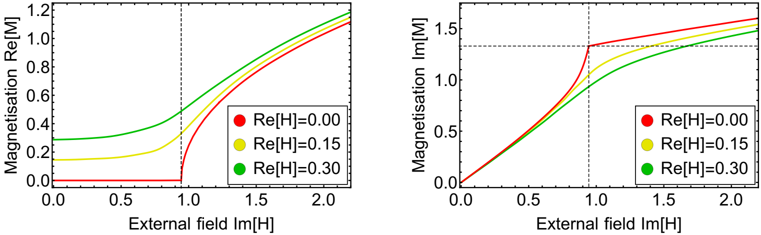
Functional Flows for Complex Actions

Friederike Ihssen
26.07.2022 - ERG 2022



Based on arXiv:2207.10057 in collaboration with J. M. Pawlowski

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Studienstiftung des deutschen Volkes

HGS-HIRe for FAIR
Helmholtz Graduate School for Hadron and Ion Research

Phase Transitions: Statistical Mechanics

Partition function :

$$\mathcal{Z} = \sum_i \exp(-\beta E_i)\,.$$

Free energy :

$$F = -\beta^{-1} \log(\mathcal{Z})\,.$$

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$$V \to \infty$$

$$N/V = \text{const.}$$

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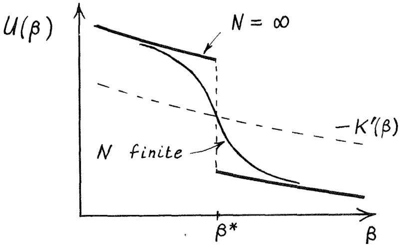

M. Fisher, Lecture Notes 1965

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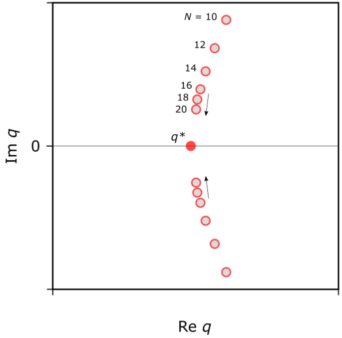

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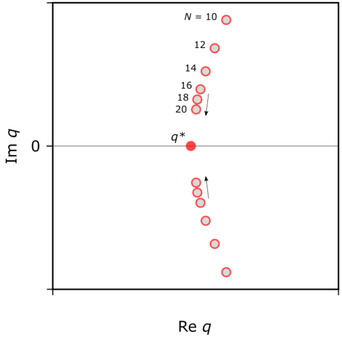

PRD.95.085001 L. Zambelli, O. Zanusso
PRL.125.191602 A. Connelly, G. Johnson, F. Rennecke, V. Skokov
arXiv:2203.16651 F. Rennecke, V. Skokov

A Functional Approach

The infrared regularized path-integral / generating functional:

$$\mathcal{Z}_k[J] = \int [d\varphi]_{\mathrm{ren},p^2 \geq k^2} \exp\Big\{ - S[\varphi] + J \cdot \varphi \Big\},$$

$$\int [d\varphi]_{\mathrm{ren},p^2 \geq k^2} = \int [d\varphi]_{\mathrm{ren}} \exp\Big\{ - \frac{1}{2} \varphi \cdot R_k \cdot \varphi \Big\}$$

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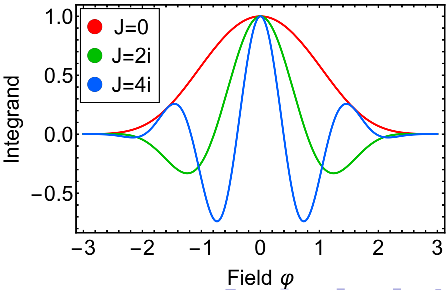
- Real function of a complex variable.
- Sample from oscillatory integrand.

arXiv 2111.12645: F. Attanasio, M. Bauer, L. Kades, J. M. Pawlowski

arXiv 2203.01243: J. M. Pawlowski, J. Urban

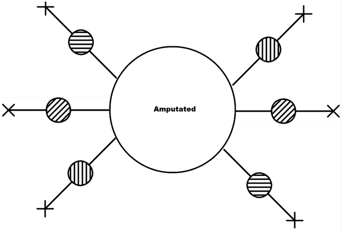


Generating functionals

- (Amputated) connected correlation functions:

$$\mathcal{W}_k[J] = \ln \mathcal{Z}_k[J]$$

$$S_{\text{eff},k}[\phi] = -W_k[S_k^{(2)}\phi]$$



T. Weigand, Lecture Notes 2011

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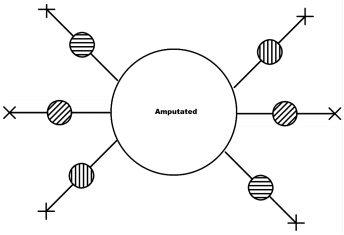
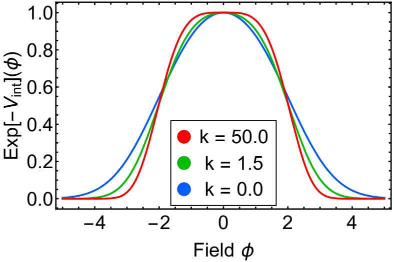


T. Weigand, Lecture Notes 2011

- 1PI correlation functions / Effective action:

$$\Gamma_k[\phi] = \sup\left\{\int_x J(x)\phi(x) - \mathcal{W}_k[J] - \Delta S_k[\phi]\right\}$$

Optimising the RG flow



Polchinski Flow:

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$$J = (S^{(2)} + R_k)[\phi_0]\,\phi\,.$$

- Remove classical mass $\Rightarrow S_{\text{int}}$

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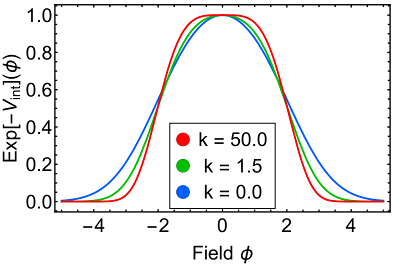

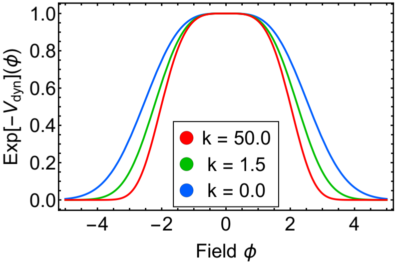

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RG-adapted Flow:

- Amputate RG-adapted propagator:

$$J = (G_k)^{-1}[\phi_0]\,\phi\,.$$

- Remove $m_k \Rightarrow S_{\text{dyn}}$

An algebraic flow

From the Schwinger Functional

Non-linear parabolic PDE:

$$\left(\partial_t + \int_x \phi\, \gamma_{\mathrm{dyn},k}\, \frac{\delta}{\delta\phi}\right) S_{\mathrm{dyn},k}[\phi] + \frac{1}{2}\int_x \phi\, \partial_t \Gamma_k^{(2)}[\phi_0]\, \phi$$

$$= \frac{1}{2}\,\mathrm{Tr}\, \mathcal{C}\left[S^{(2)}_{\mathrm{dyn}}[\phi] - \left(S^{(1)}_{\mathrm{dyn}}[\phi]\right)^2 \right]$$

- Convection terms
- Diffusion term

Discontinuous Galerkin

Computational domain:

$$\Omega \simeq \Omega_h = \bigcup_{k=1}^{K} D^k$$

Solution in each cell:

$$u_h^k(t,x) = \sum_{n=1}^{N+1} \hat{u}_n^k(t)\psi_n(x)$$

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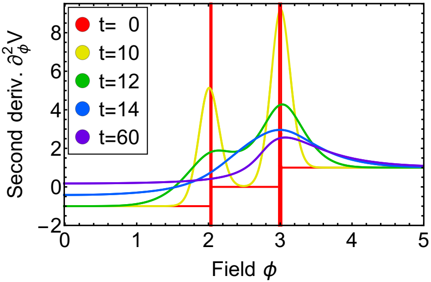

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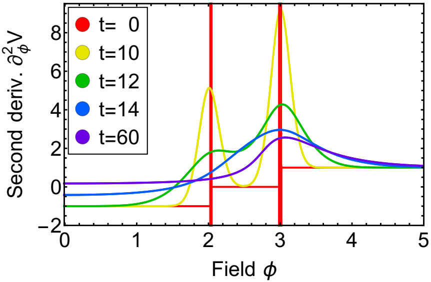

arXiv:1903.09503 E. Grossi, N. Wink
PRD.104.016028 E. Grossi, F. Ihssen, J. M. Pawlowski, N. Wink
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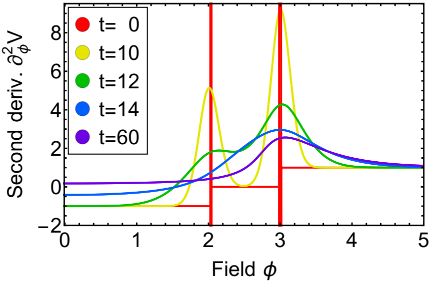

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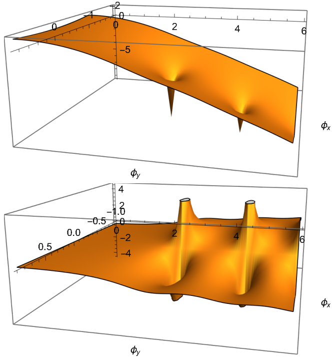
The $O(N)$ Model

$$S[\varphi] = \int_x \left\{ \frac{1}{2}\varphi(x)\left[-\partial_\mu^2 + m^2\right]\varphi(x) + \frac{\lambda}{4!}\varphi(x)^4 \right\}$$



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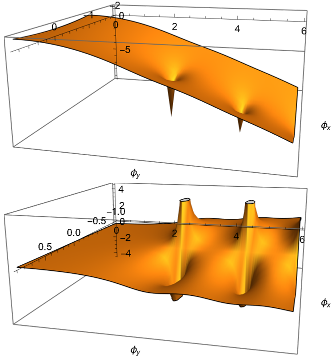

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arXiv:2108.02504 A. Koenigstein, M. J. Steil, N. Wink, E. Grossi, J.Braun, M. Buballa, D. H. Rischke

arXiv:2108.04037 M. Steil, A. Koenigstein

arXiv:2108.10085 A. Koenigstein, M. J. Steil, N. and Wink, E. Grossi, J. Braun

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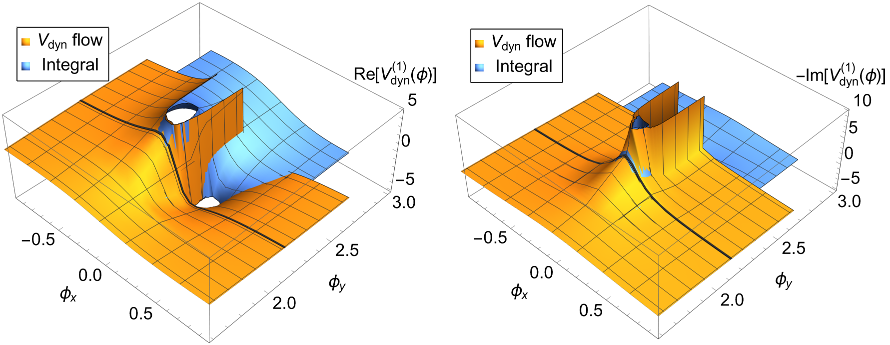

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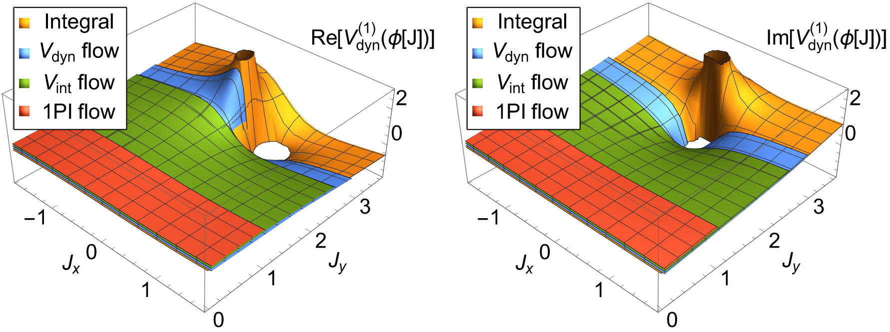

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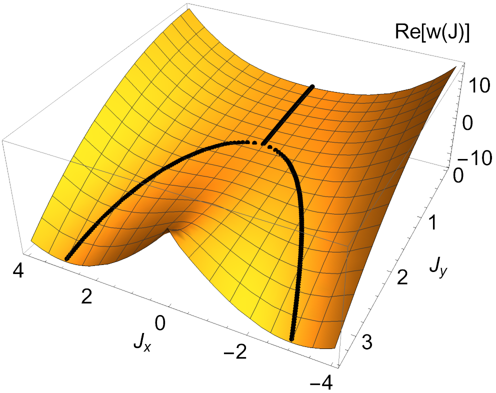

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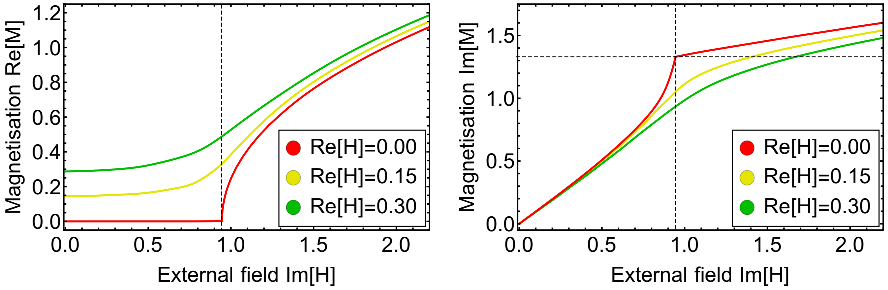

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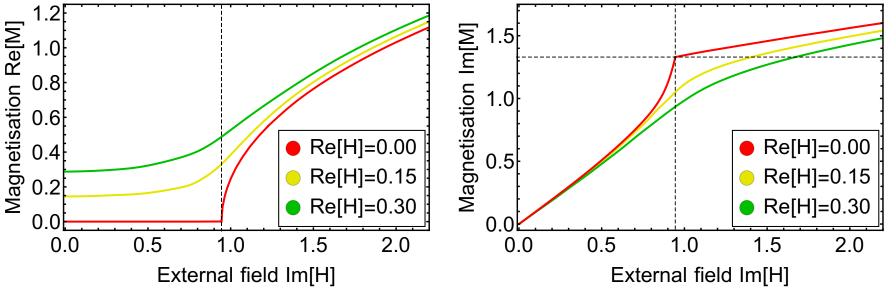

LY-location for $\mathrm{Im}[H] = 0.994$, $m^2_{\mathrm{UV}} = 1$ at $\mathrm{Im}[\langle M\rangle] = 1.33$!

Recovering a Real Phase Transition

Scaling behaviour: $\dfrac{m^2 - m_c^2}{m_c^2} = const\, H^{1/\Delta}$

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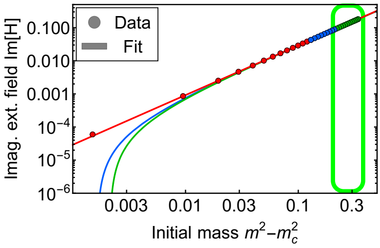

How close do we need to go to extrapolate m_c^2?

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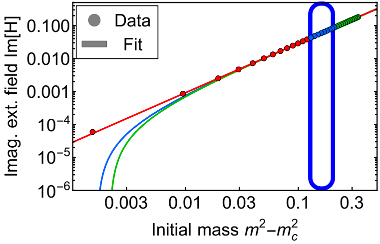

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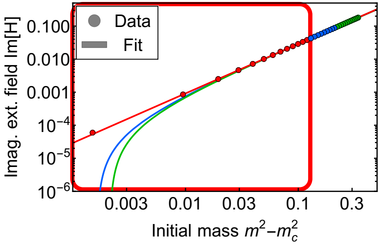

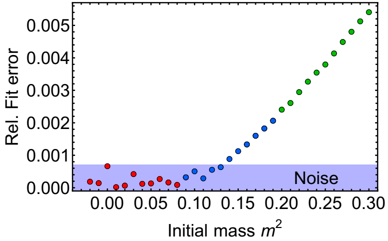

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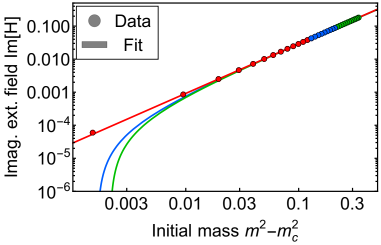
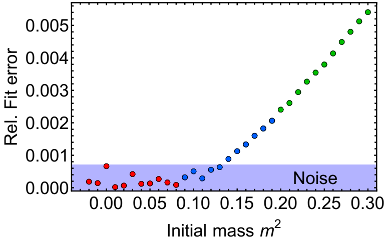




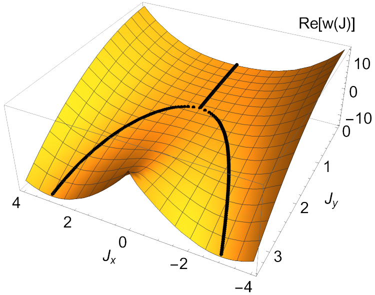
Δ	m_c^2
$1.4969(66)$	$-0.03943(17)$

$\Rightarrow$ Last computed IR-mass:

$$m_0^2(m^2 = -0.039) = 4.2\ 10^{-4}$$

Summary & Outlook

- Established a new functional flow: **the RG-adapted flow**
- Investigated the convergence of different functional flows in the complex plane using DG-methods.
- Retrieved location of the Lee-Yang singularity in $d = 0$ and $d = 4$.
- We extracted the critical mass m_c^2 of the real phase transition in $d = 4$!



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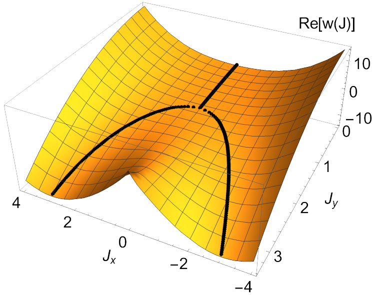

- Investigate extraction in $d \neq 4$
- Go towards QCD phase structure

Thank you for
your attention!

Dealing with complex variables

Map the complex derivative:

- $\partial_\phi \to \partial_{\phi_x}$
- $\partial_\phi \to -\mathrm{i}\partial_{\phi_y}$

Ensure positive, real diffusion:

- $\left(S_k^{(2)}[\phi_0]\right)^2 = \left(m^2 + k^2 + \phi_0 \frac{\lambda}{2}\right)^2 \in \mathbb{R}_{>0}$

$$\partial_t \left(u_x + \mathrm{i}u_y\right) + \partial_{\phi_x}\left[\frac{k^2}{\left(S_k^{(2)}[\phi_0]\right)^2}\left(\left(u_x + \mathrm{i}u_y\right)^2 - \partial_{\phi_x}\left(u_x + \mathrm{i}u_y\right)\right)\right] = 0\,.$$

General functional Flows

$$\partial_t P[\phi] = \frac{\delta}{\delta\phi(x)}\Big(\Psi[\phi]\, P[\phi]\Big)\,, \qquad P[\phi] = e^{-S_{\text{eff}}[\phi]}$$

- General reparametrisation with unchanged path integral.
- RG-Kernel:

$$\Psi[\phi] = \frac{1}{2}\mathcal{C}[\phi]\frac{\delta S_{\text{eff}[\phi]}}{\delta\phi} + \gamma_\phi \phi$$

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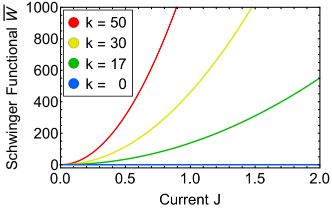
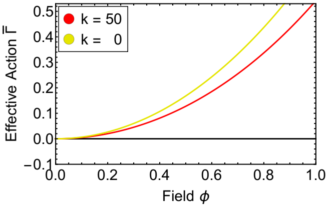
Choice of $\mathcal{C}[\phi]$:

- Field dependence
- Regulator

$$\partial_t W_k[J] = -\frac{1}{2}\,\mathrm{Tr}\,\partial_t R_k \left[W^{(2)}[J] + \left(W^{(1)}[J]\right)^2\right]$$

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{1}{\Gamma_k^{(2)}[\Phi] + R_k}\right)_{ab} \partial_t R_k^{ab}\right]$$





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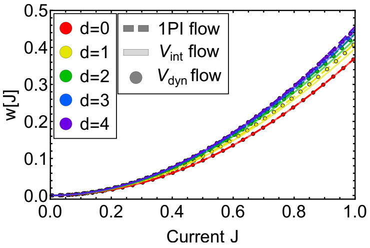

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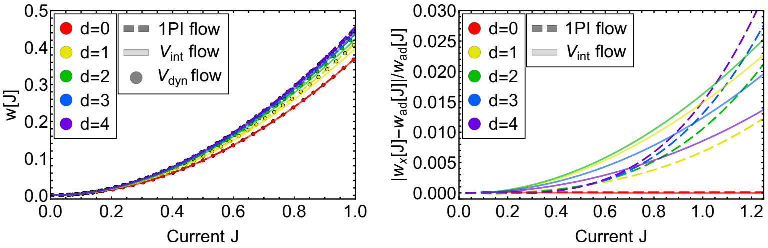
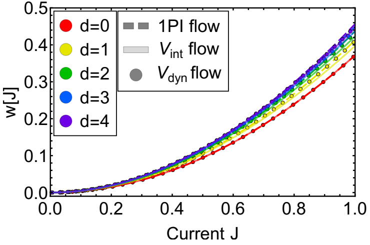
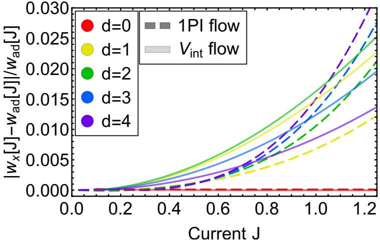


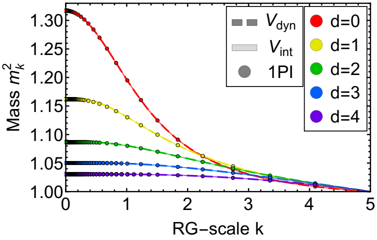
- Field dependence is not exact for $d > 0$.

Convergence beyond $d = 0$





- Field dependence is not exact for $d > 0$.
- Physics is the same!

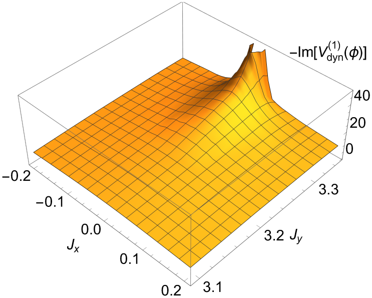


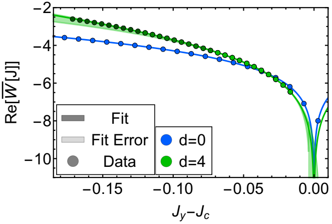
Computing the complex dynamical potential

- Resolve the complex plane in ϕ_y-slices.
- Singularity at $J_c = 3.3659(42)\, i$.
- Predicted from fixed-point analyses

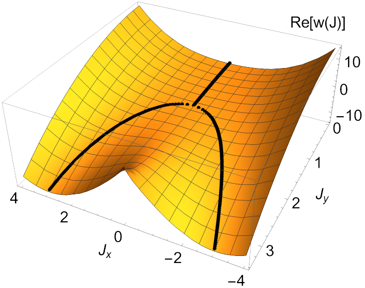
$$\mathrm{Re}[W(0, J_y)] = c\, J_y^2 + a + b \log(J_c - J_y)\,,$$

	b	J_c	c
$d = 0$	1.0008(55)	3.002104(65)	−0.281(12)
$d = 4$	1.24(41)	3.3659(42)	−0.99(91)





$$M(\phi_y, H) = \phi_{\mathrm{EoM}}\,, \quad \text{with} \quad \partial_{\phi_x} w[J(\phi_x, \phi_y)] - H|_{\phi_x = \phi_{x,\mathrm{EoM}}} = 0$$



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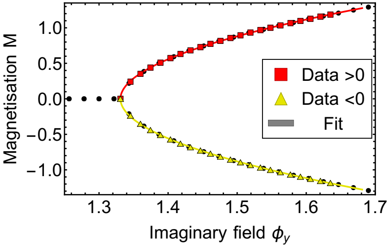


Mean Field: $\beta = 1/2$

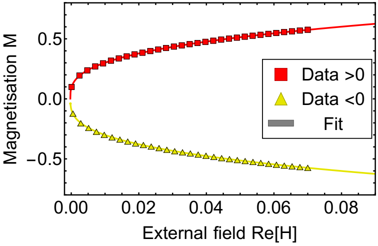
Fit: $0.505(23)$

Scaling relations:

- Preserves symmetry:

$$\mathrm{Re}\left[M(\phi_y, H=0)\right] = B\left(\frac{\phi_y - \phi_c}{\phi_c}\right)^{\beta}$$

$$M(\phi_y, H) = \phi_{\text{EoM}} , \quad \text{with} \quad \partial_{\phi_x} w[J(\phi_x, \phi_y)] - H|_{\phi_x = \phi_{x,\text{EoM}}} = 0$$



Mean Field: $\delta = 3$

Fit: $2.992(18)$

Scaling relations:

- Preserves symmetry:

$$\text{Re}\left[M(\phi_y, H = 0)\right] = B \left(\frac{\phi_y - \phi_c}{\phi_c}\right)^{\beta}$$

- Breaks symmetry:

$$\text{Re}\left[M(\phi_y = \phi_c, H)\right] = B_c H^{1/\delta}$$

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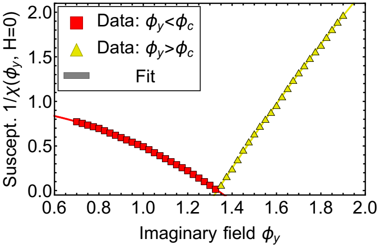

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- Susceptibility:

$$\chi(\phi_y, H=0) = C_{+/-}\left(\frac{\phi_y - \phi_c}{\phi_c} + \mathrm{subl.}\right)^{-\gamma}$$

Mean Field: $|C_+| = 2|C_-|$

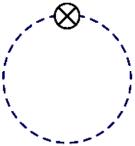
$$C_+ = -0.4369(11)\,,\; C_- = 0.2025(21)$$

$$R_\chi = \frac{C_+ B^{\delta-1}}{B_c^\delta} = 1.010(57)$$

Solving FRG-Equations

$$\partial_t u + f(\phi, \partial_\phi u) + g(\partial_\phi^2 u) = s(\phi) \ ,$$

$+\frac{1}{2}$



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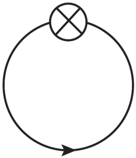

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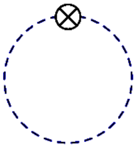
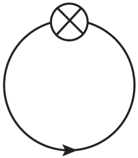
$+\frac{1}{2}$ 

$-$ 

PDE with second order derivatives:

$\Rightarrow$ Analogy to Hydrodynamics

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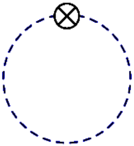
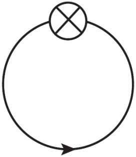
$+\frac{1}{2}$ 

$-$ 

PDE with second order derivatives:

$\Rightarrow$ Analogy to Hydrodynamics

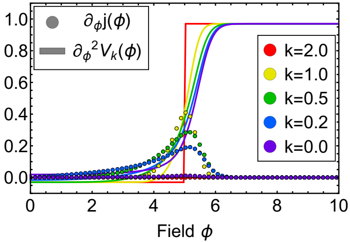
PDE with discontinuities:

$\Rightarrow$ Requires discontinuous methods

Dealing with (real) Diffusion: LDG-Methods

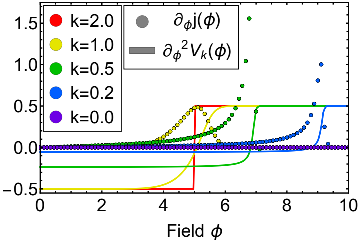
$$\partial_t u = \partial_q \left(F(t,q,u) + \sqrt{a(t,u)} v \right)$$

$$v = \sqrt{a(t,u)} \partial_q u$$



Riemann problem with positive diffusion:

- Slopes travel with convection.
- Slopes smooth out with diffusion.



Map $t \to -t$ for negative diffusion!

arXiv:2207.12266: F. Ihssen, J.M. Pawlowski, F. R. Sattler, N. Wink

$$\partial_t u = \partial_q \left(F\left(t, q, u\right) + \sqrt{a(t,u)}v \right)$$

$$v = \sqrt{a(t,u)}\partial_q u$$

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- $\sqrt{a} \propto \frac{1}{k^2+u} \Rightarrow$ Complex

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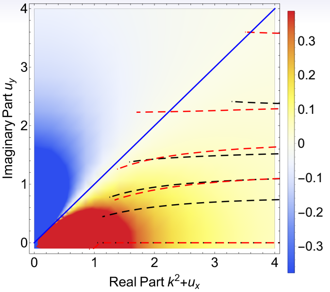

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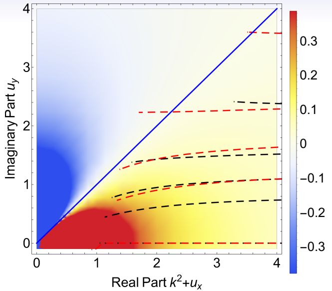

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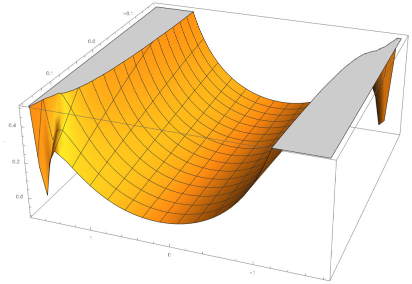

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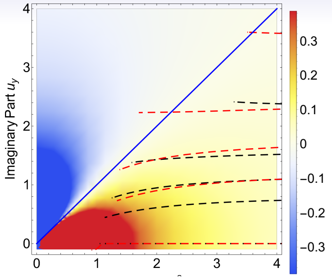

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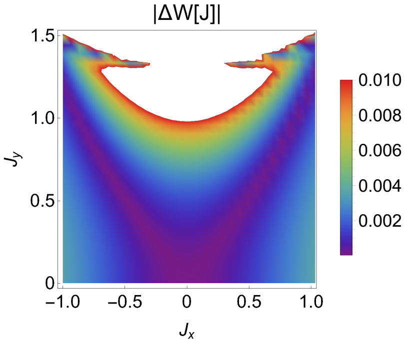

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Scalar conservation law:

$$\partial_t u + \partial_x f(u) = 0$$

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Convergence on Nodes:

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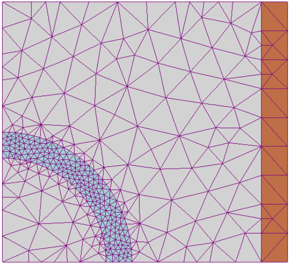
Orthogonal to test-functions:

$$\int_\Omega \mathcal{R}_h(t, x)\psi(x) = 0$$

Discontinuous Galerkin

Computational domain:

$$\Omega \simeq \Omega_h = \bigcup_{k=1}^{K} D^k$$



https://en.wikipedia.org/wiki/Finite_element_method

Solution in each cell:

$$u_h^k(t,x) = \sum_{n=1}^{N+1} \hat{u}_n^k(t)\psi_n(x)$$

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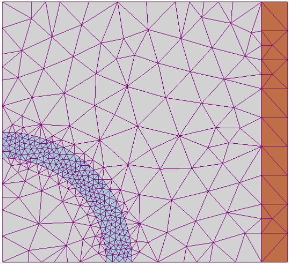

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/Finite_element_method

Best of Finite Volume

- Cell average vanishes.
- geometrically flexible
- Inherently discontinuous
- Higher order accuracy problems

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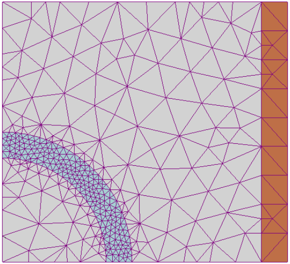

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Best of Finite Volume

- Cell average vanishes.
- geometrically flexible
- Inherently discontinuous
- Higher order accuracy problems

Best of Finite Element

- Residue vanishes weakly.
- Higher order accuracy
- Method is global

$$\int_{D^k} \Big((\partial_t u_h^k)\psi_n - \big(f_h^k(u_h^k) - \sqrt{a_h^k(u_h^k)}\, v_h^k\big)\partial_x \psi_n \Big)\, \mathrm{d}x = - \int_{\partial D^k} \Big(f_h^* + \mathrm{h}_1 \Big) \cdot \hat{n}\, \psi_n \, \mathrm{d}x \,,$$

$$\int_{D^k} \Big(v_h^k \, \psi_n + j_h^k(u_h) \partial_x \psi_n \Big)\, \mathrm{d}x = - \int_{\partial D^k} \mathrm{h}_2 \cdot \hat{n}\, \psi_n \, \mathrm{d}x$$

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- Conservative numerical flux:

$$f^*(u_h^+, u_h^-) = \frac{1}{2}\left(f_h(u_h^+) + f_h(u_h^-)\right) - \frac{C_{\mathrm{conv}}}{2}[\mathbf{u_h}]$$

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$$\mathbf{h}(u^-, u^+) = \begin{pmatrix} -\frac{j(u_h^+) - j(u_h^-)}{u_h^+ - u_h^-} \frac{v_h^+ + v_h^-}{2} \\ -\frac{j_h^+ + j_h^-}{2} \end{pmatrix} - \frac{C_{\mathrm{diff}}}{2}[\mathbf{h}] \,,$$

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$$j(s) = \int_0^s \sqrt{a(s')}\mathrm{d}s'$$

- Diffusive flux:

$$\mathbf{h}(u^-, u^+) = \begin{pmatrix} -\dfrac{j(u_h^+) - j(u_h^-)}{u_h^+ - u_h^-}\dfrac{v_h^+ + v_h^-}{2} \\ -\dfrac{j_h^+ + j_h^-}{2} \end{pmatrix} - \frac{C_{\mathrm{diff}}}{2}[\mathbf{h}]\,,$$

$$\sqrt{a(s)}\partial_x s = \partial_s j(s)\partial_x s = \partial_x j(s)$$

Implementation

Distributed and Unified Numerics Environment

- Modular toolbox (C++)
  - Variety of grids (1D, 2D ...).
  - Implementations of FEM, FVM, DG.
  - Various time-stepping modules
- Large-scale computing