Dimensional reduction along the RG flow in combinatorially non-local field theories

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Paradigmatic example: $\mathrm{O}(N)$-symmetric scalar field theory on d-dim. space M

$$\Gamma_k[\varphi] = \int_M \mathrm{d}^d x \left(\frac{1}{2} Z_k \partial\varphi_i \partial\varphi_i + U_k(\varphi_i\varphi_i) \right)$$

FRG equation $k\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\frac{\partial_k \mathcal{R}_k}{\Gamma_k^{(2)}+\mathcal{R}_k}$ with $\mathcal{R}_k = Z_k(k^2-p^2)\theta(k^2-p^2)$ in local potential approximation (LPA) for constant average field $\rho = \frac{1}{2}\varphi_i\varphi_i$:

$$k\partial_k U_k(\rho) = Z_k k^2 \left(\frac{1}{Z_k k^2 + U_k'(\rho) + 2\rho\, U_k''(\rho)} + \frac{N-1}{Z_k k^2 + U'(\rho)} \right) F_M(k)$$

- geometry of M encoded in spectral sum/threshold function $F_M(k)$
- general rescaling $U_k = F_M(k) u$ and $\rho = \frac{F_M(k)}{Z_k k^2}\tilde{\rho}$
- leads to k-dependent effective dimension $d_{\text{eff}}(k) := k\partial_k \log F_M(k)$

$$k\partial_k u + d_{\text{eff}}(k) u - (d_{\text{eff}}(k) - 2)\tilde{\rho} u' = \frac{1}{1 + u' + 2\tilde{\rho}\, u''} + \frac{N-1}{1+u'}$$

Dimensional flow in local field theory

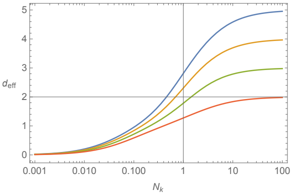
$$k\partial_k u + d_{\text{eff}}(k)u - (d_{\text{eff}}(k) - 2)\tilde{\rho}u' = \frac{1}{1 + u' + 2\rho\, u''} + \frac{N-1}{1+u'}$$

Standard case $M = \mathbb{R}^d$

- $F_{\mathbb{R}^d}(k) = \int_{|\boldsymbol{p}|<k} \mathrm{d}\boldsymbol{p} = c_d k^d$ power function
- constant dimension $d_{\text{eff}}(k) \equiv k\partial_k \log F_{\mathbb{R}^d}(k) = d$

Example of compact space: d-torus $M = \mathbb{T}_d$ of radius a

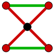
- $F_{\mathbb{T}_d}(k) = \sum_{|\boldsymbol{j}|<ak} = \sum_{s=0}^{d} 2^s \binom{d}{s} \binom{\lfloor ak \rfloor}{s}$
- for any compact M: polynomial of deg. d in $\lfloor ak \rfloor$
- d_{eff} flows from $d_{\text{eff}}(k \gg 1) = d$ to $d_{\text{eff}}(k \ll 1) = 0$
- $\rightarrow$ no Wilson-Fisher FP
- in limit $a \rightarrow \infty$ recovered (for $2 < d < 4$)

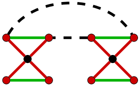
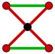


Compact M yields non-autonomous FRG equations and dimensional flow!

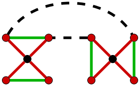
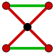
Combinatorial non-locality: from vectors...

Diagrammatic understanding of N dependence:

- green lines for index contractions at vertices, e.g. $(\varphi^2)^2 = (\varphi_i\varphi_i)^2 =$ 
- Second derivatives occuring in FRG eq.: $\delta_{\varphi_i}\delta_{\varphi_j}(\frac{1}{4}\varphi^2)^2 = \delta_{ij}\varphi^2 + 2\varphi_i\varphi_j$
- terms $\propto N$ have green line going through all "Wick contractions":

$\mathrm{Tr}[(\delta_{ij}\varphi^2)^2] \cong$  $\propto N \cdot$ 

- in all other terms, all green lines stop at some external vertex

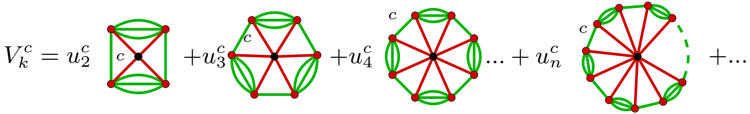
$\mathrm{Tr}[(\delta_{ij}\varphi^2)(\varphi_j\varphi_i)] \cong$  $\propto$ 

- same for higher orders φ^{2n}, and thus for expansion $U_k = \sum_n \frac{u_n}{n!}(\frac{1}{2}\varphi_j\varphi_j)^n$

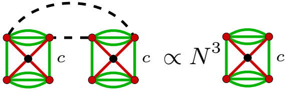
Combinatorial non-locality: ...to tensors

Interactions of tensor fields $\varphi_{j_1 j_2 \dots j_r}$ given by r-regular graph [$\to$ morning talk Gurau]

Example: "Cyclic-melonic" interactions



melonic interactions generate terms of various order in N up to N^{r-1}:



$\to$ FRG equation in cyclic-melonic potential approximation, $U_k(\rho) = \mu_k \rho + V_k(\rho)$:

$$k\partial_k U_k = Z_k k^2 \left(\frac{1}{Z_k k^2 + U_k' + 2\rho U_k''} + \frac{\binom{r}{1} N}{Z_k k^2 + U_k'} + \sum_{s=2}^{r} \frac{\binom{r}{s} N^s}{Z_k k^2 + \mu_k + \frac{r-s}{r} V_k'} \right) F_M(k)$$

Cyclic-melonic LPA at large N

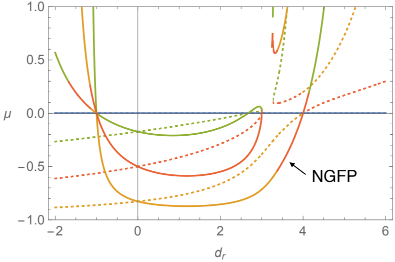
Rescaled FRG equation on $M = \mathbb{R}^d$ at large N

$$k\partial_k u + du - (d-2)\tilde{\rho}u' = \frac{1}{1 + (r-1)\tilde{\mu} + u'}$$

Modified Wilson-Fisher fixed point

only quantitative change due to r factor:

- couplings at FP have the same sign
- ex. plot $\tilde{\mu}^*$ at order ρ^4 (vertex exp.):
- again one relevant direction
- example: exponents in $d = 3$:



n	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
6	0.44448	-1.9006	-6.1670	-11.553	-16.454	-28.527				
7	0.45290	-1.8256	-4.7984	-9.8777	-13.603	-21.312	-34.652			
8	0.45314	-1.8669	-4.1832	-8.2540	-12.239	-17.179	-26.712	-41.022		
9	0.45218	-1.8834	-4.0306	-7.0618	-11.165	-14.647	-21.814	-32.301	-47.464	
10	0.45205	-1.8787	-4.0690	-6.3878	-10.063	-13.168	-18.442	-26.782	-38.014	-53.954
11	0.45214	-1.8757	-4.1043	-6.1630	-9.0992	-12.228	-16.073	-22.864	-31.940	-43.840
12	0.45217	-1.8761	-4.1011	-6.1886	-8.4649	-11.474	-14.452	-19.951	-27.551	-37.247

Combinatorially non-local field theory

Comb. non-local *field theory*: let tensor indices $j_1, j_2, ..., j_r$ propagate!

- propagator $\frac{1}{\mu+p^2+\frac{1}{a^2}\sum_c j_c^2}$
- regulator $\mathcal{R}_k$ function of $k^2 - p^2 - \frac{1}{a^2}\sum_c j_c^2$
- threshold functions $F_{d,r}$ are *combined* integrals $\int \mathrm{d}^d p$ *and* sums $\sum_{j_1,...,j_r}$, cut off by $p^2 + \frac{1}{a^2}\sum_c j_c^2 < k^2$

Ex: Cyclic-melonic potential approximation

Powers in N and threshold functions combine $N^s F_M(k) \to F_{d,s}(k)$

$$\frac{k\partial_k U}{Z_k k^2} = \frac{F_{d,0}(k)}{Z_k k^2 + U' + 2\rho U''} + \frac{\binom{r}{1} F_{d,1}(k)}{Z_k k^2 + U'} + \sum_{s=2}^{r} \frac{\binom{r}{s} F_{d,s}(k)}{Z_k k^2 + \mu_k + \frac{r-s}{r} V_k'}$$

- tensor indices contribute to scaling via $N \to N_k = ak$
- effective dimension d_{eff} flows from $d_r \equiv d + r - 1$ to d: nontrivial dim.reduction!
- also relative factor between μ and V flows from $1/r$ to 1

Motivation

Where do such theories occur?

- non-commutative QFT in matrix representation has propagator $\frac{1}{1+|j_1|+|j_2|}$
- Group-field quantum gravity on $g_c \in G$ compact subgroup of Lorentz group

$$\varphi(g_1, g_2, ..., g_r) \underset{\text{(Peter-Weyl)}}{\overset{\text{Fourier}}{\longleftrightarrow}} \varphi_{j_1, j_2, \ldots, j_r}$$

- generates $D = r$ dimensional discrete manifolds, $g \in G$ give geometry thereon
- continuum spacetime at critical points (∞ many diagrams = geometries)

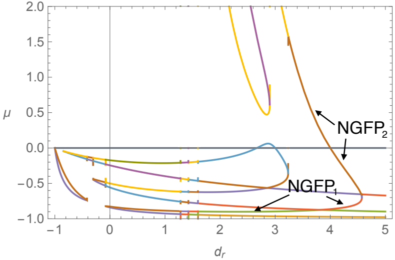
Beyond local potential approximation

Dynamic tensor d.o.f. generate derivative interactions

- already anomalous dimension η yields significant changes

$$\eta_k = \frac{d_r \bar{u}_2}{-2(1+\bar{\mu})^2 + \bar{u}_2}$$

- two branches of Wilson-Fisher-type FP
- hints for asymptotically safe FP...



Summary & Outlook

Results:

- dimensional reduction along the RG flow on compact domain
- cyclic-melonic LPA: flow $d_{\text{eff}} = d + r - 1 \to d$ and of ratio mass/couplings
- group-field QG: d_{eff} important for phase structure, but not spacetime dim. D !
- hints for new fixed points in LPA′ (flowing η)

Cyclic-melonic LPA only first step to understanding of non-local phase spaces:

- analytic methods for more precise statements about NGFPs
- rich combinatorics of tensors: FPs in other regimes (necklaces, connected graphs...)
- More realistic QG models on $\mathrm{SL}_2(\mathbb{C})$,
  Landau-Ginzburg already done [Marchetti/Oriti/Pithis/JT '22] $\to$ *talk Pithis tomorrow!*

Thanks for your attention!