

# Novel critical phenomena in compressible polar active fluids

Patrick Jentsch

Chiu Fan Lee

Department of  
Bioengineering



Imperial College  
London

ERG 2022 Berlin

26.07.2022



# Overview

1. Compressible polar active matter has a rich phase diagram
  - Interplay between **ordered** and **disordered motion** with **phase-separation**
2. There is a **multicritical point** where all phases merge
3. We discover **3 new nonequilibrium universality** classes using the FRG

# Hydrodynamic Theory

## Continuity equation for density

$$\partial_t \rho = -\nabla \cdot \mathbf{g}$$

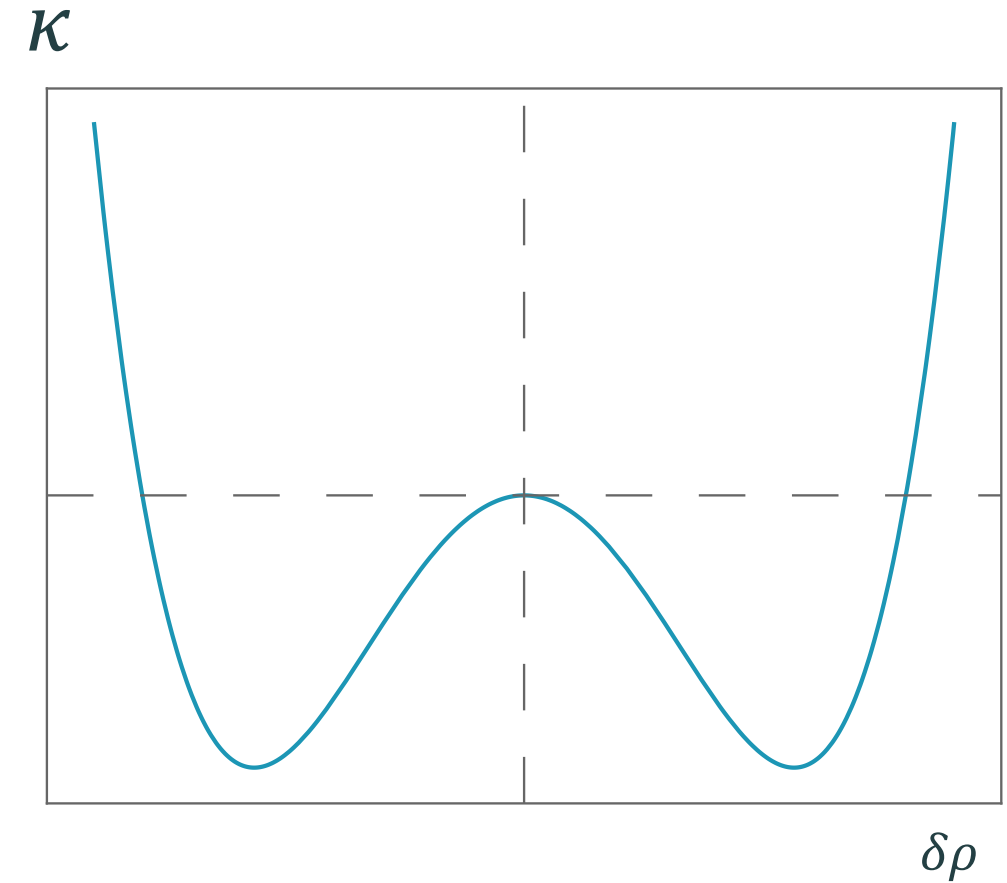
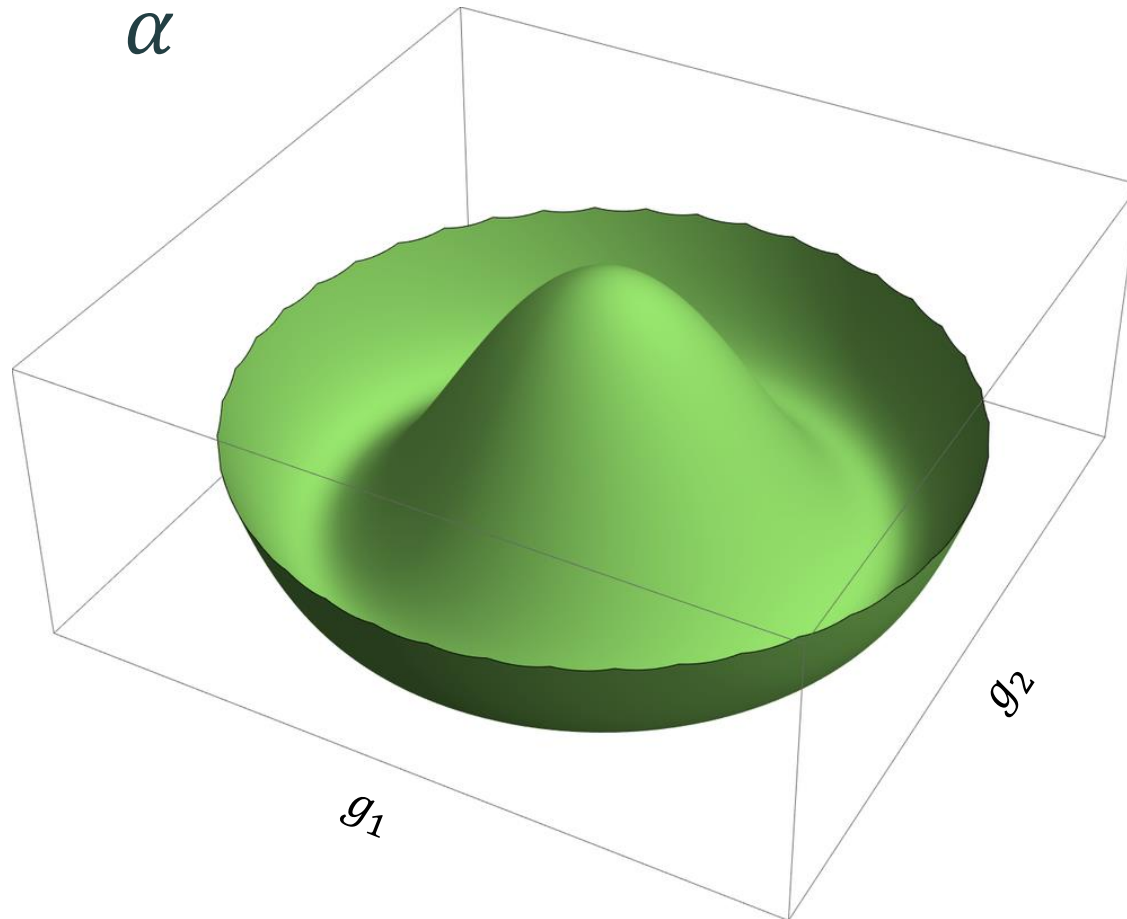
Under assumptions of:

- Mass conservation
- Translation sym.
- Rotation sym.
- Chiral sym.

## Toner-Tu equation for momentum density

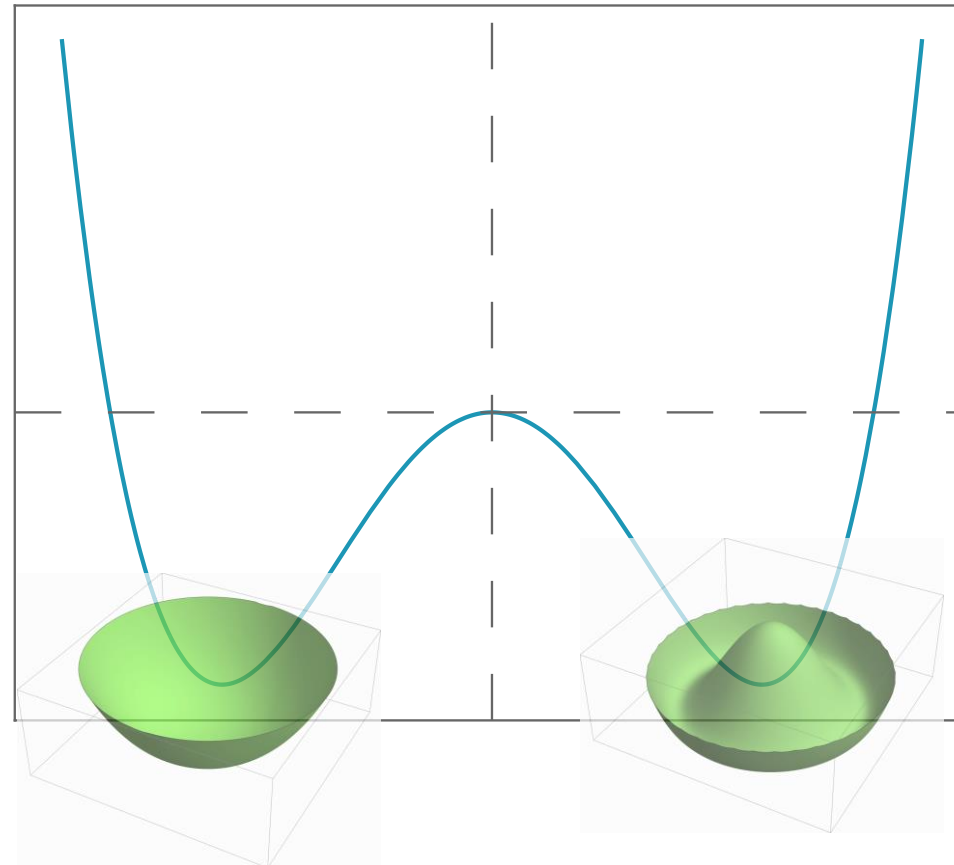
$$\begin{aligned} & \gamma \partial_t \mathbf{g} + \lambda_1 \mathbf{g} \cdot \nabla \mathbf{g} + \lambda_2 \mathbf{g} \cdot (\nabla \mathbf{g})^T + \lambda_3 \mathbf{g} \nabla \cdot \mathbf{g} \\ = & \boxed{-\alpha(\rho, g^2) \mathbf{g} - \kappa(\rho, g^2) \nabla \rho} + (\zeta - \mu) \nabla^2 \mathbf{g} + \mu \nabla (\nabla \cdot \mathbf{g}) + \eta \nabla^2 \nabla \rho \\ & + \kappa_2 \mathbf{g} (\mathbf{g} \cdot \nabla \rho) + \mathbf{f} \end{aligned}$$

# Spontaneous Symmetry Breaking

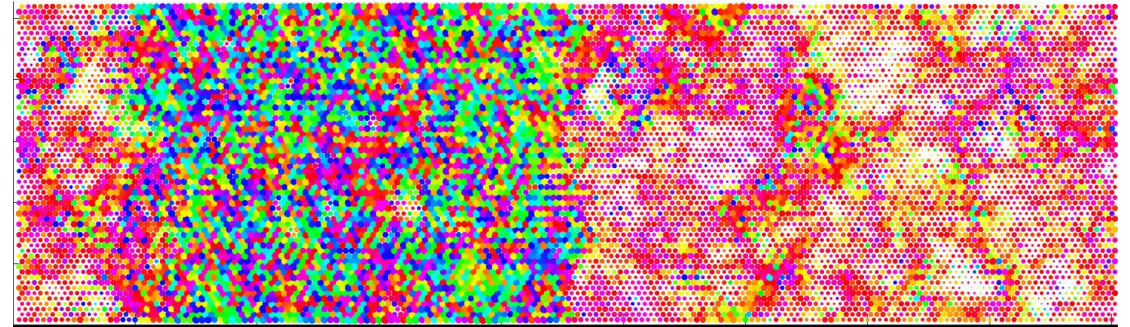
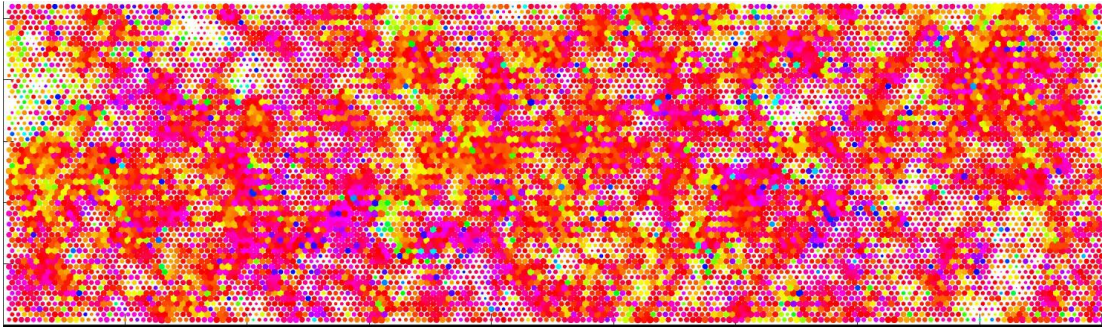
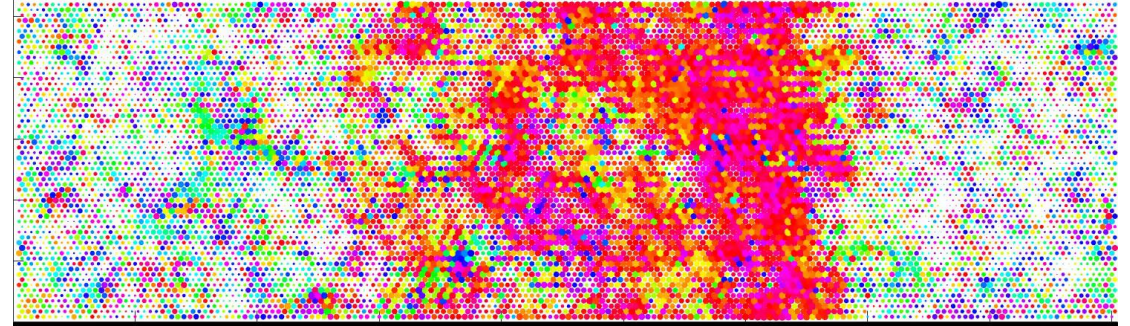
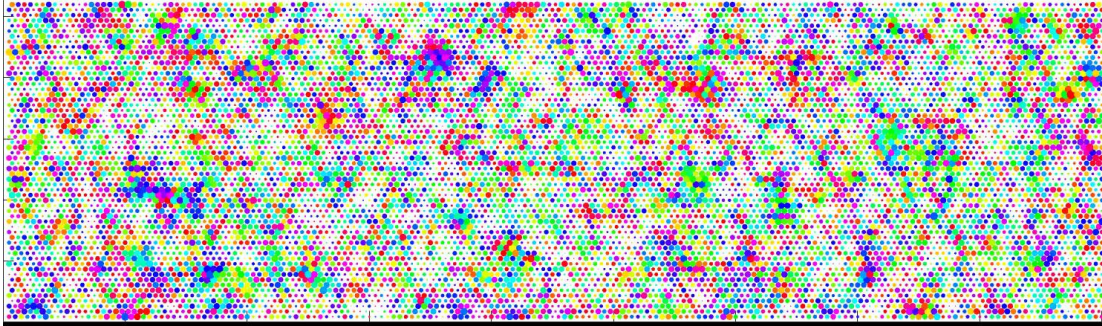


# Spontaneous Symmetry Breaking

$\mathcal{K}$

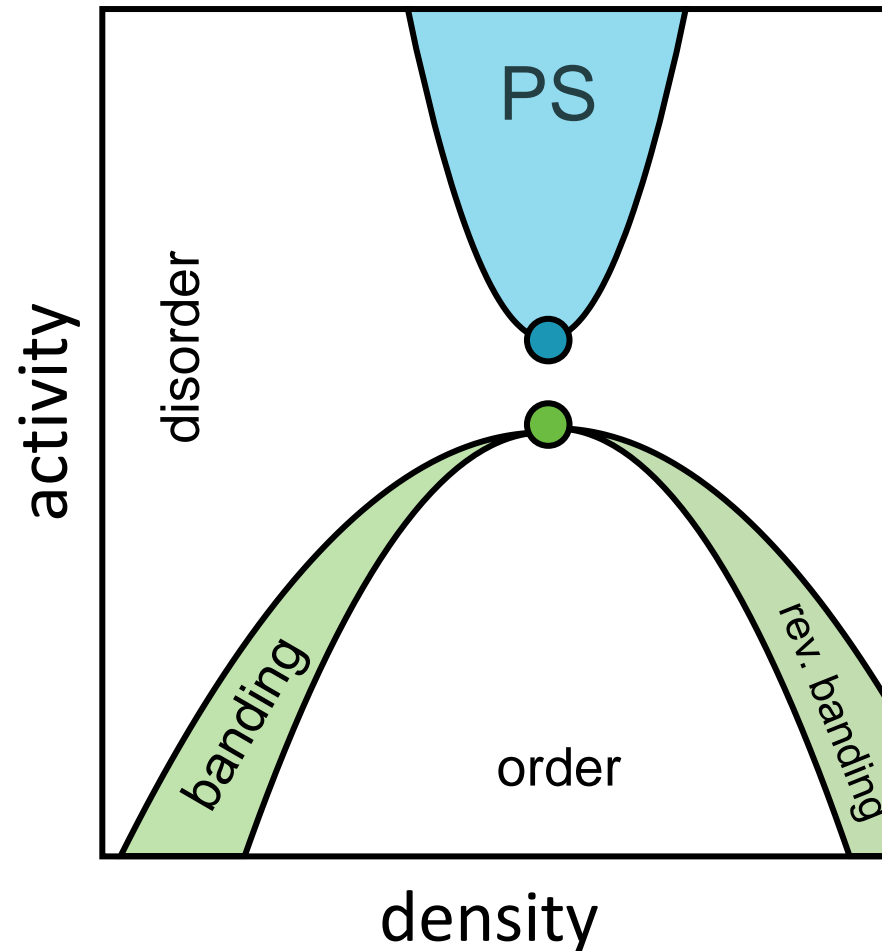


# Simulations



Nesbitt, Pruessner, Lee, New Journal of Physics (2021)

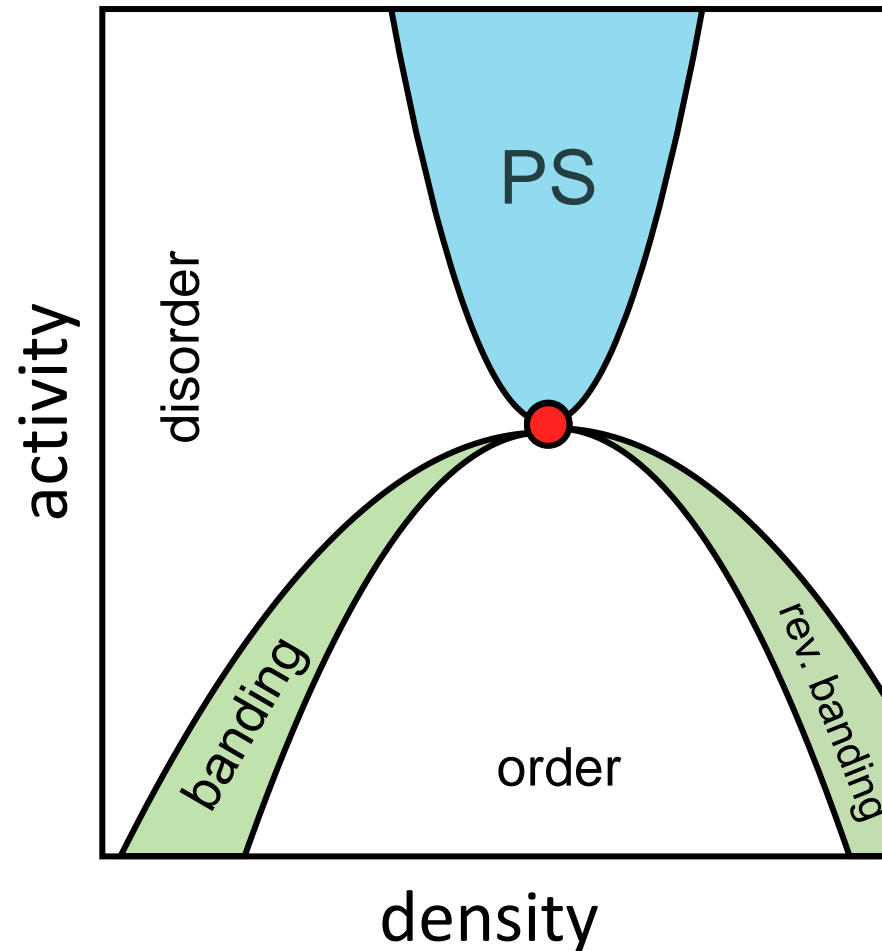
# Phase Diagram



- **Critical order-disorder**  
 Nesbitt, Pruessner, Lee, *New Journal of Physics* **23**, 43047 (2021)
- **Critical phase separation**  
 Partridge, Lee, *PRL* (2019)  
 Maggi *et al.*, *Soft Matter* (2021)  
 Siebert *et al.*, *Phys. Rev. E* (2018)  
 Caballero, Nardini, Cates, *J. Stat. Mech. Theory Exp.*, (2018)
- **Multicritical Point**  
 Bertrand, Lee, *arXiv:2012.05866* (2020)



# Phase Diagram



- Critical order-disorder  
Nesbitt, Pruessner, Lee, *New Journal of Physics* **23**, 43047 (2021)
- Critical phase separation  
Partridge, Lee, *PRL* (2019)  
Maggi *et al.*, *Soft Matter* (2021)  
Siebert *et al.*, *Phys. Rev. E* (2018)  
Caballero, Nardini, Cates, *J. Stat. Mech. Theory Exp.*, (2018)
- Multicritical Point  
Bertrand, Lee, *arXiv:2012.05866* (2020)

# EOM near the multicritical point in $d = 6 - \epsilon$

$$\partial_t \mathbf{g} = -\alpha(\rho) \mathbf{g} + (\mu_{\parallel} - \mu_{\perp}) \nabla^2 \mathbf{g} + \mu_{\perp} \nabla(\nabla \cdot \mathbf{g})$$

Momentum density

$$-\kappa(\rho) \nabla \rho + \zeta \nabla^2 \nabla \rho$$

"pressure"

$$+ f$$

Noise

$$\partial_t \rho = -\nabla \cdot \mathbf{g}$$

Continuity eq.

$$\langle f(\mathbf{r}, t) f(\mathbf{r}', t') \rangle = 2D \delta^d(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

Noise

# Critical Exponents

Anomalous dimensions  $\eta_g$  and  $\eta_\rho$

Dynamic exponent  $z$

$$\langle g(\mathbf{q}, \omega) g(-\mathbf{q}, -\omega) \rangle = q^{-2+\eta_g} S_g \left( \frac{\omega}{q^z} \right)$$

$$\langle \rho(\mathbf{q}, \omega) \rho(-\mathbf{q}, -\omega) \rangle = q^{-4+\eta_\rho} S_\rho \left( \frac{\omega}{q^z} \right)$$

Nonequilibrium exponent

$$\frac{\langle g(\mathbf{q}, \omega) g(-\mathbf{q}, -\omega) \rangle}{\frac{1}{\omega} \text{Im} \chi(\mathbf{q}, \omega)} = q^{\eta_{NE}} S_{NE} \left( \frac{\omega}{q^z} \right)$$

# Functional Renormalization Group

## Exact Wetterich equation

$$\partial_k \Gamma_k = \text{Tr} \left\{ \left( \Gamma_k^{(2)} + R_k \right)^{-1} \partial_k R_k \right\}$$

Wetterich, Phys. Lett. B (1993)  
Morris, Int. J. Mod. Phys. A (1994)  
Ellwanger, Zeitschr. f. Phys. C (1994)

# Ansatz for effective Action

Martin-Siggia-Rose-de Dominicis-Janssen formalism

Canet, Chaté, Delamotte, *J. Phys. A: Math. Theor.* (2011)

$$\partial_t \mathbf{g} = -\alpha(\rho) \mathbf{g} + (\mu_{\parallel} - \mu_{\perp}) \nabla^2 \mathbf{g} + \mu_{\perp} \nabla(\nabla \cdot \mathbf{g}) - \kappa(\rho) \nabla \rho + \zeta \nabla^2 \nabla \rho + f$$

Momentum density      "pressure"      Noise

$$\partial_t \rho = -\nabla \cdot \mathbf{g}$$

Continuity eq.

$$\langle f(\mathbf{r}, t) f(\mathbf{r}', t') \rangle = 2D \delta^d(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

Noise

$$\Gamma_k [\bar{\mathbf{g}}, \mathbf{g}, \bar{\rho}, \rho] = \int_{\mathbf{r}, t} \left\{ -2D_k \bar{\mathbf{g}}^2 \right. \quad \left. + \bar{\rho} (\partial_t \rho + \nabla \cdot \mathbf{g}) \right. \quad \left. + \bar{\mathbf{g}} (\kappa_k(\rho) \nabla \rho - \zeta_k \nabla^2 \nabla \rho) \right.$$

Noise      Continuity eq.      "pressure"

$$\left. + \bar{\mathbf{g}} (\gamma_k \partial_t \mathbf{g} + \alpha_k(\rho) \mathbf{g} - (\mu_{\parallel, k} - \mu_{\perp, k}) \nabla^2 \mathbf{g} - \mu_{\perp, k} \nabla(\nabla \cdot \mathbf{g})) \right\}$$

Momentum density

# Ansatz for effective Action

Martin-Siggia-Rose-de Dominicis-Janssen  
formalism Canet, Chaté, Delamotte, *J. Phys. A: Math. Theor.* (2011)

$$\Gamma_k [\bar{\mathbf{g}}, \mathbf{g}, \bar{\rho}, \rho] = \int_{r,t} \left\{ -2D_k \bar{\mathbf{g}}^2 + \bar{\rho}(\partial_t \rho + \nabla \cdot \mathbf{g}) + \bar{\mathbf{g}}(\kappa_k(\rho) \nabla \rho - \zeta_k \nabla^2 \nabla \rho) \right. \\ \left. + \bar{\mathbf{g}}(\gamma_k \partial_t \mathbf{g} + \alpha_k(\rho) \mathbf{g} - (\mu_{\parallel,k} - \mu_{\perp,k}) \nabla^2 \mathbf{g} - \mu_{\perp,k} \nabla(\nabla \cdot \mathbf{g})) \right\}$$

# Momentum dependent terms

Projections at non-zero density fluctuations

$$\partial_k \mu_{\parallel} = \frac{1}{2VT} \frac{1}{d-1} \frac{d^2}{dq^2} \text{Tr} \frac{\mathbf{q} \otimes \mathbf{q}}{q^2} \frac{\delta^2 \partial_k \Gamma_k}{\delta \bar{\mathbf{g}}(\mathbf{q}, 0) \delta \mathbf{g}(-\mathbf{q}, 0)} \Big|_{\rho = \rho_{\min}}$$

Nonperturbative in  $\mu_{\perp}/\mu_{\parallel}$  and  $\zeta\gamma/\mu_{\parallel}^2$

# Flow equations

$$\partial_k \Gamma_k^{(1,1,0,0)} = \tilde{\partial}_k \left( \frac{1}{2} \left( \begin{array}{cccc} \text{Diagram 1} & \text{Diagram 2} & \text{Diagram 3} & \text{Diagram 4} \end{array} \right) \right)$$

The diagram shows four circular nodes connected by horizontal arrows. Each node has a small circle at the top and a larger circle at the bottom. The nodes are connected by solid arrows pointing right. The first node has two incoming arrows from below. The second node has two outgoing arrows to the left. The third node has two outgoing arrows to the left. The fourth node has two outgoing arrows to the left.

$$\partial_k \Gamma_k^{(1,0,0,1)} = \tilde{\partial}_k \left( \frac{1}{2} \left( \begin{array}{cccc} \text{Diagram 1} & \text{Diagram 2} & \text{Diagram 3} & \text{Diagram 4} \\ \text{Diagram 5} & \text{Diagram 6} & \text{Diagram 7} & \text{Diagram 8} \end{array} \right) \right)$$

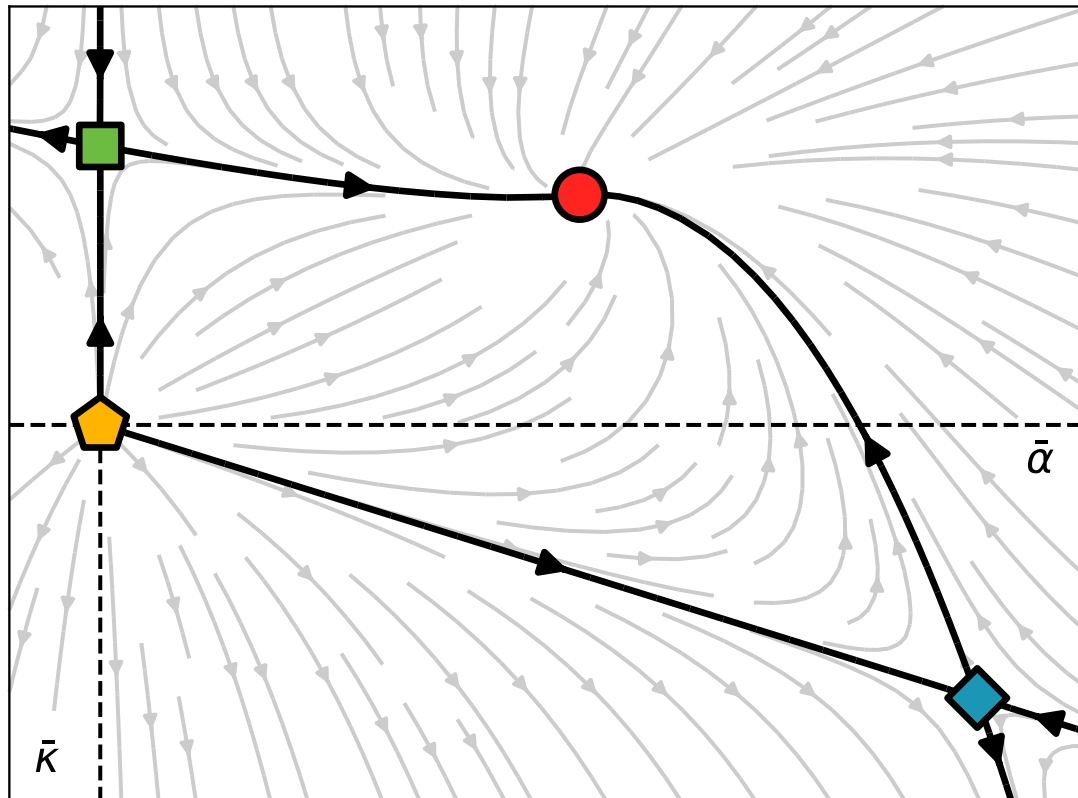
The diagram shows eight circular nodes arranged in two rows of four. Each node has a small circle at the top and a larger circle at the bottom. The nodes are connected by horizontal arrows pointing right. The top row nodes are connected by solid arrows, and the bottom row nodes are connected by solid arrows. The first node in the top row has two incoming arrows from below. The second node in the top row has two outgoing arrows to the left. The third node in the top row has two outgoing arrows to the left. The fourth node in the top row has two outgoing arrows to the left. The first node in the bottom row has two outgoing arrows to the left. The second node in the bottom row has two outgoing arrows to the left. The third node in the bottom row has two outgoing arrows to the left. The fourth node in the bottom row has two outgoing arrows to the left.

$$\partial_k \Gamma_k^{(1,1,0,0)} = \tilde{\partial}_k \left( \begin{array}{cc} \text{Diagram 1} & \text{Diagram 2} \end{array} \right)$$

The diagram shows two circular nodes connected by horizontal arrows. Each node has a small circle at the top and a larger circle at the bottom. The nodes are connected by solid arrows pointing right. The first node has two incoming arrows from below. The second node has two outgoing arrows to the left.



# Flow diagram (2D projection)



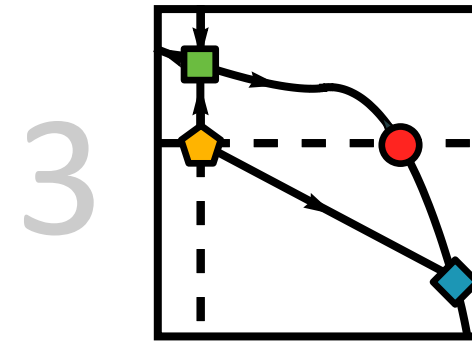
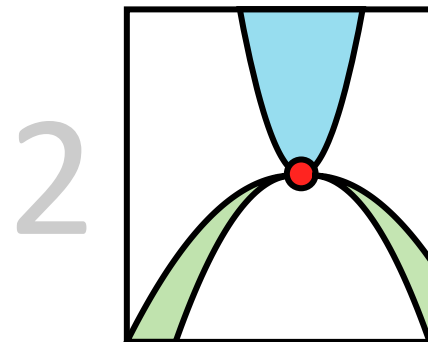
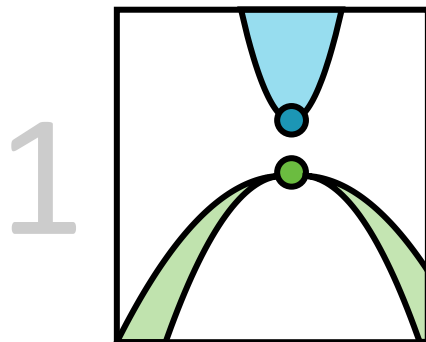
-  Gaussian
-  Unstable FP
-  Multicritical Point
-  Unstable FP

# Critical Exponents and Universal Ratios

	$\eta_\rho$	$\eta_g$	$\eta_{NE}$	$z - 2$	$\mu_\perp / \mu_\parallel$	$\zeta\gamma / \mu_\parallel^2$
⬡	0	0	0	0		
■	0	0	0	0		1.43
●	$0.066\epsilon^2$	$0.043\epsilon^2$	$0.076\epsilon^2$	$0.011\epsilon^2$	1.45	1.37
◆	$0.223\epsilon^2$	$-0.026\epsilon^2$	$0.039\epsilon^2$	$0.084\epsilon^2$	0.31	0

# Conclusion

1. Compressible polar active matter has a rich phase diagram
  - Interplay between **ordered** and **disordered motion** with **phase-separation**
2. There is a **multicritical point** where all phases merge
3. We discover **3 new nonequilibrium universality** classes using the FRG



# Thank you!

Preprint available:  
[arXiv:2205.01610](https://arxiv.org/abs/2205.01610)

Universality in Biology Group

Chiu Fan Lee (PI)

Shalabh Anand (Postdoc)

Andrew Killeen (PhD)

John-Antonio Argyriadis (PhD)

Sam Whitby (PhD)

Sulaimaan Lim (PhD)



**Imperial College  
London**