3-simplex/tetrahedron τ



$\sum_{t \in \tau} B_t = 0$ (closure)

e.g. for $G = \mathrm{SL}(2, \mathbb{C})$ recover metric information:

$$(B_1, B_2, B_3) \longmapsto g_{ij} = e_{iA} e_j^A = \frac{1}{8\mathrm{tr}(B_1 B_2 B_3)} \varepsilon_i^{kl} \varepsilon_{jmn} \left(B_k^{AB} B_{AB}^m\right) \left(B_l^{CD} B_{CD}^n\right)$$

bivectors $B_i^{AB} = \varepsilon_i^{\ jk} e_j^A e_k^B$ $\quad e_i^A$ tetrads

Lorentz index $A \in \{0, 1, 2, 3\}$

$B_i \in \mathfrak{sl}(2, \mathbb{C}),\ i \in \{1, 2, 3\}$

Group Field Theory

classical theory (dynamics):

$$S_{\mathrm{GFT}} = \int (\mathrm{d}g)^r \bar{\varphi}(g_I)\mathcal{K}\varphi(g_I) + \mathcal{V}[\bar{\varphi}(g_I), \varphi(g_I)]$$

$\mathcal{K}$: kinetic operator, $\mathcal{V}$: non-linear and non-local interaction term

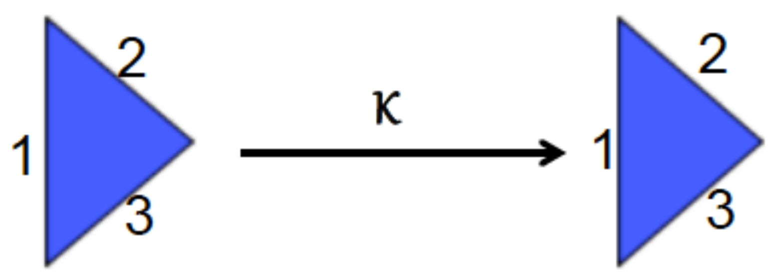
model specified by: G, dimension d, $\mathcal{K}$, $\mathcal{V}$ and symmetries of φ

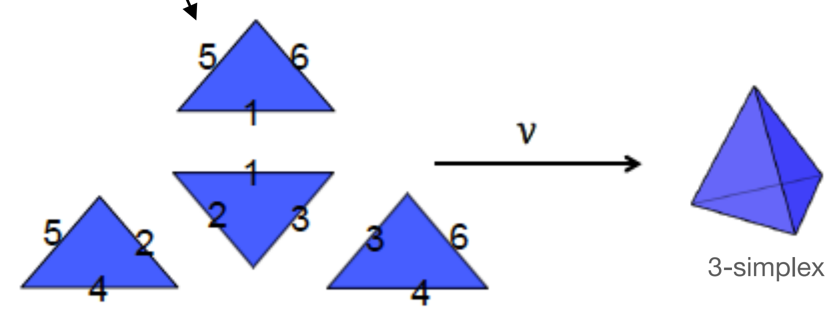
crucial feature of GFT models: combinatorially non-local interaction

example in 3d: (Boulatov)

$$S = \int (\mathrm{d}g)^3 |\varphi_{123}|^2 + \frac{\lambda}{4!}\int (\mathrm{d}g)^6 \varphi_{123}\varphi_{145}\varphi_{256}\varphi_{364} + \mathrm{c.c.}, \qquad \varphi_{123} \equiv \varphi(g_1, g_2, g_3)$$

combinatorics of a 3-simplex

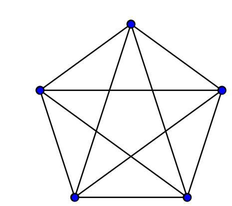




example in 4d: (Ooguri)

$$S = \int (\mathrm{d}g)^4 |\varphi_{1234}|^2 + \frac{\lambda}{5!}\int (\mathrm{d}g)^{10} \varphi_{1234}\varphi_{4567}\varphi_{7389}\varphi_{962(10)}\varphi_{(10)851} + \mathrm{c.c.}$$

combinatorics of a 4-simplex



4-simplex

$$Z_{\mathrm{GFT}} = \int \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\, e^{-S_{\mathrm{GFT}}[\varphi,\bar{\varphi}]} = \sum_{\Gamma} \frac{\lambda^{V_\Gamma}}{\mathrm{Sym}(\Gamma)}\mathcal{A}_\Gamma$$

GFT Feynman amplitude $\mathcal{A}_\Gamma$

graph corresponds to discrete geometries

Boulatov and Ooguri model provide GFT quantizations of BF-theory in 3d & 4d

→ **build models starting with BF-theory (TFT)**

BF-theory and Plebanski action

$$S[\omega, B, \mu] = \int \left[B_{IJ} \wedge F^{IJ}(\omega) + \frac{1}{2}\mu_{IJKL} B^{IJ} \wedge B^{KL} \right]$$

Lagrange multiplier

$\mathfrak{sl}(2,\mathbb{C})$ – valued 1-form

$\mathfrak{sl}(2,\mathbb{C})$ – valued 2-form

field strength: $F^{IJ}(\omega) = \mathrm{d}\omega^{IJ} + \omega^I_{\ J} \wedge \omega^{KJ}$

variation wrt μ $\longrightarrow$ "simplicity constraint" on B:

$$B^{IJ} \wedge B^{KL} = e\epsilon^{IJKL}, \quad e = \frac{1}{4!}\epsilon_{IJKL} B^{IJ} \wedge B^{KL}$$

solve for B $\longrightarrow$ solutions in two sectors: (1) topological sector vs.

(2) gravitational sector (Palatini)

first-order formulation

$$\longrightarrow S_{\text{Palatini}}[e, \omega] = \frac{1}{2} \int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}$$

tetrad field

$\delta_e S = 0 \rightarrow$ Einstein field eqns. $\quad \delta_\omega S = 0 \rightarrow$ 1st Cartan: $\mathrm{d}_\omega e^I + \omega^I_{\ J} \wedge e^J = 0$

spin connection

Quantization of BF-theory via GFT [Ooguri]

$$Z = \int \mathcal{D}\omega \mathcal{D}B \mathrm{e}^{iS[\omega,B]} = \int \mathcal{D}\omega \delta(F(\omega))$$ **(integral over flat connections, i.e. no local dof)**

(=volume of space of flat connection, infinitely large?!)

ill-defined in the continuum ⟶ **quantization on a lattice**

⟶ **agrees with GFT path integral quantization of Boulatov and Ooguri model via**

$$Z_{\mathrm{GFT}} = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} e^{-S_{\mathrm{GFT}}[\varphi,\bar{\varphi}]} = \sum_{\Gamma} \frac{\lambda^{V_\Gamma}}{\mathrm{Sym}(\Gamma)} \mathcal{A}_\Gamma$$

How to impose simplicity constraints at GFT level to render this model one for gravitational dof?

⟶ **Example: Barrett-Crane model**

complete Barrett-Crane GFT model

[Barrett, Crane; Perez, Rovelli; Oriti, Baratin; Jercher, Oriti, Pithis]

a model for Lorentzian quantum gravity in 4d

- start with Ooguri model: GFT model for BF-theory in 4d (topological theory)
- impose so-called simplicity constraints to turn it into a theory of gravity (first-order Palatini)
- **add non-dynamical timelike, spacelike and light like normal vector X to domain**

⟶ **allows to impose closure and simplicity covariantly and commutatively**

⟶ **unique model**

$$\varphi(g_1, ..., g_4; X_\alpha) : \mathrm{SL}(2,\mathbb{C})^4 \times \mathrm{SL}(2,\mathbb{C})/U^{(\alpha)} \to \mathbb{C} \qquad \alpha \in \{+, 0, -\}$$

$U^{(+)} = \mathrm{SU}(2),\ U^{(-)} = \mathrm{SU}(1,1),\ U^{(0)} = \mathrm{ISO}(2)$ **stabilizers of** $X_+ = (1,0,0,0),\quad X_0 = \frac{1}{\sqrt{2}}(1,0,0,1),\quad X_- = (0,0,0,1)$

timelike, lightlike, spacelike

symmetries:

$$\varphi(g_1, ..., g_4; X_\alpha) = \varphi(g_1 h^{-1}, ..., g_4 h^{-1}; h \cdot X_\alpha), \quad \forall h \in \mathrm{SL}(2,\mathbb{C}) \qquad \textbf{(closure)}$$

$$\varphi(g_1, ..., g_4; X_\alpha) = \varphi(g_1 u_1, ..., g_4 u_4; X_\alpha), \quad \forall u_1, ..., u_4 \in \mathrm{U}_{X_\alpha} \qquad \textbf{(simplicity)}$$

⟶ **fields correspond to spacelike, timelike and lightlike tetrahedra**

$$S[\varphi,\bar\varphi] = K[\varphi,\bar\varphi] + V[\varphi,\bar\varphi]$$

$$K[\varphi,\bar\varphi] = \sum_\alpha \int_{\mathrm{SL}(2,\mathbb{C})^4} (\mathrm{d}g)^4 \int_{\mathrm{SL}(2,\mathbb{C}/\mathrm{U}^{(\alpha)}} \mathrm{d}X_\alpha \bar\varphi(g_1, ...g_4; X_\alpha)\varphi(g_1, ...g_4; X_\alpha)$$

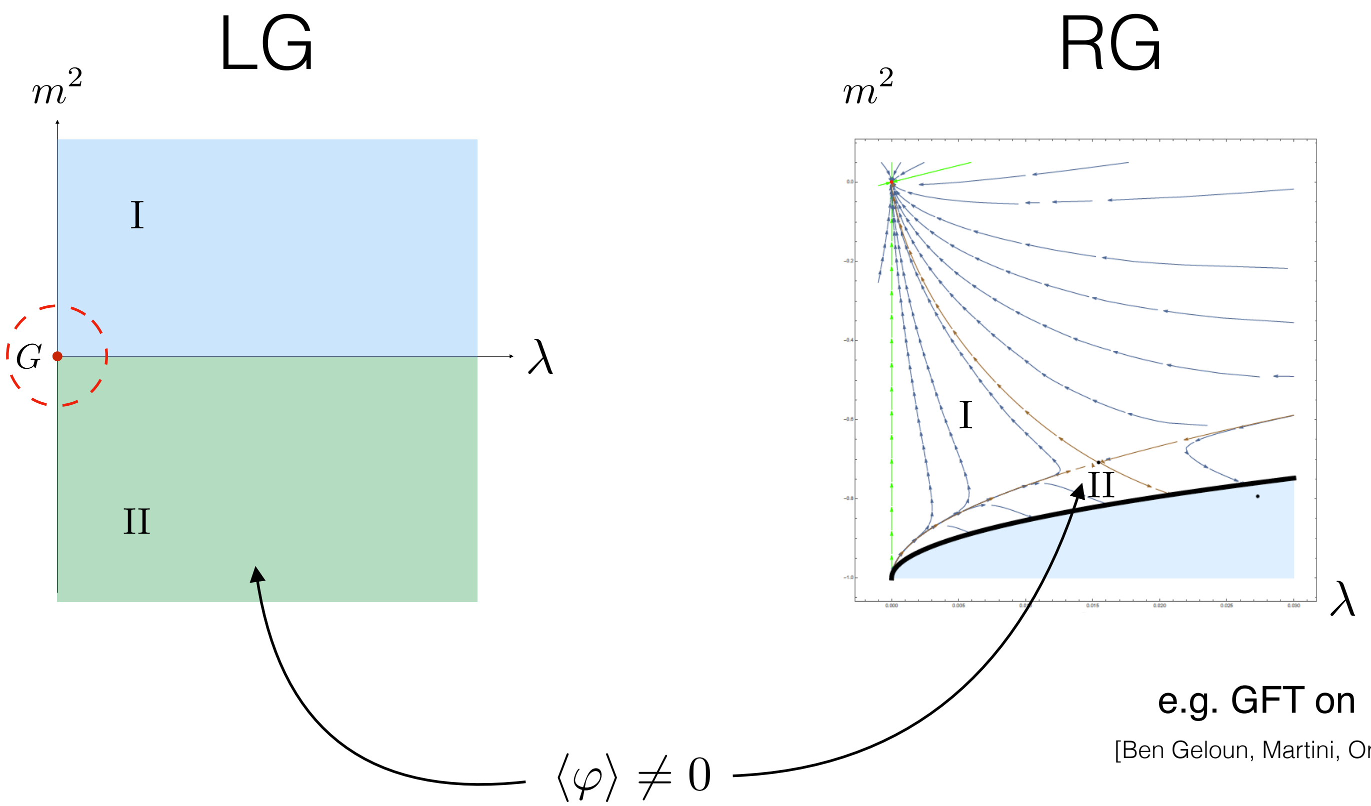
$$V[\varphi,\bar\varphi] = \int (\mathrm{d}g)^{10} \sum_{\alpha_1 ... \alpha_5} \int \mathrm{d}X_{\alpha_1} ... \int \mathrm{d}X_{\alpha_5} \varphi_{1234}(X_{\alpha_1})\varphi_{4567}(X_{\alpha_2})\varphi_{7389}(X_{\alpha_3})\varphi_{962(10)}(X_{\alpha_4})\varphi_{(10)851}(X_{\alpha_5}) + \mathrm{c.c}$$

Back to Landau-Ginzburg method

Applicable in GFT?

Why bother in GFT? Applicable?

- transition to condensate phase in GFT with non-trivial VEV?!



e.g. GFT on $\mathbb{R}^3$

[Ben Geloun, Martini, Oriti]

- problem of the continuum limit in GFT/spin foam models
- mapping phases/phase structure of such models
- to this aim: exploit field theory character of GFTs

- condensate remains **hypothesis** for realistic models (but getting there); **test with LG theory applied to GFT**

- important for group field theory condensate cosmology: condensate phase is important pillar

- upshot: LG MFT applicable in GFT in spite of non-locality of its interactions, gauge invariance and simplicity

Landau-Ginzburg mean-field theory of GFTs

[Thürigen, Pithis; Marchetti, Oriti, Thürigen,Pithis]

(goal: determine ingredients to realize phase transition)

local scalar field theory: **LG theory gives coarse picture of phase structure** thus sufficient to point to the formation of a condensate phase; fully accurate only above critical dimension

⟶ **method works also for GFTs** (non-local)

(shown for simplified models on Abelian compact/non-compact group with/out closure constraint with/out additional local dof; **wip** on Lorentz group and simplicity constraints imposed)

mean-field analysis for $\varphi(\boldsymbol{g}) : G^r \to \mathbb{R}, \mathbb{C}$ take $G = \mathbb{R}$

- devise regularisation scheme due to non-locality together with projection onto uniform fields: $G \to \mathrm{U}(1)$

specifies combinatorics

- extract critical dimension via Ginzburg quantity $Q \sim \lambda_\gamma \xi^{4-(r-s_0)}$

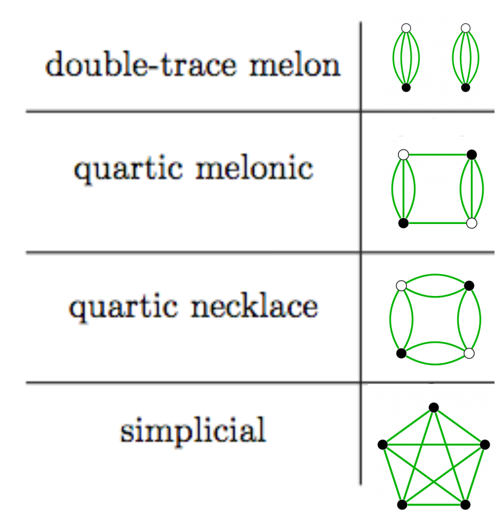
coupling

rank of group field

correlation length

quartic interaction

non-local contribution



- various interactions (power and combinatorics) checked and result generalized

- impose closure constraint $r \to r - 1$

More realistic scenario - kinematics

[wip: Marchetti, Oriti, Thürigen, Pithis]

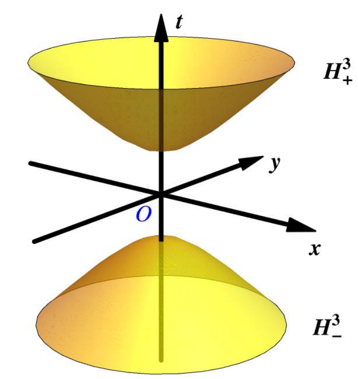
- work within context of the complete Barrett-Crane model

Here:

- **caveat: restrict to spacelike tetrahedra/timelike normals**

$$\varphi(\boldsymbol{g}, X) = \varphi(g_1, g_2, g_3, g_4, X) = \mathrm{SL}(2,\mathbb{C})^4 \times \mathbb{H}^3 \to \mathbb{R}, \mathbb{C}$$

$$\mathbb{H}^3 \longrightarrow \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2) \cong \mathbb{H}^3$$



- decomposition of the field in terms of irreducible representations

$$\varphi(\boldsymbol{g}, X) = \prod_{i=1}^{4} \left(\int \mathrm{d}\rho_i \rho_i^2 \sum_{j_i, m_i} D^{\rho_i, 0}_{j_i m_i 0 0}(g_i X) \right) \varphi^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4}$$

Wigner matrices of $\mathrm{SL}(2,\mathbb{C})$ in the so-called unitary principal series

- integration over normal to get rid of irrelevant information on embedding

$$\varphi(\boldsymbol{g}) = \int_{\mathbb{H}^3} \mathrm{d}X \varphi(\boldsymbol{g}, X) = \prod_{i=1}^{4} \left(\int \mathrm{d}\rho_i \rho_i^2 \sum_{j_i, m_i, l_i, n_i} D^{\rho_i, 0}_{j_i m_i 0 0}(g_i) \right) B^{\rho_1\rho_2\rho_3\rho_4}_{l_1 n_1 l_2 n_2 l_3 n_3 l_4 n_4} \varphi^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4}$$

Barrett-Crane intertwiner $B^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4} \equiv \int \mathrm{d}X \prod_{i=1}^{4} D^{(\rho_i, 0)}_{j_i m_i 0 0}(X)$

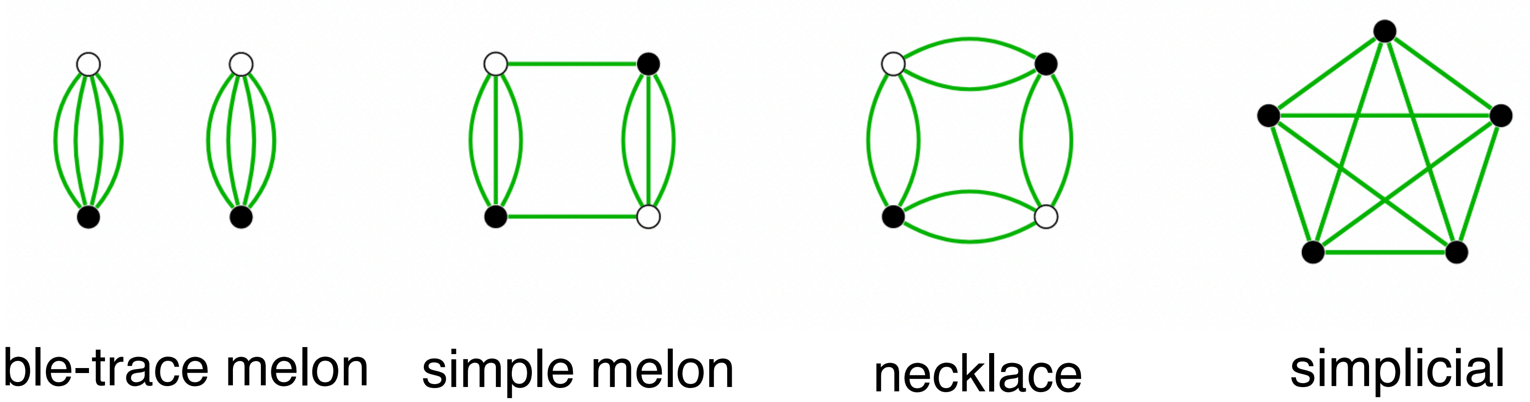
More realistic scenario - dynamics

$$S[\varphi,\bar{\varphi}] = S_0[\varphi,\bar{\varphi}] + S_{\mathrm{IA}}[\varphi,\bar{\varphi}]$$

GFT action   kinetic term   interaction(s)

$$S_0[\varphi,\bar{\varphi}] = \int_{\mathrm{SL}(2,\mathbb{C})^4} \mathrm{d}\boldsymbol{g} \int_{\mathbb{H}^3} \mathrm{d}X \, \bar{\varphi}(\boldsymbol{g},X) \left(-\sum_{i=1}^{4} \Delta_i + \mu \right) \varphi(\boldsymbol{g},X)$$

consider interactions of type:



e.g.

$$S_{\mathrm{IA,simplex}}[\varphi,\bar{\varphi}] = \frac{\lambda}{5!} \int_{\mathrm{SL}(2,\mathbb{C})^{10}} [\mathrm{d}g]^{10} \int_{\mathbb{H}^{3\cdot 5}} [\mathrm{d}X]^5 \varphi_{1234}(X_1)\varphi_{4567}(X_2)\varphi_{7389}(X_3)\varphi_{9620}(X_4)\varphi_{0851}(X_5) + \mathrm{c.c.}$$

More realistic scenario - regularization

- due to closure constraint together with projection onto uniform fields Φ_0 one has infinite volume factors as $\mathrm{SL}(2,\mathbb{C})$ is non-compact

⟶ have to regularize models: done by analytic continuation and compactification of $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{Spin}(4)$ [Dona, Gozzini, Nicotra]

concretely:

- at local level it amounts to **map** between corresponding **Lie algebras** $\mathfrak{spin}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \leftrightarrow \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$

- at global level it amounts to map between corresponding Lie groups via **mapping respective Cartan decompositions** into each other:

$\mathrm{SL}(2,\mathbb{C})$

$$\mathrm{SU}(2) \times \mathrm{A}^+ \times \mathrm{SU}(2) \to \mathrm{SL}(2,\mathbb{C})$$
$$(u, \mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3}, v) \mapsto u\mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3}v^{-1}$$

$\mathrm{Spin}(4)$

$$\mathrm{SU}(2) \times \mathrm{T}^+ \times \mathrm{SU}(2) \to \mathrm{Spin}(4)$$
$$(u, \mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3}, v) \mapsto (u\mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3}v^{-1}, u\mathrm{e}^{i\frac{1}{2}\frac{t}{a}\sigma_3}v^{-1})$$

$$\mathrm{A}^+ = \{\mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3} | \eta \in \mathbb{R}_+\} \xrightarrow[\Lambda]{\text{introduce regulator}} \mathrm{A}^+_\Lambda = \{\mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3} | \eta \in [0,\Lambda)\} \xrightarrow[\eta \to -it]{\text{Wick rotate}} \mathrm{T}^+_\Lambda = \{\mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3} | t \in [0,\Lambda)\} \xrightarrow[\Lambda \to 2\pi a]{\text{compactify}} \mathrm{T}^+ = \{\mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3} | t \in [0, 2\pi a)\}$$

- essentially amounts to mapping of respective homogeneous spaces into each other

$$\mathbb{H}^3 \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$$
$$\mathrm{d}H^2 = a^2 \left(\left(\frac{\mathrm{d}\eta}{a} \right)^2 + \sinh^2\left(\frac{\eta}{a}\right) \mathrm{d}\Omega_2 \right)$$

skirt radius (a)

$$S^3 \cong \mathrm{Spin}(4)/\mathrm{SU}(2)$$
$$\mathrm{d}S^2 = a^2 \left(\left(\frac{\mathrm{d}t}{a} \right)^2 + \sin^2\left(\frac{t}{a}\right) \mathrm{d}\Omega_2 \right)$$

- map representation labels $\rho \to -ip$ ⟶ work with $\mathrm{Spin}(4)$-representation theory instead

More realistic scenario - correlation function and length

Starting from regularized action:

→ **linearize equations of motion** over non-trivial background Φ_0

→ solve for regularized correlation function:

$$C(\boldsymbol{g}) = \prod_{i=1}^{4}\left(\sum_{p_i}\frac{p_i^2}{\mathrm{vol}(\mathrm{T}^+)}\sum_{\substack{j_i,m_i;\\ l_i,n_i}} D^{(p_i,0)}_{j_i m_i l_i n_i}(g_i)\right) B^{p_1p_2p_3p_4}_{l_1n_1l_2n_2l_3n_3l_4n_4}\,\hat{C}^{p_1p_2p_3p_4}_{j_1m_1j_2m_2j_3m_3j_4m_4}$$

$$\hat{C}^{p_1p_2p_3p_4}_{j_1m_1j_2m_2j_3m_3j_4m_4} = \frac{1}{\frac{1}{a^2}\sum_i(-\mathrm{Cas}_{1,p_i}) + b_{\boldsymbol{p},\boldsymbol{j},\boldsymbol{m}}}$$

encapsulates remaining non-locality of interactions after projection onto Φ_0

→ analyze correlation function **mode-by-mode**

→ turns out that only the zero-mode behaviour of the correlator is important for us; there we can Wick rotate back and decompactify to $\mathrm{SL}(2,\mathbb{C})$

→ **only** these **zero-modes contribute to** the **correlation length** and **determine** the behaviour of the **Ginzburg Q**-parameter

→ **result for correlation length** (via asymptotic analysis or second-moment-method): $\xi^2 \sim \frac{1}{a^2\mu^2} + \frac{1}{\mu}$

modification due to hyperbolicity of domain

(see also Benedetti, 1403.6712)

More realistic scenario - Ginzburg Q

- results for local scalar field theory on one 3-hyperboloid

$$Q \sim \lambda_\gamma \xi^2 e^{-2\cdot 1 \frac{\xi}{a}}$$

(Benedetti, 1403.6712)

coupling

exponential suppression due to hyperbolicity of domain

for finite skirt radius a :

rank of the group field

$$Q \sim \lambda_\gamma \xi^2 e^{-2(4-s_0)\frac{\xi}{a}}$$

combinatorics of interaction $s_0 \leq 4$

flat limit:
$a \to \infty$

dimension of 3-hyperboloid

$$Q \sim \lambda_\gamma \xi^{4-3(4-s_0)}$$

(agrees with our results 2110.15336)

impact of closure constraint via the BC intertwiner:

$s_0 \to s_0 + 1$ (i.e. one more zero-mode, or one rank less $r \to r-1$)

- can be generalized to arbitrary interactions
- local degrees of freedom can be added straightforwardly

⟶ **Ginzburg Q always very small**

⟶ **LG mean-field theory can self-consistently describe phase transition** $(\Phi_0 = 0 \leftrightarrow \Phi_0 \neq 0)$

Conclusions

- LG theory is also applicable to GFT models in spite of their non-local interactions
- it informs us about the coarse phase structure of different models
- apply it to Barrett-Crane model for Lorentzian first order Palatini gravity
- from there we can extract the rescaling relations of couplings needed for RG studies

Extensions

- consider all the bare causal structure (spacelike, timelike and lightlike tetrahedra)
- extension to other relevant models
- conduct full-fledged (functional) **RG and 1/N analyses** of these models
- devise **observables** & tools to characterize different phases wrt their geometric properties

Thank you for your attention!