(Coordinate-) invariant Renormalization-Group improvement

Aaron Held

DAAD PRIME Fellow, currently at **The Princeton Gravity Initiative**

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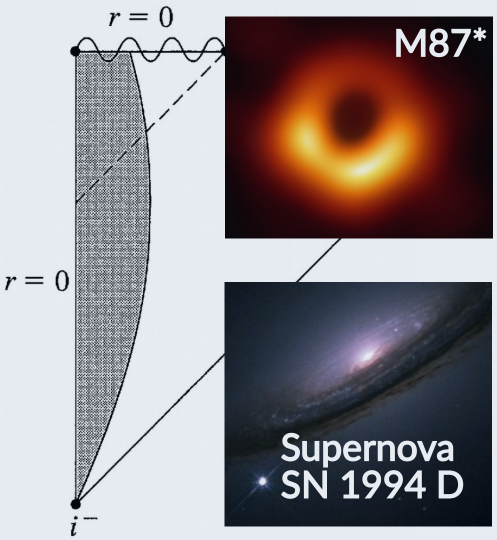
Motivation:

RG-improvement **could** give **qualitative** insights into a more complete (asymptotically safe) description of black holes.



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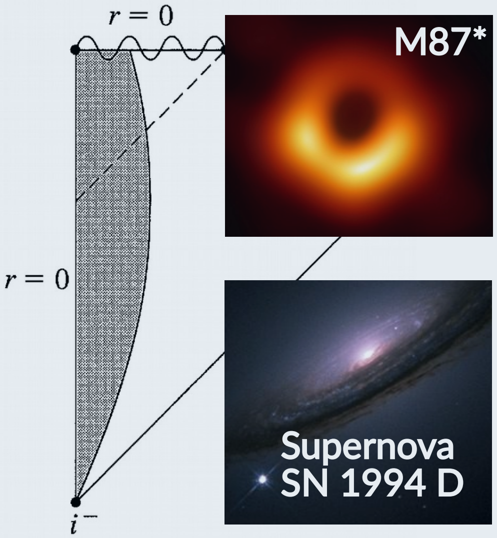

Key Question:

Is RG-improvement **coordinate independent?**

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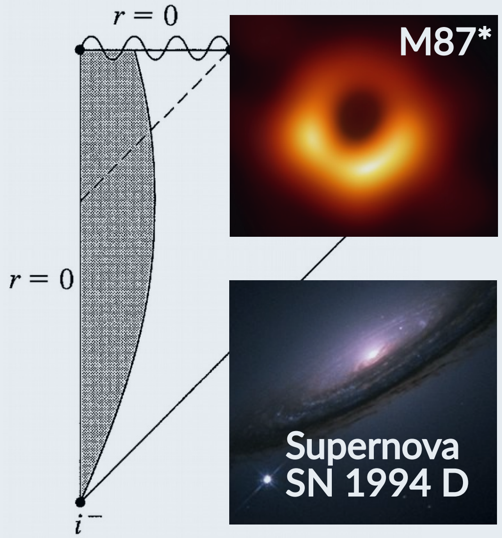

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Methodology:

How to tell two spacetimes apart?

$$\mathfrak{K}_1 = \frac{1}{8}\left(\mathcal{K}_5^2 - 2\,\mathcal{K}_6\right)^2 - \left(\mathcal{K}_6^2 - 2\,\mathcal{K}_8\right)$$

$$\mathfrak{K}_2 = \frac{1}{8}\mathcal{K}_5\left(\mathcal{K}_5^2 - 6\,\mathcal{K}_6\right)^2 + \mathcal{K}_7$$

Zakhary, McIntosh ‘97
Carminati, McLenaghan ‘91
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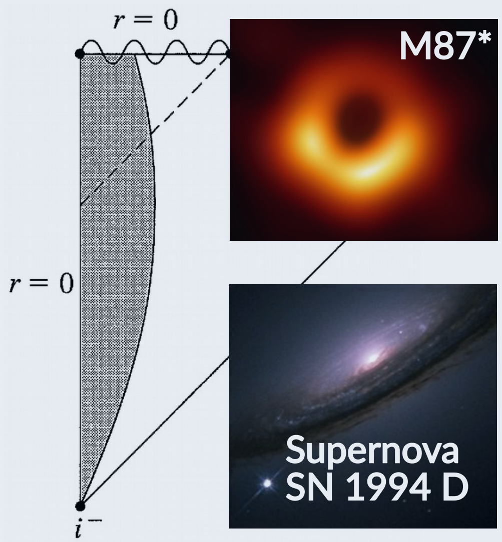

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Key Results:

Metric RG-improvement is **coordinate dependent**

$$\begin{array}{ccc} g(G_0, X) & \xrightarrow[X\mapsto \underline{X}=F(X)]{\text{coordinate trafo}} & g(G_0, \underline{X}) \\ \downarrow \text{RG} & & \downarrow \text{RG} \\ \tilde{g}(G(\mathcal{K}(X)), X) & \neq & \tilde{\underline{g}}(G(\mathcal{K}(\underline{X})), \underline{X}) \end{array}$$

Invariant RG-improvement is **coordinate independent**

$$\begin{array}{ccc} \mathcal{K}(G_0, X) & \xrightarrow[X\mapsto \underline{X}=F(X)]{\text{coordinate trafo}} & \mathcal{K}(G_0, \underline{X}) \\ \downarrow \text{RG} & & \downarrow \text{RG} \\ \widetilde{\mathcal{K}}(G(\mathcal{K}(X)), X) & \xrightarrow[X\mapsto \underline{X}=F(X)]{\text{coordinate trafo}} & \widetilde{\mathcal{K}}(G(\mathcal{K}(\underline{X})), \underline{X}) \end{array}$$

Brief review:
Renormalization-Group improvement of black-hole spacetimes

RG improvement: Coulomb potential

- classical Coulomb potential $V_{\text{classical}}(r) = -\frac{e^2}{4\pi r}$

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$$e^2(k) = \frac{e^2(k_0)}{1 - b\,\text{Log}(k/k_0)} \quad \text{with} \quad b = \frac{e^2(k_0)}{6\pi^2}$$

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$$k = \frac{1}{r}$$ (no other physical scale; unique by dimensional analysis)

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$$k = \frac{1}{r}$$

(no other physical scale; unique by dimensional analysis)

matches conventional perturbative result, cf. **Dittrich, Reuter '85**

- RG-improved Coulomb potential $V_{\text{RG-improved}}(r) = -\frac{e^2}{4\pi r}\left(1 + b\,\text{Log}(r/r_0)\right)$

RG improvement: Schwarzschild metric Reuter, Bonanno '99

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \qquad \text{with} \qquad f(r) = 1 - \frac{2\,M\,\bar{G}_N}{r}$$

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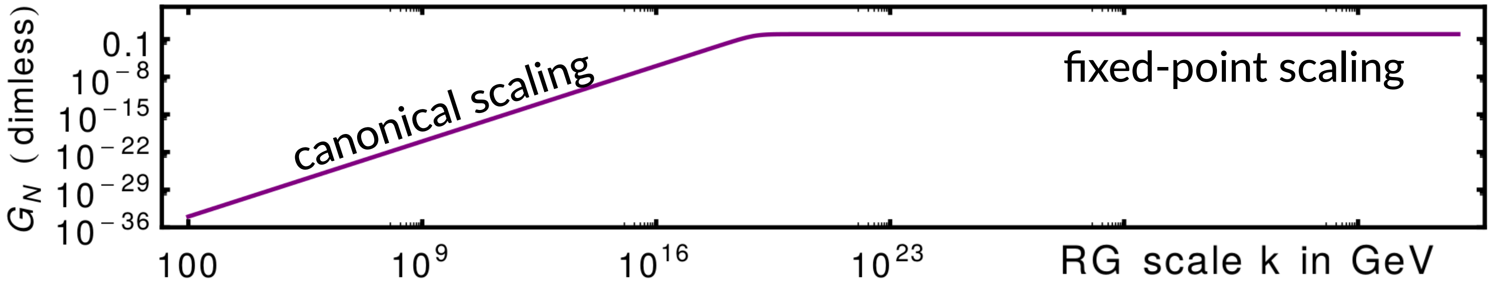

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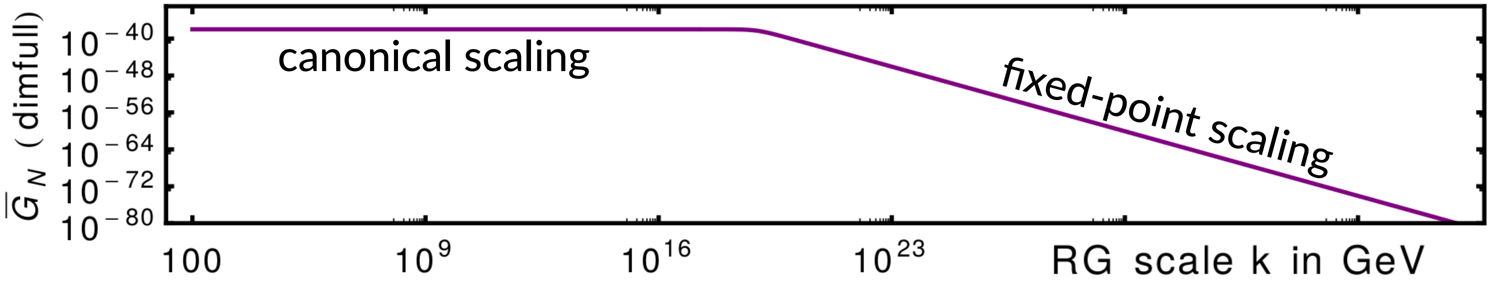

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- **dim'ful** running

$$\bar{G}_N(k^2) = \frac{\bar{G}_0}{1 + \gamma\,\bar{G}_0 k^2} = \begin{cases} \bar{G}_0 \text{ for } k \to 0 \\ 1/(\gamma\,k^2) \text{ for } k \to \infty \end{cases}$$

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$$k^4 = \text{curvature} \sim \text{Riem}^2 \sim \text{Weyl}^2 \sim \frac{\bar{G}_0^2}{r^6}$$

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without singularity

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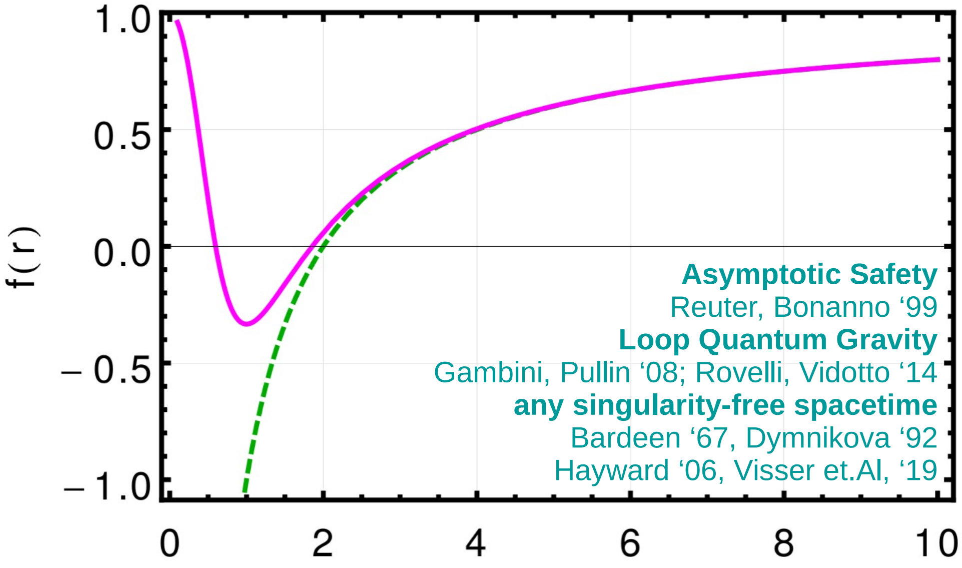

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RG improvement of the Kerr metric

$$ds_{BL} = -\frac{\Delta - a^2\,\sin(\theta)^2}{\Sigma}\,dt^2 + \frac{\Sigma}{\Delta}\,dr^2 + \rho^2\,d\theta^2 + \frac{(a^2+r^2)^2 - a^2\,\Delta\,\sin(\theta)^2}{\Sigma}\,\sin(\theta)^2\,d\phi^2 - \frac{2(a^2+r^2-\Delta)}{\Sigma}a\,\sin(\theta)^2\,dt\,d\phi$$

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Reuter, Tuiran '06, '10
Pawlowski, Stock '18

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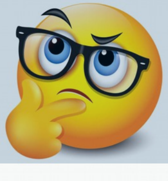
$$M\,\bar{G}_N(r) = M \frac{r^3/r_g^3}{r^3/r_g^3 + \tilde{\gamma}}$$

regular* ✔
Newtonian limit ✔

no longer based on local curvature scales



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Schwarzschild → Kerr (Newman-Janis algorithm; Newman, Janis '65; Drake, Szekeres '00)

- **Algebraically special** (Killing tensor)
- Geodesic motion **separable**
- **Hidden symmetry** (Carter constant)

regular Schwarzschild → regular non-local Kerr (Newman-Janis algorithm)

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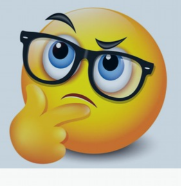
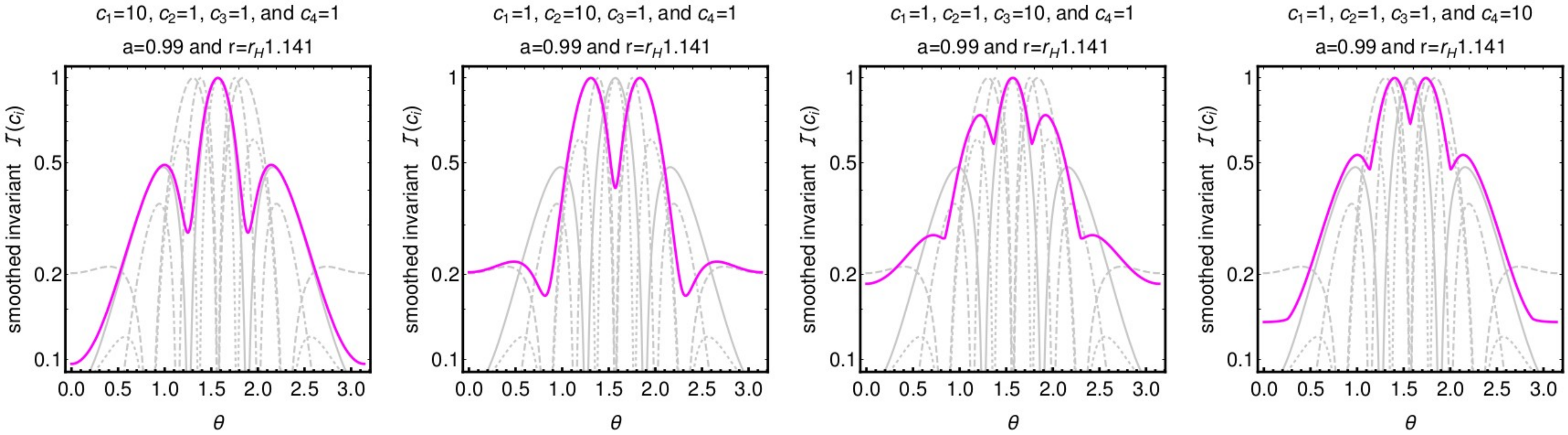


RG improvement of the Kerr metric

- distinct invariants imply **non-unique** scale identification

cf. **Lake '03; Overduin et.Al '19** for Kerr and **Zakhary, McIntosh '97** for general set of invariants



scale identification with a **weighted average**

$$k^2 = K_{GR} \equiv \left[c_1\,\mathcal{K}_1^2 + c_2\,\mathcal{K}_2^2 + c_3(\mathcal{K}_3^2)^{2/3} + c_4(\mathcal{K}_4^2)^{2/3}\right]^{1/4}$$

Eichhorn, Held, '21

RG improvement of the **Kerr metric**

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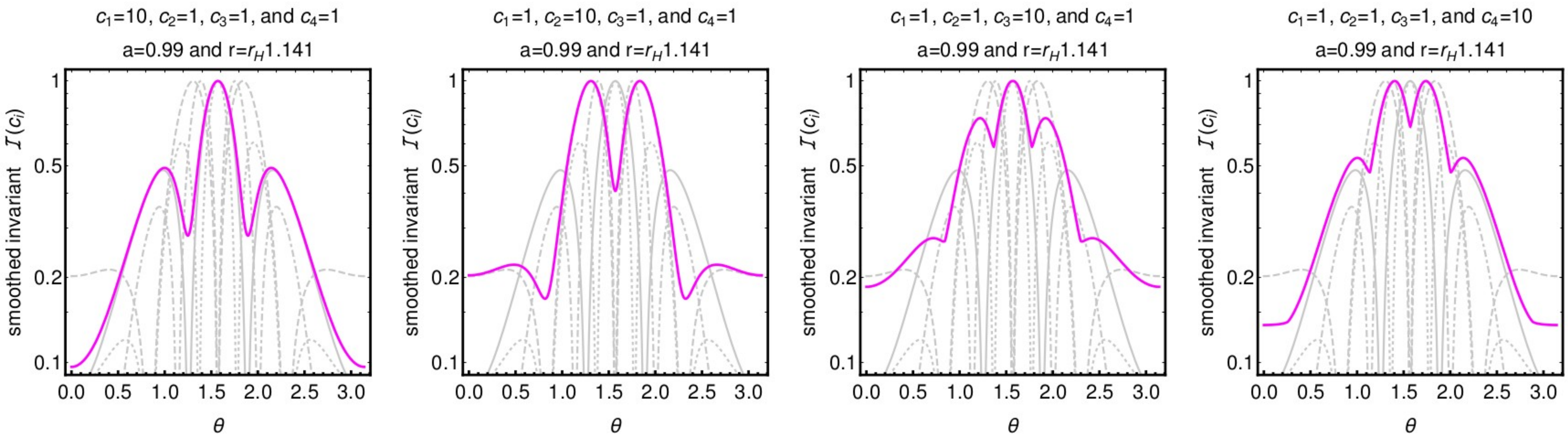

scale identification with an **envelope to the maximum**

$$k^2 = K_{GR} \equiv \left[\mathcal{K}_1^2 + \mathcal{K}_2^2\right]^{1/4} = \frac{48\, G_0^2 M^2}{(r^2 + a^2 \cos^2\theta)^3}$$

Eichhorn, Held, '21

RG improvement of the **Kerr metric**

$$ds_{BL} = -\frac{\Delta - a^2 \sin(\theta)^2}{\Sigma} dt^2 + \frac{\Sigma}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{(a^2+r^2)^2 - a^2 \Delta \sin(\theta)^2}{\Sigma} \sin(\theta)^2 d\phi^2 - \frac{2(a^2+r^2-\Delta)}{\Sigma} a \sin(\theta)^2 \, dt \, d\phi$$

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$M \, \bar{G}_N(k(r))$

regular at r=0* ✔

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RG-improvement tied to local curvature scales

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coordinate singularity at the horizon

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↓

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RG improvement of the Kerr metric

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RG-improvement, tied to local curvature scales

RG improvement of the Kerr metric

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$$ds_{\mathrm{Kerr}} = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}du^2 + 2\,du\,dr - \frac{2a\sin^2\theta\,(a^2-\Delta+r^2)}{\Sigma}du\,d\varphi - 2a\sin^2\theta\,dr\,d\varphi + \Sigma\,d\theta^2 + \frac{\sin^2\theta\left((a^2+r^2)^2 - a^2\Delta\sin^2\theta\right)}{\Sigma}d\varphi^2$$

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Methodology:
How to tell two spacetimes apart?

$$g_{\mu\nu}(X) \overset{?}{\longleftrightarrow} \underline{g}_{\mu\nu}(\underline{X})$$

Zakhary-McIntosh invariants

Zakhary, McIntosh ‘97
Carminati, McLenaghan ‘91

Weyl invariants

$$\mathcal{K}_1 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma},$$
$$\mathcal{K}_2 = C_{\mu\nu\rho\sigma}\overline{C}^{\mu\nu\rho\sigma},$$
$$\mathcal{K}_3 = C_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\alpha\beta}C_{\alpha\beta}{}^{\mu\nu},$$
$$\mathcal{K}_4 = \overline{C}_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\alpha\beta}C_{\alpha\beta}{}^{\mu\nu},$$

Ricci invariants

$$\mathcal{K}_5 = R,$$
$$\mathcal{K}_6 = R_\mu{}^\nu R_\nu{}^\mu,$$
$$\mathcal{K}_7 = R_\mu{}^\nu R_\nu{}^\rho R_\rho{}^\mu,$$
$$\mathcal{K}_8 = R_\mu{}^\nu R_\nu{}^\rho R_\rho{}^\sigma R_\sigma{}^\mu,$$

Mixed invariants

$$\mathcal{K}_9 = R^{\mu\nu}R^{\rho\sigma}C_{\mu\nu\rho\sigma},$$
$$\mathcal{K}_{10} = R^{\mu\nu}R^{\rho\sigma}\overline{C}_{\mu\nu\rho\sigma},$$
$$\mathcal{K}_{11} = R^{\nu\rho}R_{\gamma\delta}\left(C_{\mu\nu\rho\sigma}C^{\mu\gamma\delta\sigma} - \overline{C}_{\mu\nu\rho\sigma}\overline{C}^{\mu\gamma\delta\sigma}\right),$$
$$\mathcal{K}_{12} = 2R^{\nu\rho}R_{\gamma\delta}C_{\mu\nu\rho\sigma}\overline{C}^{\mu\gamma\delta\sigma},$$
$$\mathcal{K}_{13} = R_\mu{}^\gamma R_\gamma{}^\rho R_\nu{}^\delta R_\delta{}^\sigma C^{\mu\nu}{}_{\rho\sigma},$$
$$\mathcal{K}_{14} = R_\mu{}^\gamma R_\gamma{}^\rho R_\nu{}^\delta R_\delta{}^\sigma \overline{C}^{\mu\nu}{}_{\rho\sigma},$$
$$\mathcal{K}_{15} = \frac{1}{16}R^{\nu\rho}R_{\gamma\delta}\left(C_{\mu\nu\rho\sigma}C^{\mu\gamma\delta\sigma} + \overline{C}_{\mu\nu\rho\sigma}\overline{C}^{\mu\gamma\delta\sigma}\right),$$
$$\mathcal{K}_{16} = \frac{1}{32}R^{\rho\sigma}R^{\gamma\delta}C^{\mu\kappa\lambda\nu}\left(C_{\mu\rho\sigma\nu}C_{\kappa\gamma\delta\lambda} + \overline{C}_{\mu\rho\sigma\nu}\overline{C}_{\kappa\gamma\delta\lambda}\right),$$
$$\mathcal{K}_{17} = \frac{1}{32}R^{\rho\sigma}R^{\gamma\delta}\overline{C}^{\mu\kappa\lambda\nu}\left(C_{\mu\rho\sigma\nu}C_{\kappa\gamma\delta\lambda} + \overline{C}_{\mu\rho\sigma\nu}\overline{C}_{\kappa\gamma\delta\lambda}\right),$$

Zakhary-McIntosh invariants

Zakhary, McIntosh '97
Carminati, McLenaghan '91

Weyl invariants

$$\mathcal{K}_1 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma},$$
$$\mathcal{K}_2 = C_{\mu\nu\rho\sigma}\overline{C}^{\mu\nu\rho\sigma},$$
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Ricci invariants

$$\mathcal{K}_5 = R,$$
$$\mathcal{K}_6 = R_\mu{}^\nu R_\nu{}^\mu,$$
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Mixed invariants

$$\mathcal{K}_9 = R^{\mu\nu}R^{\rho\sigma}C_{\mu\nu\rho\sigma},$$
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- polynomially complete basis of Riemann invariants

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Cartan '28
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$$\mathbb{I} = \frac{48\,G_0^2 M^2}{(r - i\,a\cos\theta)^6}$$

Kerr spacetime

- polynomially complete basis of Riemann invariants
- usually sufficient to fully characterize spacetimes
- otherwise: Cartan-Karlhede algorithm

Cartan '28
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- otherwise: Cartan-Karlhede algorithm
  Cartan '28
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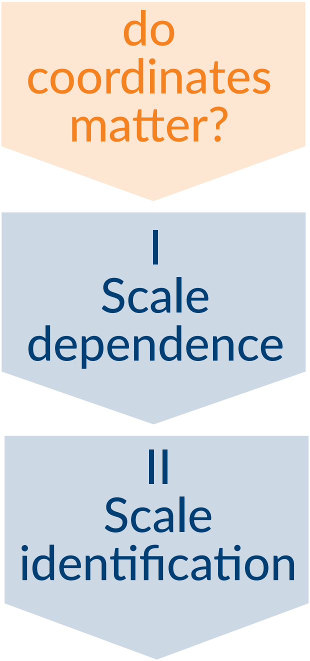
- spacetimes are inequivalent if polynomial relations (**syzygies**) among the ZM-invariants disagree

1st key result:
Metric RG-improvement is coordinate dependent

Metric RG-improvement ...



... is coordinate dependent.

Metric RG-improvement ...

✔ **Coulomb-potential** is a scalar quantity

do coordinates matter?

I
Scale dependence

II
Scale identification

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Metric RG-improvement ...

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Schwarzschild metric
is a tensorial quantity ✗

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Metric RG-improvement ...

coordinate transformation $\quad F : X^a \mapsto \underline{X}^{\underline{a}} = F^{\underline{a}}(X)$

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of the metric
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such that $f^{\underline{a}}{}_a(X, \overline{G}_0) \equiv \frac{\partial F^{\underline{a}}(X, \overline{G}_0)}{\partial X^a}$

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Metric RG-improvement ...

any RG-improved coordinate transformation $F : X^a \mapsto \underline{X}^{\underline{a}} = F^{\underline{a}}(X)$

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$$\equiv g_{\underline{ab}}(F(X))\;\; f^{\underline{a}}{}_a \;\; f^{\underline{b}}{}_b \;\; dX^a\, dX^b$$

upsets $f^{\underline{a}}{}_a(X, \overline{G}_N(X)) \neq \dfrac{\partial F^{\underline{a}}(X, \overline{G}_N(X))}{\partial X^a}$

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Metric RG-improvement ...

The two Ricci syzygies ...

$$\mathfrak{K}_1 = \frac{1}{8}\left(\mathcal{K}_5^2 - 2\,\mathcal{K}_6\right)^2 - \left(\mathcal{K}_6^2 - 2\,\mathcal{K}_8\right)$$

$$\mathfrak{K}_2 = \frac{1}{8}\mathcal{K}_5\left(\mathcal{K}_5^2 - 6\,\mathcal{K}_6\right)^2 + \mathcal{K}_7$$

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… which **distinguish metric-RG improved Schwarzschild spacetime** rooted in …

- Schwarzschild or Eddington-Finkelstein coordinates $\quad \mathfrak{K}_1 = 0, \ \mathfrak{K}_2 = 0$
- from modified horizon-penetrating coordinates. $\quad \mathfrak{K}_1 \neq 0, \ \mathfrak{K}_2 \neq 0$

$$x = \frac{r^n}{(MG_0)^{n-1}}$$

… is coordinate dependent.

Metric RG-improvement ...

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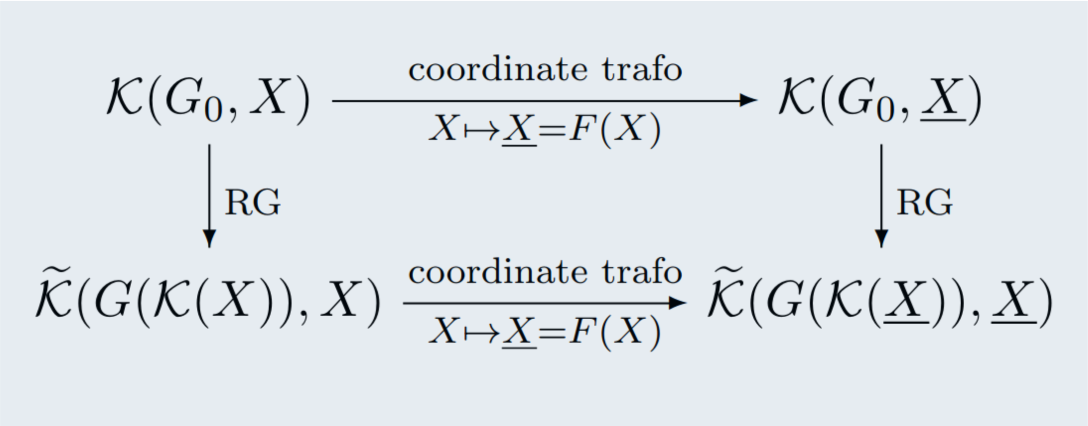
... which **distinguish metric-RG improved Kerr spacetime** rooted in ...

- ingoing Kerr coordinates $\quad \mathfrak{K}_1 = 0,\ \mathfrak{K}_2 = 0$
- from Boyer-Lindquist coordinates $\quad \mathfrak{K}_1 \neq 0,\ \mathfrak{K}_2 \neq 0$

... is coordinate dependent.

2nd key result:
Invariant RG-improvement is coordinate independent

Invariant RG-improvement ...



... is coordinate independent.

Invariant RG-improvement ...

Schwarzschild spacetime $\left(\frac{\mathcal{K}_1}{48}\right)^3 = \left(\frac{\mathcal{K}_3}{96}\right)^2 = \left(\frac{G_0\, M}{r^3}\right)^6$ and $\mathcal{K}_{i \neq 1,3} = 0$

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I: Scale dependence
$$\overline{G}_N(k^2) = \frac{\overline{G}_0}{1+\ell_{NP}^2 k^2}$$

II: Scale identification
$$k^4 = \mathcal{K}_1 = \frac{\overline{G}_0^2 M^2}{r^6}$$

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Invariant-RG-improved Schwarzschild spacetime

$$\left(\frac{\mathcal{K}_1}{48}\right)^3 = \left(\frac{\mathcal{K}_3}{96}\right)^2 = \left(\frac{\overline{G}_N(\mathcal{K}_1(r))\, M}{r^3}\right)^6$$

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requires some of the $\mathcal{K}_{i\geqslant 5} \neq 0$ Birkhoff's theorem

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Can we reconstruct the metric?

$$ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega$$

Invariant-RG-improved Schwarzschild spacetime $\left(\frac{\mathcal{K}_1}{48}\right)^3 = \left(\frac{\mathcal{K}_3}{96}\right)^2 = \left(\frac{\overline{G}_N(\mathcal{K}_1(r))\,M}{r^3}\right)^6$

requires some of the $\mathcal{K}_{i\geqslant 5} \neq 0$ Birkhoff's theorem

... is coordinate independent.

Invariant RG-improvement ...

Schwarzschild spacetime $\left(\frac{\mathcal{K}_1}{48}\right)^3 = \left(\frac{\mathcal{K}_3}{96}\right)^2 = \left(\frac{G_0\, M}{r^3}\right)^6$ and $\mathcal{K}_{i\neq 1,3} = 0$

I: Scale dependence

$$\overline{G}_N(k^2) = \frac{\overline{G}_0}{1 + \ell_{NP}^2 k^2}$$

II: Scale identification

$$k^4 = \mathcal{K}_1 = \frac{\overline{G}_0^2 M^2}{r^6}$$

Can we reconstruct the metric?

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega$$

simplifying assumption

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Invariant RG-improvement ...

Schwarzschild spacetime
$$\left(\frac{\mathcal{K}_1}{48}\right)^3 = \left(\frac{\mathcal{K}_3}{96}\right)^2 = \left(\frac{G_0\, M}{r^3}\right)^6 \quad \text{and} \quad \mathcal{K}_{i\neq 1,3} = 0$$

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$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega$$

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Invariant-RG-improved Schwarzschild spacetime
$$\frac{\left(r^2 A'' - 2rA' + 2A - 2\right)^2}{3r^4} = \mathcal{K}_1 = \frac{48 G(\mathcal{K}_1(r))^2 M^2}{r^6}$$

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Invariant RG-improvement ...

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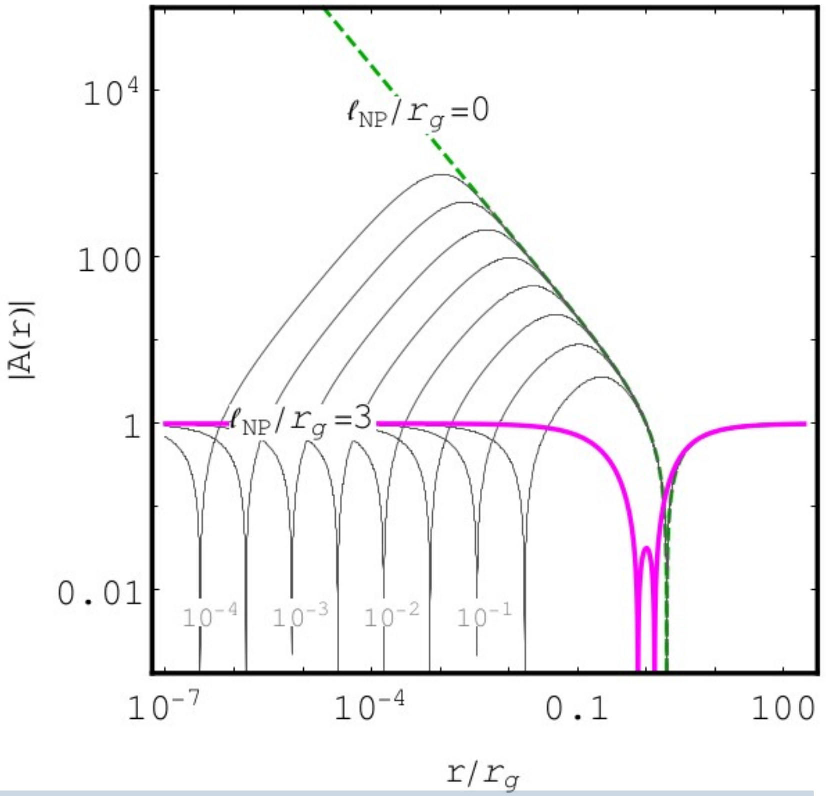

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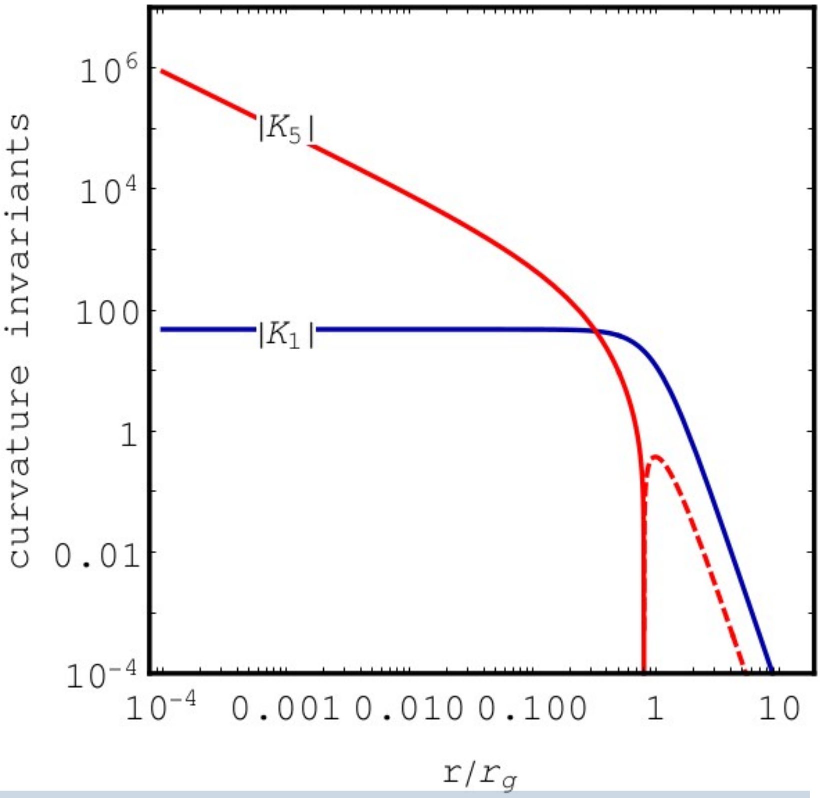
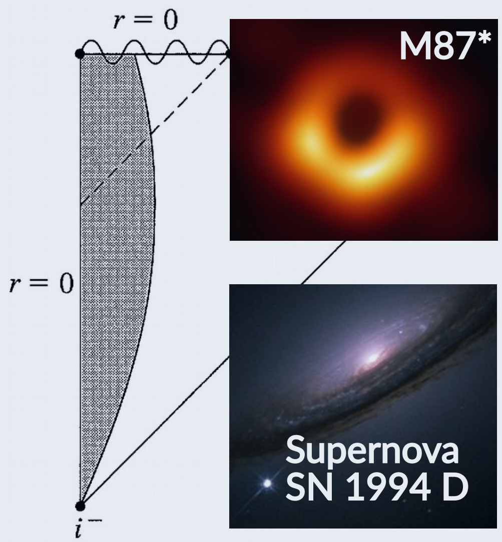


... is coordinate independent.

Motivation:

RG-improvement **could** give **qualitative** insights into a more complete (asymptotically safe) description of black holes.



Key Question:

Is RG-improvement **coordinate independent?**

Methodology:

How to tell two spacetimes apart?

$$\mathfrak{K}_1 = \frac{1}{8}\left(\mathcal{K}_5^2 - 2\,\mathcal{K}_6\right)^2 - \left(\mathcal{K}_6^2 - 2\,\mathcal{K}_8\right)$$

$$\mathfrak{K}_2 = \frac{1}{8}\mathcal{K}_5\left(\mathcal{K}_5^2 - 6\,\mathcal{K}_6\right)^2 + \mathcal{K}_7$$

Zakhary, McIntosh '97
Carminati, McLenaghan '91
Karlhede '80
Cartan '28

Key Results:

Metric RG-improvement is **coordinate dependent**

$$\begin{array}{ccc} g(G_0, X) & \xrightarrow[X\mapsto \underline{X}=F(X)]{\text{coordinate trafo}} & g(G_0, \underline{X}) \\ \downarrow \text{RG} & & \downarrow \text{RG} \\ \tilde{g}(G(\mathcal{K}(X)), X) & \neq & \tilde{\underline{g}}(G(\mathcal{K}(\underline{X})), \underline{X}) \end{array}$$

Invariant RG-improvement is **coordinate independent**

$$\begin{array}{ccc} \mathcal{K}(G_0, X) & \xrightarrow[X\mapsto \underline{X}=F(X)]{\text{coordinate trafo}} & \mathcal{K}(G_0, \underline{X}) \\ \downarrow \text{RG} & & \downarrow \text{RG} \\ \widetilde{\mathcal{K}}(G(\mathcal{K}(X)), X) & \xrightarrow[X\mapsto \underline{X}=F(X)]{\text{coordinate trafo}} & \widetilde{\mathcal{K}}(G(\mathcal{K}(\underline{X})), \underline{X}) \end{array}$$