Renormalization group and probability theory: distribution of order parameter at criticality

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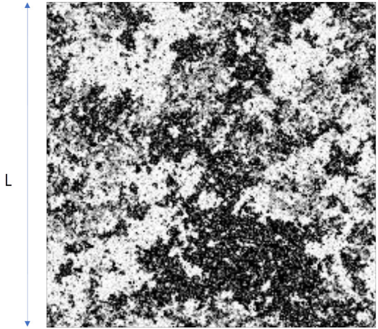
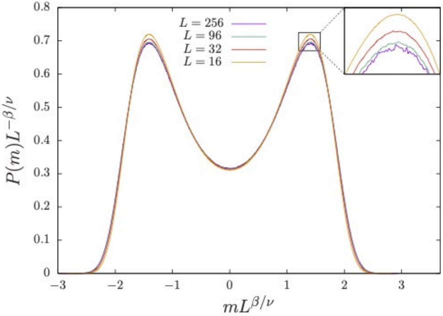
The rate function





3d Ising Xu etal. '20

$P_L(m) = e^{-I_L(m)}$, $I_L(m)$ - "the rate function"

Point 1: Generalization of the Central limit theorem

- Uncorrelated random variables with finite variance $\Rightarrow$ CLT

a) Weakly correlated random variables (Botet etal. '02; Dedecker etal. '07)

b) random variables with infinite variance (Levy '54; Ibe '13)

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$\Rightarrow$ FRG

Point 2: What is the rate function?

Is it the fixed point potential?

Perturbative RG calculations: Bruce '79; Eisenriegler '87; Chen & Dohm '95; Rudnick '98

- constraint action; effective Hamiltonian; constrained free energy
- rate function related to fixed point potential
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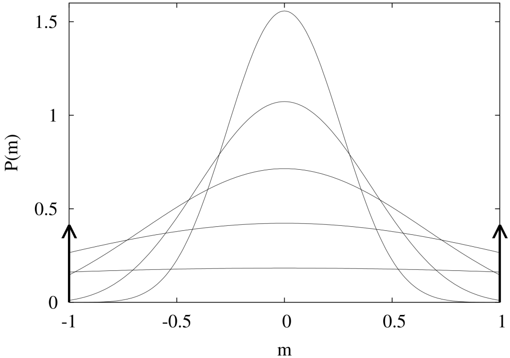
Problem 1: rate function should be universal!
Problem 2: dependence on L is crucial

Point 3: Is the rate function unique?

One or many rate functions?



1d Ising Antal, Droz & Racz '03

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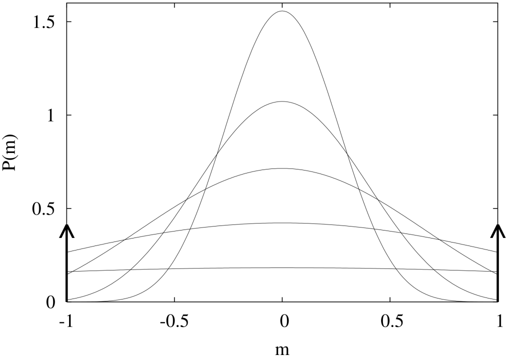

1d Ising Antal, Droz & Racz '03

Universal family of functions parametrized by $\frac{L}{\xi_\infty}$

Setup of the problem

$$P(s) \propto \int D\hat{\phi}\delta\left(s - \frac{1}{L^d}\int_x \hat{\phi}_x\right)\exp(-\mathcal{H}[\hat{\phi}]) \qquad (1)$$

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Exponentiate the "δ":

$\Rightarrow \mathcal{H}_M(\hat{\phi}) = \mathcal{H}(\hat{\phi}) + \frac{M^2}{2}\int_x \left(\hat{\phi} - s\right)^2$; with $M \to \infty$

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Use $\mathcal{H}_M$ as the bare action in the Wetterich formalism:

$$\partial_k \Gamma_{M,k}[\phi] = \frac{1}{2}\int_{x,y} \partial_k R_{M,k}(x,y)\left(\Gamma^{(2)}_{M,k} + R_{M,k}\right)^{-1}(x,y), \tag{2}$$

Setup of the problem

a) if $M = 0$: $\Gamma_{0,k}[\phi]|_{\phi=s} = L^d U_k(s)$

b) if $M \to \infty$: $\Gamma_{\infty,k}[\phi]|_{\phi=s} = L^d I_k(s)$

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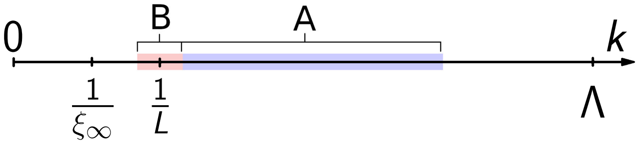
Example - LPA

$$\partial_k U_k(m) = \frac{1}{2L^d} \sum_q \frac{\partial_k R_k(q)}{q^2 + R_k(q) + U_k''(m)}, \tag{3}$$

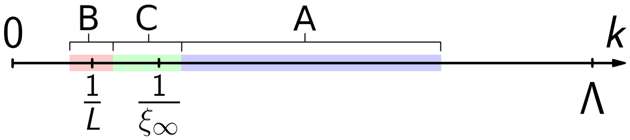
$$\partial_k I_k(s) = \frac{1}{2L^d} \sum_{q \neq 0} \frac{\partial_k R_k(q)}{q^2 + R_k(q) + I_k''(s)}. \tag{4}$$

$\xi_\infty > L$



- A- Flow of I_k approaches the plateau; scaling function approaches fixed point potential
- B- Flow abruptly terminates leaving I_0 convex; if $\xi_\infty \to \infty$ scaling function similar to fixed point potential

$\xi_\infty < L$



A- Flow of I_k approaches the plateau; scaling function approaches fixed point potential

C- The shape of I_k tends to change to convex; flow tends to slow drastically due to mass

B- Flow abruptly terminates; I_0 is either convex or concave depending on L/ξ_∞

FRG and Monte Carlo comparison @LPA

- fix 2 arbitrary scales (at a given ζ!) between FRG and MC
- obtain universal scaling functions for all $\zeta = sign(T - T_c)\frac{L}{\xi_\infty(T - T_C)}$

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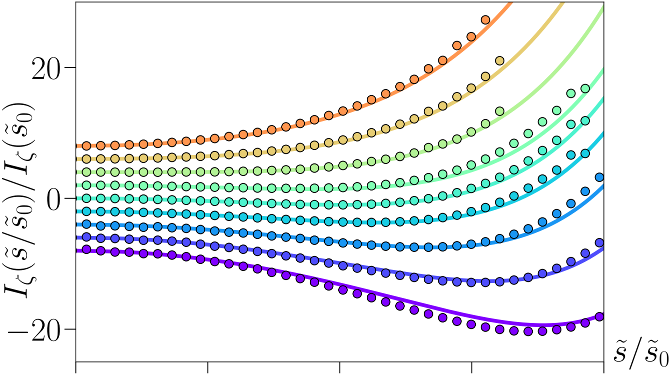
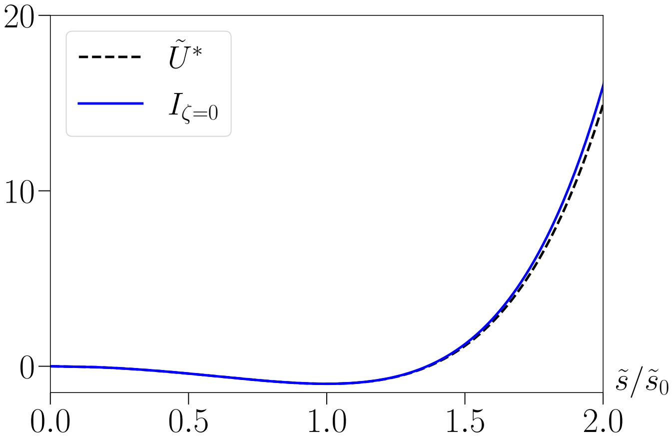


3d Ising, $\zeta \in [-4, 4]$

I_0 vs U^*

- I_0 and U^* closely resemble when $\zeta \to 0$



I_0 vs U^* for $\zeta = 0$

I_0 is universal

Case-in-point: $N \to \infty$ limit (to appear :I.Balog, B.Delamotte, A.Rançon)

- LPA is exact
- $\rho = \frac{\phi}{2}$; τ a function of L/ξ
- $I = I_0$ is universal (ϑ is elliptic theta function!)

$$\frac{\rho - \tau}{N} = -\frac{1}{2}\int_0^\infty dse^{-I(\rho)'s}(\vartheta(\frac{4\pi s}{L^2})^d - 1) + \frac{1}{2(4\pi)^{\frac{d}{2}}}\int_0^\infty ds\frac{ds}{s^{\frac{d}{2}}}. \quad (5)$$

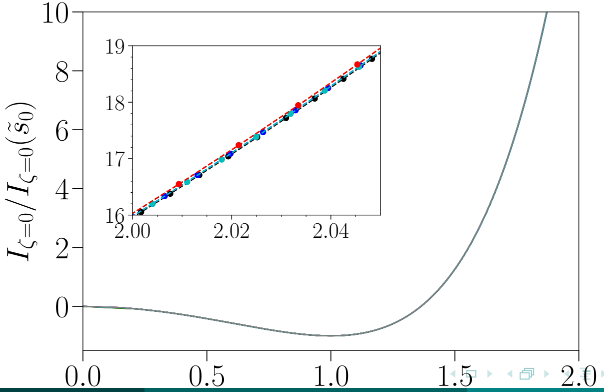
- U^* is regulator, $r(y)$ dependent (see also Knorr '19)

$$\frac{\rho}{N} = v_d \int dyy^{\frac{d}{2}-1}\Big(\frac{yr(y) + U^{*\prime}(\rho)}{y(yr(y) + y + U^{*\prime}(\rho))}\Big). \quad (6)$$

I_0 is universal

For finite N

- LPA is not exact
- regulator dependence order of magnitude smaller than difference between I_0 and U^*
- regulator dependence of U^* much larger than of I_0



Conclusions

- We construct the rate function I from the FRG
- It is a universal function parametrized by $\zeta = \frac{L}{\xi_\infty}$
- LPA calculation gives extraordinary collapse with MC data for the 3d Ising model (and for the 3d O(N) models as well)
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To do:

- rate function for different models
- probability distributions near the lower (talk of Lucija Nora Farkaš on Thursday) and upper critical dimension
- extension to momentum dependent quantities

Check our paper on arXiv:2206.03769