From Random Tensors to Tensor Field Theory

Răzvan Gurău (Berlin, 2022)

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STRUCTURES CLUSTER OF EXCELLENCE

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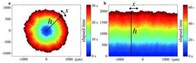
Random matrices

[Wishart ’28, Wigner ’55]

- Theory of strong interactions [’t Hooft, etc.]
- Random surfaces [David, Kazakov, Fröhlich, etc.]
- Growing interfaces fluctuations [Kardar, Parisi, Zhang, etc.]



- Free probability theory [Voiculescu, Guionnet, etc.]
- Machine learning...

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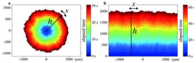

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Size of the matrices is a parameter leading to a “$1/N$ expansion”:

- take matrices of large size $N \to \infty$
- compute order by order in $1/N$

Random tensors

Generalize random matrices (M_{ab}) to random higher order tensors (T_{abc}) [Ambjørn Durhuus Jonsson '90, Boulatov '92, Ooguri '92, etc.] and [2010: RG, Rivasseau, Oriti, Bonzom, Carrozza, Benedetti, Lionni, Tanasa, Ben Geloun, Dartois, Kolanowski ...]

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But why?

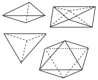
Random (quantum) spaces

Random tensors → theory of random infinitely refined spaces

GR + QFT ∼ geometry + random

Idea - build the geometry by gluing discrete "space time quanta"

$$\sum_{\text{topologies}} \int \mathcal{D}g_{(\text{metrics})} \to \sum_{\text{random discretizations}}$$



Tensor invariants as Colored Graphs

$$T'_{b^1\dots b^D} = \sum_a U^{(1)}_{b^1 a^1} \dots U^{(D)}_{b^D a^D} T_{a^1\dots a^D} \ , \quad \bar{T}'_{p^1\dots p^D} = \sum_q \bar{U}^{(1)}_{p^1 q^1} \dots \bar{U}^{(D)}_{p^D q^D} \bar{T}_{q^1\dots q^D}$$

Invariant “traces” $\sum_{a^1,q^1} \delta_{a^1 q^1} \dots T_{a^1\dots a^D} \bar{T}_{q^1\dots q^D} \dots \to$ colored graphs $\mathcal{B}$.

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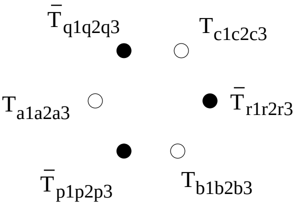

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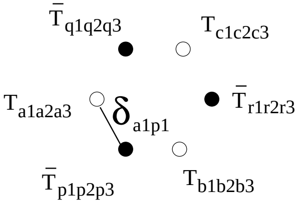

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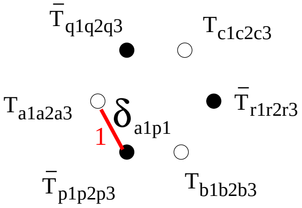

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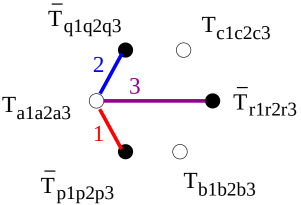

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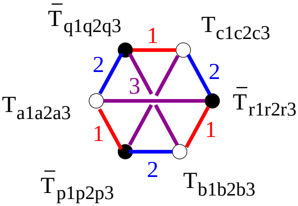

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$$\mathrm{Tr}_{\mathcal{B}}(T,\bar{T}) = \sum \prod_v T_{a^1_v\dots a^D_v} \prod_{\bar{v}} \bar{T}_{q^1_{\bar{v}}\dots q^D_{\bar{v}}} \prod_{c=1}^{D} \prod_{e^c=(w,\bar{w})} \delta_{a^c_w q^c_{\bar{w}}}$$

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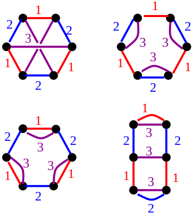

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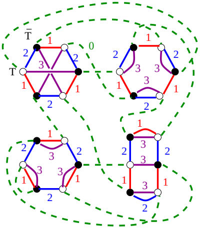

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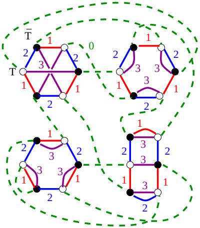

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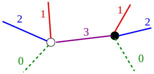
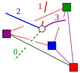
Each graph $\mathcal{G}$ is Poincaré dual to a triangulation

Colored graphs and vertex colored triangulations

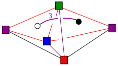
White and black $D+1$ valent vertices connected by edges with colors $0, 1 \dots D$.



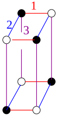
Vertex $\leftrightarrow$ D simplex with colored vertices .

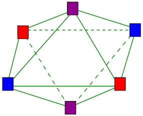


Edges $\leftrightarrow$ gluings along $D-1$ simplices respecting all the colorings



Invariants $\mathrm{Tr}_{\mathcal{B}}$: graphs with D colors $\leftrightarrow$ $D-1$ dimensional boundary triangulations.





Observables and expectations

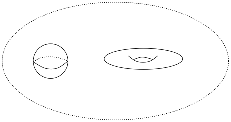
Observables: invariants $\mathrm{Tr}_{\mathcal{B}}$ encoding boundary triangulations.

Expectations:

$$\left\langle \mathrm{Tr}_{\mathcal{B}_1}\mathrm{Tr}_{\mathcal{B}_2}\ldots\mathrm{Tr}_{\mathcal{B}_q}\right\rangle = \frac{1}{\mathcal{Z}(t_{\mathcal{B}})}\int [d\bar{T}dT]\;\mathrm{Tr}_{\mathcal{B}_1}\mathrm{Tr}_{\mathcal{B}_2}\ldots\mathrm{Tr}_{\mathcal{B}_q}\; e^{-N^{D-1}S(T,\bar{T})}$$

correlations between boundary states given by sums over all bulk triangulations compatible with the boundary states

- $\left\langle \mathrm{Tr}_{\mathcal{B}_1}\mathrm{Tr}_{\mathcal{B}_2}\right\rangle_c = \left\langle \mathrm{Tr}_{\mathcal{B}_1}\mathrm{Tr}_{\mathcal{B}_2}\right\rangle - \left\langle \mathrm{Tr}_{\mathcal{B}_1}\right\rangle\left\langle \mathrm{Tr}_{\mathcal{B}_2}\right\rangle$: transition amplitude between the boundary states $\mathcal{B}_1$ and $\mathcal{B}_2$



The $1/N$ expansion

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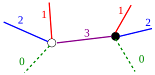
- perturbative expansion indexed by graphs embedded in D-dimensional spaces

$$\ln Z = \sum_{\substack{\text{connected} \\ D+1 \text{ colored graphs } \mathcal{G}}} \mathcal{A}(\mathcal{G})$$



- the scaling with N is:

$$\mathcal{A}(\mathcal{G}) \sim N^{D - \frac{2}{(D-1)!}\omega(\mathcal{G})}$$

with $\omega(\mathcal{G}) \geq 0$ a non negative integer associated to $\mathcal{G}$, the degree

- reorganize the perturbative series in powers of $1/N$:

$$\frac{1}{N^D} \ln Z = \underbrace{\sum_{\omega \geq 0} \left(\frac{1}{N}\right)^{\frac{2}{(D-1)!}\omega}}_{\text{expansion in 1/N}} \underbrace{\sum_{\substack{\text{connected} \\ D+1 \text{ colored graphs } \mathcal{G}}}^{\omega(\mathcal{G})=\omega} \prod t_{\mathcal{B}}}_{\text{convergent sum at fixed degree}}$$

Amplitudes and Dynamical Triangulations

$$\ln Z = \sum_{\substack{\text{connected} \\ D+1 \text{ colored graphs } \mathcal{G}}} \mathcal{A}(\mathcal{G}) \;, \quad \mathcal{A}(\mathcal{G}) = \underbrace{\lambda^{n(\mathcal{G})} \; N^{D - \frac{2}{(D-1)!}\omega(G)}}_{e^{\kappa_{D-2}(\lambda,N)\left[\#(D-2)-\text{simplices}\right] - \kappa_D(\lambda,N)\left[\#D-\text{simplices}\right]}}$$

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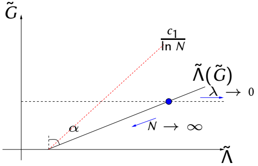
Discretized Einstein Hilbert action on equilateral triangulation dual to $\mathcal{G}$.

$$\sum_{\text{topologies}} \int [\mathcal{D}g] \ e^{-\frac{1}{16\pi G}\int d^Dx\sqrt{g}(2\Lambda - R)} \to \sum_{\substack{\text{Triangulations} \\ \text{edge length } a}} e^{-S_{\text{EH}}^{\text{discr.}}(G,\Lambda;a)} = \frac{1}{N^D} \ln Z(\lambda; N) \ ,$$

$$\frac{G}{a^{D-2}} \equiv \tilde{G} = c_1 \frac{1}{\ln N} \ , \qquad \Lambda a^2 \equiv \tilde{\Lambda} = c_2 \ \tilde{G} \ \ln\left(\frac{1}{\lambda}\right) + c_3 \ , \qquad c_1, c_2, c_3 > 0 \ (\sim O(1)) \ .$$



Quantum field theory meets gravity?

- AdS/CFT $\rightarrow$ weakly coupled bulk gravity / strongly coupled boundary CFT
- asymptotic safety $\rightarrow$ weakly coupled infrared gravity, strongly coupled ultraviolet fixed point

Strongly coupled QFT is complicated...

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Strongly coupled QFT is complicated...

Tensor field theories are non trivial but solvable strongly coupled quantum field theories.

Vectors – large N limit solvable but too restrictive in any dimension

[Berlin, Kac '52; Stanley '68;...]

Matrices – planar limit very difficult in more than zero dimensions

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QFT AT LARGE N

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Tensors – **new, melonic** large N limit

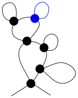
[Gurau '10 '11; Gurau Rivasseau '11, Bonzom et al. '11,...]

New family of strongly coupled CFTs

[Witten '16; Klebanov et al. '17,'18; Minwalla et al. '17; Tseytlin et al. '17; Ferrari et al. '17 ...]

The surprise

Vectors – simple
"snails": local insertions



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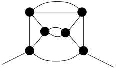

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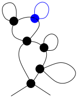
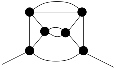
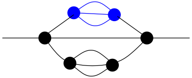


Matrices – complicated
planar



Tensors – in between
"melonic": recursive bilocal insertions



The self energy and the four point kernel

Schwinger Dyson equation (self energy Σ)

$$G = C + C\,\Sigma\,C + C\,\Sigma\,C\,\Sigma\,C + \ldots\ldots = \frac{1}{C^{-1} - \Sigma}$$

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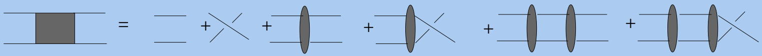
Four point Dyson equations (four point kernel $K = GG\frac{\delta\Sigma}{\delta G}$):



$$\langle\phi\phi\phi\phi\rangle_{12\rightarrow34} = 2GG + K(2GG) + KGGK(2GG)\cdots = \frac{1}{1-K}(2GG)$$

Conformal Field Theory ($x_{ij} = x_i - x_j$)

– dimensions h of primary operators fix the two point functions:

$$\langle O_{h,J}(x_1) O_{h,J}(x_2) \rangle = |x_{12}|^{-2h} \; I_{h,J}(x_{12})$$

– OPE coefficients fix the rest:

$$\phi(x_1)\phi(x_2) = \sum_{(h,J)} C_{h,J} \; P^{h,J}[O_{h,J}(x_2)] \;, \quad \langle \phi(x_1)\phi(x_2) O_{h,J}(x_3) \rangle = C_{h,J} \; Z_{h,J}(x_{ij})$$

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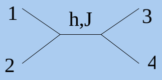

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{(h,J)} (C_{h,J})^2 \quad \underbrace{B_{h,J}(x_i)}_{\text{conformal blocks}}$$

Four point kernel in a CFT

$\{B_{\nu,J}|\mathrm{Re}(\nu) = d/2, J \geq 0\}$ – conformal blocks basis in the space of two particle to two particle operators:

$$K(2GG) = \sum_{J\geq 0} \int_{\frac{d}{2}-\mathrm{i}\,\infty}^{\frac{d}{2}+\mathrm{i}\,\infty} \frac{d\nu}{2\pi\,\mathrm{i}} \overbrace{k(\nu,J)}^{K\langle\phi\phi O_{\nu,J}\rangle = k(\nu,J)\langle\phi\phi O_{\nu,J}\rangle} \mu(\nu,J) B_{\nu,J}(x_i)$$

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$$\sum_{(h,J)} (C_{h,J})^2 B_{h,J}(x_i) = \sum_{J\geq 0} \int_{\frac{d}{2}-\mathrm{i}\,\infty}^{\frac{d}{2}+\mathrm{i}\,\infty} \frac{d\nu}{2\pi\,\mathrm{i}} \frac{1}{1-k(\nu,J)} \mu(\nu,J) B_{\nu,J}(x_i)$$

- physical spectrum – poles of the form

$$k(h,J) = 1\,, \qquad \mathrm{Re}(h) \geq d/2$$

- OPE coefficients

$$(C_{h,J})^2 = -\mathrm{Res}\left(\frac{\mu(\nu,J)}{1-k(\nu,J)}\right)_{\nu=h}$$

Non normalizable contributions

But $k(h, J) = k(\tilde{h}, J)$ with $\tilde{h} = d - h \Rightarrow$ we miss the physical primaries with dimension $h_i < d/2$ but pick their shaddows $\tilde{h}_i = d - h_i$.

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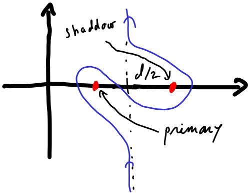
$$\sum_{(h,J)} (C_{h,J})^2 B_{h,J}(x_i) = \sum_{J\geq 0} \int_{\frac{d}{2}-\mathrm{i}\,\infty}^{\frac{d}{2}+\mathrm{i}\,\infty} \frac{d\nu}{2\pi\,\mathrm{i}} \frac{\mu(\nu,J)}{1-k(\nu,J)} B_{\nu,J}(x_i)$$
$$- \mathrm{Res}\left(\frac{\mu(\nu,\mathrm{J})}{1-\mathrm{k}(\nu,\mathrm{J})}\mathrm{B}_{\nu,\mathrm{J}}(\mathrm{x_i})\right)_{\mathrm{h}=\mathrm{h_i}}$$
$$+ \mathrm{Res}\left(\frac{\mu(\nu,\mathrm{J})}{1-\mathrm{k}(\nu,\mathrm{J})}\mathrm{B}_{\nu,\mathrm{J}}(\mathrm{x_i})\right)_{\mathrm{h}=\mathrm{d}-\mathrm{h_i}}$$

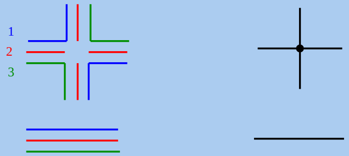


Field and Action

[Carrozza Tanasa '15, Giombi Klebanov Tarnopolsky '16 '17 '18]

Rank 3 tensor $\phi_{b_1 b_2 b_3} = O^{(1)}_{b_1 a_1} O^{(2)}_{b_2 a_2} O^{(3)}_{b_3 a_3} \phi_{a_1 a_2 a_3}$, invariant action

$$S = \frac{1}{2} \int \phi_{a_1 a_2 a_3} (-\partial^2) \phi_{a_1 a_2 a_3} + \frac{\lambda}{4 N^{3/2}} \int \phi_{a_1 a_2 a_3} \phi_{b_1 b_2 b_3} \phi_{c_1 c_2 c_3} \phi_{d_1 d_2 d_3} \underbrace{\delta_{a_1 b_1} \delta_{c_1 d_1} \delta_{a_2 c_2} \delta_{b_2 d_2} \delta_{a_3 d_3} \delta_{b_3 c_3}}_{\delta^t}$$



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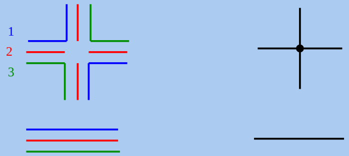
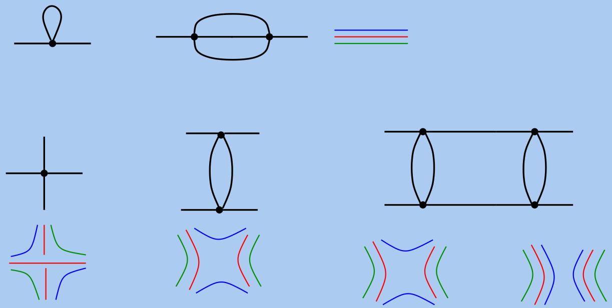


Indices follow the strands – one sum per closed colored cycle, pairwise identifications of external indices:

$$N^{-\frac{3}{2}V + F} \prod \delta_{a_i b_i}$$

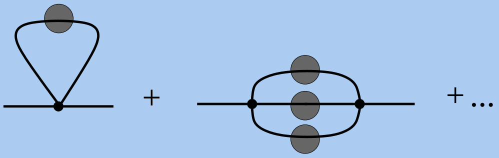
Two and four point functions



Tetrahedron, *pillow* and *double trace* four point functions

Simplest field theory

Solve Σ in terms of G (four point kernel $K = GG\frac{\delta\Sigma}{\delta G}$)



First non trivial is the **melon** (tadpole is mass)

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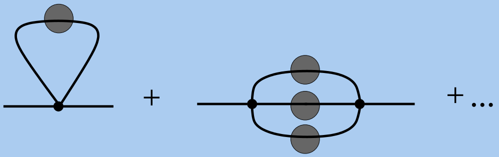
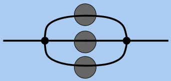


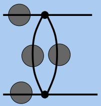
First non trivial is the **melon** (tadpole is mass)

Life would be simple if this truncated to the first non trivial term....

The large N limit does this for you!



$$\Sigma(x,y) = \lambda^2 G(x,y)^3$$



$$K(xy;zt) = 3\lambda^2 G(x,z)G(y,t)G(z,t)^2$$

Formal conformal limit

[Giombi Klebanov Tarnopolsky '17]

ϕ primary operator with dimension $\Delta_\phi = d/4$:

$$1 = C^{-1}G - G\underbrace{\left(\lambda^2 G^{q-1}\right)}_{\Sigma} \qquad G \sim \frac{1}{|x-y|^{2\frac{d}{4}}}$$

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Melonic CFT – simplest CFT with a non trivial four point kernel K!

Compute $k(\nu, J)$, spectrum $k(h, J) = 1$, etc.

Fixed point of an RG flow?

$$S = \frac{1}{2}\int \phi(-\partial^2 + \underbrace{m^2}_{\text{mass}})\phi + \int \phi\phi\phi\phi\left(\frac{\lambda}{4N^{3/2}}\delta^{\mathrm{t}} + \underbrace{\frac{\lambda_p}{4N^2}\delta^{\mathrm{p}}}_{\text{pillow}} + \underbrace{\frac{\lambda_d}{4N^3}\delta^{\mathrm{d}}}_{\text{double trace}}\right)$$

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$4-\epsilon$ dimensions [Giombi Klebanov Tarnopolsky '17]

$$\beta_g = -\epsilon g + 2g^3 \,, \qquad \beta_{g_p} = -\epsilon g_p + \left(6g^2 + \frac{2}{3}g_p^2\right) - 2g^2 g_p$$

$$\beta_{g_d} = -\epsilon g_d + \left(\frac{4}{3}g_p^2 + 4g_p g_d + 2g_d^2\right) - 2g^2(4g_p + 5g_d) \,,$$

$$g_\star = (\epsilon/2)^{1/2} \,, \qquad g_{p\star} = \pm\, \mathrm{i}\, 3(\epsilon/2)^{1/2} \,, \qquad g_{p\star} = \mp \mathrm{i}(3 \pm \sqrt{3})(\epsilon/2)^{1/2} \,,$$

Wilson Fisher like fixed point – only limit cycles!

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Is there a model which flows to a melonic CFT?

Conformal scaling

Use form the onset the (long range) infrared scaling of the covariance

$$S = \frac{1}{2}\int \phi\Big[\underbrace{(-\partial^2)^\zeta}_{\zeta=d/4} + m^2\Big]\phi + \int \phi\phi\phi\phi\left(\frac{\lambda}{4N^{3/2}}\delta^{\mathrm{t}} + \frac{\lambda_p}{4N^2}\delta^{\mathrm{p}} + \frac{\lambda_d}{4N^3}\delta^{\mathrm{d}}\right)$$

Preserves positivity Källén–Lehmann spectral representation:

$$\frac{1}{p^{2\zeta}} = \frac{1}{\Gamma(\zeta)\Gamma(1-\zeta)}\int_0^\infty dx\, \frac{x^{-\zeta}}{p^2+x}\,.$$

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Wilsonian RG – no wave function renormalization, four point couplings are marginal.

Beta functions at all orders

[Benedetti, Gurau, Harribey '19]

$\lambda_1 = \frac{\lambda_p}{3}, \lambda_2 = \lambda_d + \lambda_p$

The β functions truncate at quadratic order!

$$k\frac{\partial g}{\partial k} = \beta_g = 0 \ ,$$
$$k\frac{\partial g_1}{\partial k} = \beta_{g_1} = \beta_0^g - 2\beta_1^g \, g_1 + \beta_2^g \, g_1^2 \ ,$$
$$k\frac{\partial g_2}{\partial k} = \beta_{g_2} = \beta_0^{\sqrt{3}g} - 2\beta_1^{\sqrt{3}g} \, g_2 + \beta_2^{\sqrt{3}g} \, g_2^2 \ ,$$

with β_0^g, β_1^g, β_2^g power series in the tetrahedral coupling g

Fixed points

$$g_{1\pm} = \frac{\beta_1^g \pm \sqrt{(\beta_1^g)^2 - \beta_0^g \beta_2^g}}{\beta_2^g} = \pm \mathrm{i}\, g + O(g^2)\ ,$$

Fixed points

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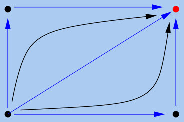
Tetrahedral invariant does not have a definite sign, pillow and double trace do – take $g = -\mathrm{i}\,|g|$!



- 4 real fixed points $(g_{1\pm}, g_{2\pm})$
- (g_{1+}, g_{2+}) **is infrared attractive**

The CFTs

[Benedetti, Gurau, Harribey, Suzuki '19]

ϕ primary operator with dimension $\Delta_\phi = \frac{d}{4}$, kernel

$$k(h,J) = 3g^2\Gamma(d/4)^4 \frac{\Gamma(-\frac{d}{4}+\frac{h+J}{2})\Gamma(\frac{d}{4}-\frac{h-J}{2})}{\Gamma(\frac{3d}{4}-\frac{h-J}{2})\Gamma(\frac{d}{4}+\frac{h+J}{2})}$$

Spectrum $h_{m,J}$ with $m, J \in \mathbb{N}$ and OPE coefficients

$$h_{0,0} \approx \frac{d}{2} \pm \alpha\sqrt{-g^2}\ , \qquad (C_{0,0})^2 \approx c_{0,0}^2 \pm \alpha'\sqrt{-g^2}$$

$$h_{m,J} \approx \frac{d}{2} + J + 2m + \gamma g^2\ , \qquad (C_{m,J})^2 = c_{m,J}^2 + \gamma' g^2$$

The OPE coefficients are real for $g = -\mathrm{i}\,|g|$

There shuld exist some (finite) quantity decreasing along the RG flow, or at least between UV and IR fixed points.

$c-$, $F-$, $a-$ THEOREMS

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Unitarity of the theory crucial for the proof of the $F-$theorem!

$F-$theorem in the long range $O(N)^3$ model

The $O(N)^3$ model is *not unitary* at finite N so does our RG flow obey the $F-$theorem in $d = 3$?

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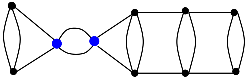
Perturbative – ring graphs, g_1 chanes sign between g_{1-} and $g_{1+} = -g_{1-}$



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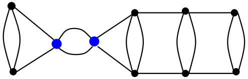

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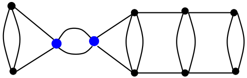

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The UV theory has an extra non normalizable state $-$ physical primary operator with dimension $h_{0,0} \approx \frac{d}{2} - \alpha\sqrt{-g^2} < d/2$.

In the IR $h_{0,0} \approx \frac{d}{2} + \alpha\sqrt{-g^2}$.

Conclusions

$O(N)^3$ model with $\zeta = d/4$ and $g = -\mathrm{i}\,|g|$:

- β functions at all orders, fixed points
- conformal invariance for correlations with only ϕ (arguments for composite operators)
- melonic CFT, kernel
- flow respects the F theorem

To do (short selection):

- resum the coefficients β_i^g
- unitarity of CFTs at leading order in N?
- scaling dimensions from RG, flow

- relax the unitarity requirement for the $F-$theorem
- $F-$theorem without dimensional regularization?