The QED four–photon amplitudes off–shell

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Abstract

The QED four-photon amplitude has been well-studied by many authors, and on-shell is treated in many textbooks. However, a calculation with all four photons off -shell is presently still lacking, despite of the fact that this amplitude appears off-shell as a subprocess in many different contexts, in vacuum as well as with some photons connecting to external fields. In the present talk we discuss the techniques used within the worldline formalism to obtain this amplitude explicitly for both scalar and spinor QED.

1. Introduction

- The prediction of the positron (Dirac 1928) gave as a particular consequence the need to include non-linear corrections to Maxwell's theory.
- Light-by-light scattering appears as a purely non-linear process in QED.
- The four-photon amplitude appears in many processes for example: Delbrück scattering, photon splitting, photon merging, birefringence, dichroism, the electron (or muon) anomalous magnetic moment and the beta function.
- 1936 H. Euler and W. Heisenberg presented the one-loop effective action for spinor QED in a background of constant strength from which is possible to obtain the four-photon amplitude at low energies for spinor QED [1].
- 1936 V. Weisskopf presented the one-loop effective action for scalar QED in a background of constant strength from which is possible to obtain the four-photon amplitude at low energies for scalar QED [2].
- 1951 R. Karplus and M. Neuman computed for the first the four-photon amplitude for arbitrary kinematics [3].
- 1971 V. Costantini, B De Tollis and G. Pistoni calculated the same amplitude with two photons on-shell and two off-shell [4].
- 1953 R. R. Wilson reported the first measurement of Delbrück scattering [5].
- 2017 LHC collaboration reported the first measurement of light-by-light scattering [6].

3. The four-photon amplitude for scalar and spinor QED

Within the worldline formalism scalar case is simpler. The four-photon amplitude for scalar QED can be written as

$$\Gamma_{\rm scal}[k_1,\varepsilon_1;\cdots,k_4,\varepsilon_4]=(-ie)^4\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,T^4\,{\rm e}^{-m^2T}\int_0^1\prod_{i=1}^4 du_i\,Q_{\rm scal}(\dot G_{ij})\exp\left[\sum_{i,j=1}^4\frac{T}{2}G_{ij}k_i\cdot k_j\right]. \tag{1}$$

Here $G_{ij}=|u_i-u_j|-(u_i-u_j)^2$ is the bosonic Green's function, $\dot G_{ij}=\partial_{u_i}G_{ij}$ its first derivative, and $Q_{\rm scal}=Q^4_{\rm scal}+Q^3_{\rm scal}+Q^2_{\rm scal}+Q^{22}_{\rm scal}$ is a polynomial function of only $\dot G_{ij}$'s

$$\begin{aligned}Q^4_{\rm scal}&=\dot G(1234)+\dot G(2314)+\dot G(3124),\\ Q^3_{\rm scal}&=\dot G(123)T(4)+\dot G(234)T(1)+\dot G(314)T(2)+\dot G(124)T(3),\\ Q^2_{\rm scal}&=\dot G(12)T(34)+\dot G(23)T(14)+\dot G(31)T(24)+\dot G(34)T(12)+\dot G(14)T(23)+\dot G(24)T(31),\\ Q^{22}_{\rm scal}&=\dot G(12)\dot G(34)+\dot G(23)\dot G(14)+\dot G(31)\dot G(24),\end{aligned} \tag{2}$$

where

$$\dot G(i_1i_2\cdots i_n):=\dot G_{i_1i_2}\dot G_{i_2i_3}\cdots\dot G_{i_ni_1}Z_n(i_1\cdots i_n),\qquad Z_n(i_1\cdots i_n)\equiv\left(\frac12\right)^{\delta_{n,2}}{\rm tr}(f_{i_1}f_{i_2}\cdots f_{i_n}), \tag{3}$$

$f_i^{\mu\nu}=k_i^\mu\varepsilon_i^\nu-k_i^\nu\varepsilon_i^\mu$ is the field strength tensor of external photons and $\dot G(i_1i_2\cdots i_n)$ is called "bicycle".

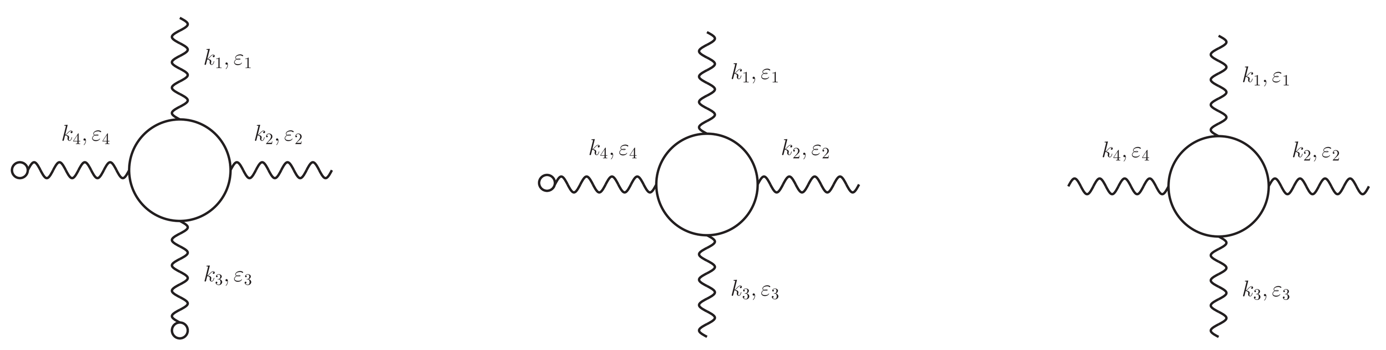

Figure: Feynman diagrams for the one-loop four-photon amplitude. Here, the small bullets indicates the low energy photons.

$T(i)$ and $T(ij)$ are called the one- and two-tails. In the "Q-representation" we have (see [7])

$$T(i):=\sum_r\dot G_{ir}\varepsilon_i\cdot k_r,\qquad T(ij):=\sum_{r,s}\left\{\dot G_{ir}\varepsilon_i\cdot k_r\dot G_{js}\varepsilon_j\cdot k_s+\frac12\dot G_{ij}\varepsilon_i\varepsilon_j\left[\dot G_{ir}k_i\cdot k_r-\dot G_{jr}k_j\cdot k_r\right]\right\}. \tag{4}$$

These one- and two-tails, by some IBP, can be written in terms of tensor f_i making the gauge invariance manifest

$$T_r(i):=\sum_j\dot G_{ij}\,\frac{r_i\cdot f_i\cdot k_j}{r_i\cdot k_i},\qquad T_H(ij):=\sum_{r,s}\dot G_{ri}\dot G_{js}\;k_r\cdot H_{ij}\cdot k_s,\qquad H^{\mu\nu}_{ij}=\frac{(f_if_j)^{\mu\nu}\,k_i\cdot k_j-k_i^\mu k_j^\nu\,{\rm tr}(f_if_j)}{(k_i\cdot k_j)^2}. \tag{5}$$

This is called the "H-representation". The r_i's are arbitrary vectors with the only condition $r_j\cdot k_i\neq0$.
Using IBP, we can also find a better representation of the two-tail, the "short-representation"

$$T_{sh}(ij)=\sum_{r,s\neq i,j}\dot G_{ri}\dot G_{js}\,\frac{k_r\cdot f_i\cdot f_j\cdot k_s}{k_i\cdot k_j}. \tag{6}$$

The previous representation allow us to write the four-photon amplitude with a minimal basis of five tensors

$$\begin{aligned}&T^{(1)}_{(1234)}=Z_4(1234)\,, &&T^{(2)}_{(12)(34)}=Z_2(12)Z_2(34)\,,\\ &T^{(3)r_4}_{(123)i}=Z_3(123)\frac{r_4\cdot f_4\cdot k_i}{r_4\cdot k_4},\quad i=1,2,3\,, &&T^{(4)}_{(12)ii}=Z_2(12)\,\frac{k_i\cdot f_3\cdot f_4\cdot k_i}{k_3\cdot k_4},\quad i=1,2,\\ &T^{(5)}_{(12)ij}=Z_2(12)\,\frac{k_i\cdot f_3\cdot f_4\cdot k_j}{k_3\cdot k_4},\quad (i,j)=(1,2),(2,1),\end{aligned} \tag{7}$$

which correspond exactly to the one presented in [4] where they used Ward identities.
For the spinor case, we use the "Bern-Kosower replacement rules" to obtain the spinor amplitude from the scalar one. In this case, the replacement rule consist in multiplying the scalar amplitude by a global factor of "−2" and applying the following "cycle replacement rule"

$$\dot G_{i_1i_2}\dot G_{i_2i_3}\cdots\dot G_{i_ni_1}\to\dot G_{i_1i_2}\dot G_{i_2i_3}\cdots\dot G_{i_ni_1}-G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1} \tag{8}$$

where $G_{Fi_1i_2}\equiv{\rm sign}(u_{i_1}-u_{i_2})$ denotes the 'fermionic' worldline Green's function.

4. Two low-energy photons

In this section, we discuss the one-loop off-shell four-photon amplitude for spinor QED with legs 3 and 4 in the low energy limit. This amplitude is given by

$$\Gamma_{\rm spin(34)}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4]=-\frac{2e^4}{(4\pi)^{\frac D2}}\,\hat\Gamma_{\rm spin(34)}(Q_{\rm spin}), \tag{9}$$

$$\hat\Gamma_{\rm spin(34)}=\int_0^1du_1\int_0^1du_2\int_0^\infty dT\;T^{3-\frac D2}{\rm e}^{-T(m^2-G_{12}k_1\cdot k_2)}Q_{\rm spin(34)}(G_{12})\;\;,\quad i=4,3,2,22. \tag{10}$$

Here, the integrand $Q_{\rm spin(34)}(G_{12})$ is obtain by truncating equation (6) to the terms linear in k_3 and k_4, and performing the polynomial integration in u_3 and u_4.
From these expressions, we computed the explicit result in terms of $Y_{nl}(x_1)$, Eq. (16). We fix $k_1=-k_2$ to re-calculate the two-loop β-function and to compare this amplitude against the one obtained from the photon propagator in a constant external field.

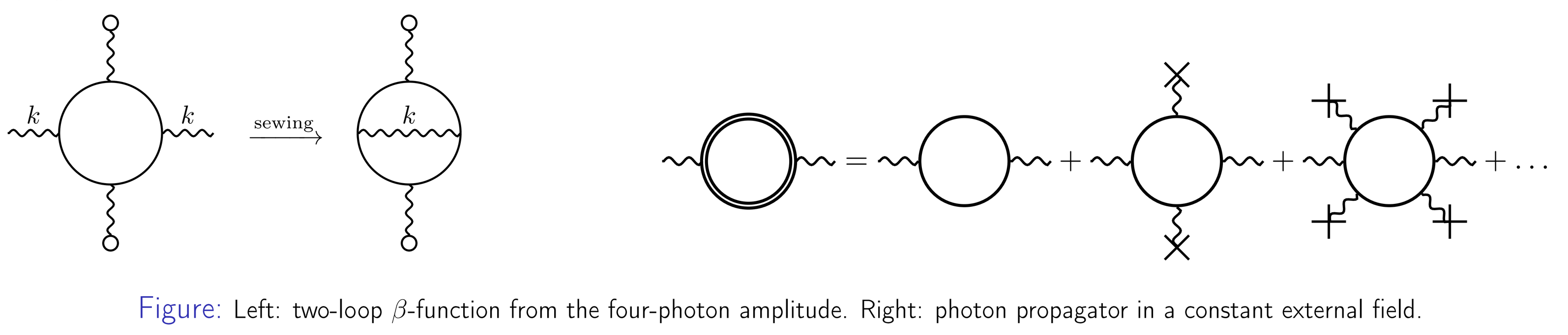

Figure: Left: two-loop β-function from the four-photon amplitude. Right: photon propagator in a constant external field.

4.1 Delbrück scattering at low energies

In this section, we follow [4] conventions in order to reproduce their result for Delbrück scattering at low energies. We consider k_3 and k_4 to have low-energy, on-shell and with circular polarization.
For scalar QED, we obtain

$$\begin{aligned}d\sigma_{\rm scal(++)}=d\sigma_{\rm scal(--)}&=(Z\alpha)^4\left(\frac{\alpha}{m}\right)^2\left(\frac{3}{16}\right)^2\left(\frac{1}{32}\right)^2\left(\frac{\omega}{m}\right)^4\cos^4\frac\theta2d\Omega,\\ d\sigma_{\rm scal(+-)}=d\sigma_{\rm scal(-+)}&=(Z\alpha)^4\left(\frac{\alpha}{m}\right)^2\left(\frac{15}{16}\right)^2\left(\frac{1}{32}\right)^2\left(\frac{\omega}{m}\right)^4\sin^4\frac\theta2d\Omega.\end{aligned} \tag{11}$$

For spinor QED, we obtain

$$\begin{aligned}d\sigma_{\rm spin(++)}=d\sigma_{\rm spin(--)}&=(Z\alpha)^4\left(\frac{\alpha}{m}\right)^2\left(\frac{73}{72}\right)^2\left(\frac{1}{32}\right)^2\left(\frac{\omega}{m}\right)^4\cos^4\frac\theta2d\Omega,\\ d\sigma_{\rm spin(+-)}=d\sigma_{\rm spin(-+)}&=(Z\alpha)^4\left(\frac{\alpha}{m}\right)^2\left(\frac{5}{8}\right)^2\left(\frac{1}{32}\right)^2\left(\frac{\omega}{m}\right)^4\sin^4\frac\theta2d\Omega.\end{aligned} \tag{12}$$

Spinor cross section is in agreement with [4]. Scalar cross section, to the best of our knowledge, is a new result.

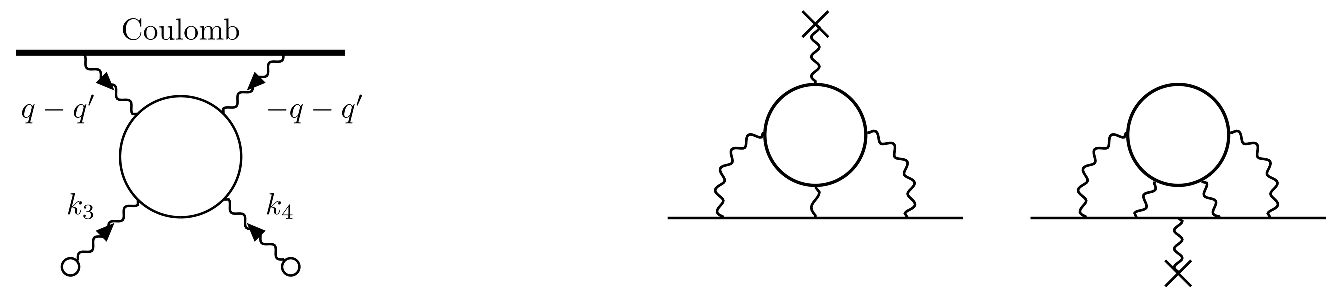

Figure: Left: Delbrück scattering. Right: three- and four-loop $g-2$ corrections.

6. One low-energy photon

In this section, we present the one-loop off-shell four-photon amplitude for spinor QED with leg 4 in the low energy limit

$$\begin{aligned}\Gamma_{\rm spin(4)}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4]&=-\frac{2e^4}{(4\pi)^{\frac D2}}\hat\Gamma_{\rm spin(4)}(Q_{\rm spin}),\\ \hat\Gamma_{\rm spin(4)}(Q_{\rm spin})&=\hat\Gamma^4_{\rm spin(4)}(1234)+\hat\Gamma^3_{\rm spin(4)}(123;4_1)+\hat\Gamma^3_{\rm spin(4)}(234;1)+\hat\Gamma^2_{\rm spin(4)}(12;34)\\ &\quad+\hat\Gamma^2_{\rm spin(4)}(34;12)+\hat\Gamma^{22}_{\rm spin(4)}(12,34)+(1\to2\to3\to1)+(1\to3\to2\to1).\end{aligned} \tag{13}$$

The explicit result for each of these $\hat\Gamma^i_{\rm spin(4)}$ is

$$\begin{aligned}\hat\Gamma^4_{\rm spin(4)}(1234)&=Z_4(1234)\Bigg[\left(\frac13-2\partial_{\hat k_{31}}\right)\left(\mathcal D_2K_4+c_2K_3\right)-\frac13\left(\frac{Y_{30}(x_1)}{k_2\cdot k_1}+\frac{Y_{30}(x_3)}{k_2\cdot k_3}\right)+2\left(\frac{Y_{31}(x_1)}{k_2\cdot k_1}+\frac{Y_{31}(x_3)}{k_2\cdot k_3}\right)\\ &\quad+\left(\mathcal D_1+\mathcal D_3+4\partial_{\hat k_{31}}-1\right)K_4+(c_1+c_3)K_3-\left(\frac{Y_{30}(x_2)}{k_1\cdot k_2}+\frac{Y_{30}(x_3)}{k_1\cdot k_3}\right)-\left(\frac{Y_{30}(x_1)}{k_3\cdot k_1}+\frac{Y_{30}(x_2)}{k_3\cdot k_2}\right)\Bigg],\\ \hat\Gamma^3_{\rm spin(4)}(123;4_1)&=-\frac13Z_3(123)k_2\cdot f_4\cdot k_1\Bigg\{\left(\partial_{\hat k_{12}}-4\partial^2_{\hat k_{12}}\right)\left[\left(\mathcal D_3-1\right)K_5+c_3K_4\right]-\left(\frac{Y_{41}(x_1)}{k_3\cdot k_1}+\frac{Y_{41}(x_2)}{k_3\cdot k_2}\right)+4\left(\frac{Y_{42}(x_1)}{k_3\cdot k_1}+\frac{Y_{42}(x_2)}{k_3\cdot k_2}\right)\\ &\quad+\left(\mathcal D_2+\mathcal D_1\right)\partial_{\hat k_{12}}K_5+\left(c_2+c_1\right)\partial_{\hat k_{12}}K_4-\frac{Y_{41}(x_1)}{k_2\cdot k_1}-\frac{Y_{41}(x_2)}{k_1\cdot k_2}\Bigg\},\\ \hat\Gamma^3_{\rm spin(4)}(234;1)&=-Z_3(234)\Bigg\{\frac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1}\Bigg[\left(\frac23+2\partial_{\hat k_{23}}\right)\left(\mathcal D_2K_4+c_2K_3\right)-\frac23\left(\frac{Y_{40}(x_1)}{k_2\cdot k_1}+\frac{Y_{40}(x_3)}{k_2\cdot k_3}\right)-2\frac{Y_{41}(x_3)}{k_2\cdot k_3}\Bigg]-(2\leftrightarrow3)\Bigg\},\\ \hat\Gamma^2_{\rm spin(4)}(12;34)&=\frac43Z_2(12)\Bigg\{k_1\cdot f_3\cdot f_4\cdot k_1\Bigg[\frac{k_2\cdot k_4}{k_3\cdot k_4}\partial^2_{\hat k_{12}}\left(\mathcal D_1K_5+c_1K_4\right)-\frac{k_2\cdot k_4}{k_3\cdot k_4}\frac{Y_{42}(x_2)}{k_1\cdot k_2}+\partial_{\hat k_{12}}\left(1-4\partial_{\hat k_{31}}\right)\partial_{\hat k_{31}}K_5\Bigg]\\ &\quad-k_1\cdot f_3\cdot f_4\cdot k_2\Bigg[\frac{k_1\cdot k_4}{k_3\cdot k_4}\partial^2_{\hat k_{12}}\left(\mathcal D_1K_5+c_1K_4\right)-\frac{k_1\cdot k_4}{k_3\cdot k_4}\frac{Y_{42}(x_2)}{k_1\cdot k_2}-\partial_{\hat k_{12}}\partial_{\hat k_{23}}\left(\mathcal D_3K_5+c_3K_4\right)+\frac{Y_{42}(x_2)}{k_3\cdot k_2}\Bigg]+(1\leftrightarrow2)\Bigg\},\\ \hat\Gamma^2_{\rm spin(4)}(34;12)&=-\frac23Z_2(34)\frac{k_3\cdot f_1\cdot f_2\cdot k_3}{k_1\cdot k_2}\left(\mathcal D_3K_4+c_3K_3-\frac{Y_{40}(x_1)}{k_3\cdot k_1}-\frac{Y_{40}(x_2)}{k_3\cdot k_2}\right),\\ \hat\Gamma^{22}_{\rm spin(4)}(12,34)&=\frac83Z_2(12)Z_2(34)\;\partial_{\hat k_{12}}K_4.\end{aligned} \tag{14}$$

Permutations do not apply to sub-indices in K_n or Y_{nl}. The previous compact form of the amplitudes is possible due to the following definitions: The three-point function

$$K_n\equiv\frac{\Gamma\left(n-\frac D2\right)}{m^{2n-D}}\int_0^1\prod_{i=1}^3du_i\frac{1}{\left(1-G_{12}\hat k_{12}-G_{13}\hat k_{13}-G_{23}\hat k_{23}\right)^{n-\frac D2}},\qquad \partial^l_{\hat k_{12}}K_n\equiv\frac{\Gamma\left(n-\frac D2-l\right)}{m^{2n-D}}\frac{\partial^l}{\partial\hat k^l_{12}}\prod_{i=1}^3\int_0^1du_i\frac{1}{\left(1-G_{12}\hat k_{12}-G_{13}\hat k_{13}-G_{23}\hat k_{23}\right)^{n-\frac D2-l}}$$

where $\hat k_{ij}=k_i\cdot k_j/m^2$. It is important to point out that K_n defined before has been already worked out in [8].
Furthermore, new techniques to compute these kind of integrals have emerge very recently [9].
The operators $\mathcal D_i$ and "constants" c_i are

$$\mathcal D_1\equiv-\frac12\left[\frac{k_1\cdot k_2}{k_1\cdot k_3}\left(1-4\partial_{\hat k_{12}}\right)+\frac{k_1\cdot k_3}{k_1\cdot k_2}\left(1-4\partial_{\hat k_{31}}\right)\right],\qquad c_1\equiv\frac{1}{k_1\cdot k_3}+\frac{1}{k_1\cdot k_2}. \tag{15}$$

The Y_{nl} functions are defined as a derivative of order l of the hypergeometric function ${}_2F_1$

$$Y_{nl}(x_i)\equiv\frac{\Gamma\left(n-\frac D2-l\right)}{m^{2n-D}}\frac{d^l}{dx_i^l}\,{}_2F_1\left(1,n-\frac D2-l;\frac32;\frac{x_i}{4}\right),\qquad x_i=\sum_{j=1}^3\hat k_{ij},\;\;j\neq i,\qquad \hat k_{ij}=k_i\cdot k_j/m^2. \tag{16}$$

7. Conclusions

- We study the off-shell four-photon amplitude with one, two or all legs in the low energy limit using the worldline formalism.
- In the case of one low-energy photon, we managed to write down the final results in terms of generalized hypergeometric functions. This is an unknown amplitude and our presentation is written in a very compact way ready to be used for higher loop construction.
- The final result for the case with two low-energy photons is much simpler, and can be written in terms of only ${}_2F_1$. Here the special case where $k_1=-k_2$ can be obtained from the vacuum polarization diagram in a constant field after taking two photons from the background field, which provided a useful check on our calculation. We have also used the same special case for constructing the two-loop β-functions for scalar and spinor QED, recuperating the known results.
- We compute the differential cross section for Delbrück scattering at low energies obtaining new results and exact match with the known ones.
- Our final goal in this project is the off-shell four-photon amplitude with arbitrary momenta, which is still in process.
- We expect these results to become useful not only for one loop processes in external fields but also for certain multi-loop calculations such as LBL contributions to $g-2$.

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