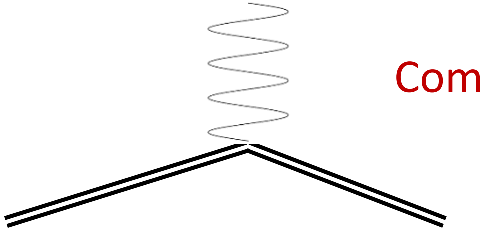
LCFA for radiation in a time-dependent electric field: applicability and corrections

E.G. Gelfer, A.M. Fedotov, A.A. Mironov, S.Weber

Original idea of LCFA



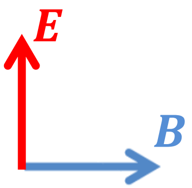
Compton scattering (photon emission) in a strong field

A.I. Nikishov and V.I. Ritus, *Sov. Phys. JETP* **19**, 529 (1964),
V.I. Ritus, *Journal of Soviet Laser Research* **6,** 497-617 (1985).

- If $a_0 = \frac{eE}{m\omega} \gg 1$, then the external field can be considered as constant,

since the formation length of the photon emission $\sim\lambda/a_0$

- If the particle is fast ($\gamma \gg 1$), then in its proper reference frame the external field is crossed ($\boldsymbol{E} \perp \boldsymbol{B}, E = B$)



- $W^{e\to e\gamma} \approx W_{CCF}(\chi)$, the probability can be approximated by CCF probability, which depends on the single QED parameter χ

$$\chi = \frac{e\sqrt{-\left(F_{\mu\nu}p^{\nu}\right)^2}}{m^3}$$

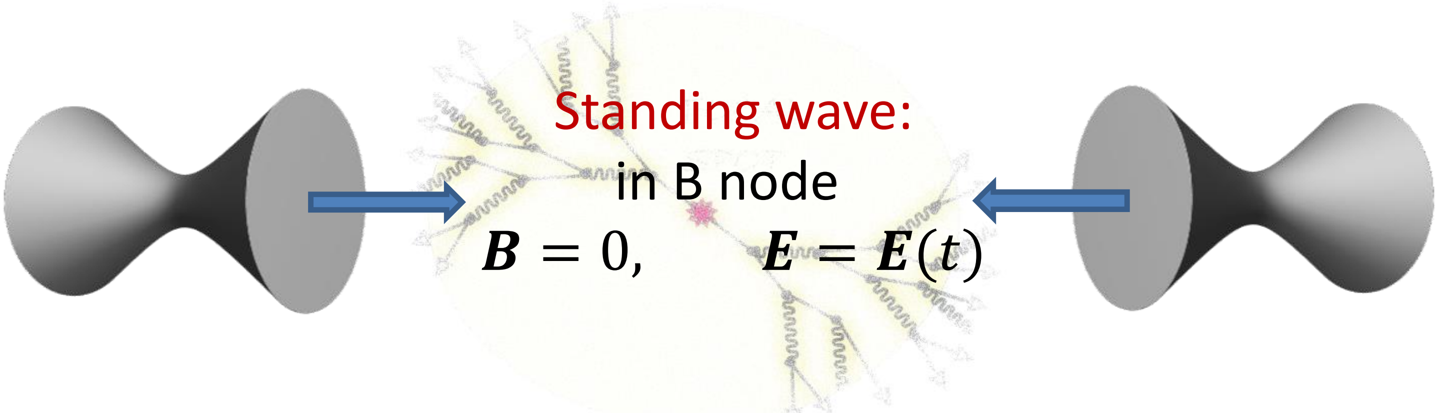
Motivation

- LCFA is used in almost all the codes, where strong field QED is implemented

$$\frac{dP^{LCFA}}{du} = -\frac{e^2 m^2 c}{4(1+u)^2}\left[\mathrm{Ai}_1(z) + \frac{2}{z}\mathrm{Ai}'(z)\right], \qquad z = \left(\frac{\varkappa}{\chi\chi'}\right)^{\frac{2}{3}}, \qquad u = \frac{\varkappa}{\chi'}$$

- It is crucial to determine, when it fails and how to correct it
- LCFA has been studied mainly for plain wave like fields

Colliding pulses are beneficial for cascades



Standing wave:

in B node

$\boldsymbol{B} = 0, \qquad \boldsymbol{E} = \boldsymbol{E}(t)$

- We consider arbitrary time-dependent electric field $\boldsymbol{E}(t) = -\boldsymbol{A}'(t)$

NCS probability in scalar QED

WKB wave functions

$$\Phi^{(\pm)}_{WKB}(t) = \frac{C}{\sqrt{2\mathcal{E}(t)}}\, e^{\mp\frac{i}{\omega}\int_{-\infty}^{t}\mathcal{E}(t')dt' + i\boldsymbol{pr}}$$

$$\boxed{\mathcal{E}(t) = \sqrt{m^2 + \boldsymbol{P}^2(t)}, \qquad \boldsymbol{P}(t) = \boldsymbol{p} - e\boldsymbol{A}(t)}$$

$$\frac{W}{V_4} = -\frac{e^2}{4\pi^2\omega}\int \mathrm{d}\mathbf{k} \sum_s M^{\mu}_{s} M^{*}_{s,\mu}\,\delta\left(\frac{k}{\omega} + \mathcal{K}_{\mathrm{p}'} - \mathcal{K}_{\mathrm{p}} - s\right)$$

emission probability per 4-volume

photon wave vector

$$\mathcal{K}_{\mathrm{p}} = \frac{1}{2\pi\omega}\int_0^{2\pi} \mathcal{E}(t)dt$$

particle energy averaged over field period

$$\boxed{M^{\mu}_{s} = \frac{1}{2\pi\sqrt{2k}}\int_0^{2\pi} dt\, h^{\mu}(t) e^{f(t)}}$$

partial emission amplitude (absorption of s photons) – we need its square to obtain the probability distribution

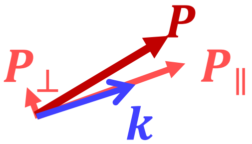
$$f(t) = \frac{i}{\omega}\left[kt + \int_{-\infty}^{t}\mathcal{E}'(t')dt' - \int_{-\infty}^{t}\mathcal{E}(t')dt'\right], \qquad h^{\mu}(t) = \frac{P^{\mu}(t)}{\sqrt{\mathcal{E}(t)\mathcal{E}'(t)}}$$

$$P^{\mu}(t) = \{\mathcal{E}(t), \boldsymbol{P}(t)\}$$

Stationary phase approximation

$$\dot{f}(t_0) = 0 \Longrightarrow \mathcal{E}(t_0) - \mathcal{E}'(t_0) - k = 0$$

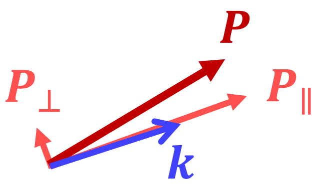
expand all vectors to parallel and transverse components (with respect to $\boldsymbol{k}$)



$$\sqrt{m^2 + P_{\parallel}^2(t_0) + P_{\perp}^2(t_0)} - \sqrt{m^2 + (P_{\parallel}(t_0) - k)^2 + P_{\perp}^2(t_0)} - k = 0 \Longrightarrow \boxed{\boldsymbol{P}_{\perp}^2(t_0) = -m^2}$$

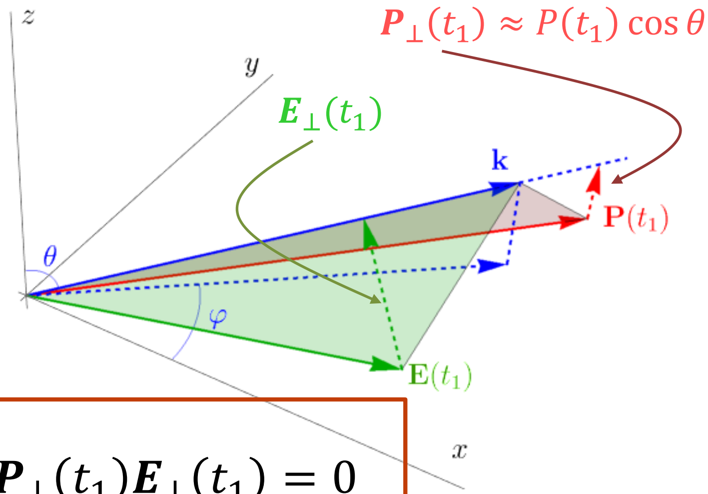
Amplitude calculation

- Stationary point $\boldsymbol{P}_\perp^2(t_0) = -m^2$



$$t_0 = t_1 + it_2, \qquad t_2 \ll 1$$

$$\boldsymbol{P}_\perp(t_0) \approx \boldsymbol{P}_\perp(t_1) + it_2 \frac{e\boldsymbol{E}_\perp(t_1)}{\omega}$$



$$\boxed{t_2 = \frac{\sqrt{\sigma}}{a_\perp}, \qquad \sigma = 1 + \frac{P_\perp^2(t_1)}{m^2}, \qquad a_\perp = \frac{eE_\perp(t_1)}{m\omega}, \qquad \boldsymbol{P}_\perp(t_1)\boldsymbol{E}_\perp(t_1) = 0}$$

- Phase expansion $f(t) \approx f(t_0) + \frac{1}{2}\ddot{f}(t_0)(\Delta t)^2 + \frac{1}{6}\dddot{f}(t_0)(\Delta t)^3 + \cancel{\frac{1}{24}\ddddot{f}(t_0)(\Delta t)^4} + \cdots$

- LCFA result ($(\Delta t)^2$ and $(\Delta t)^3$ lead to Airy functions)

$$M_s^\mu \approx \frac{1}{\sqrt{2k}}\left[\frac{1}{a_\perp}\left(\frac{2\chi\chi'}{\varkappa}\right)^{\frac{1}{3}} h^\mu(t_1)\mathrm{Ai}(y) - \frac{i}{a_\perp^2}\left(\frac{2\chi\chi'}{\varkappa}\right)^{\frac{2}{3}} \dot{h}^\mu(t_1)\mathrm{Ai}'(y) - \frac{y}{2a_\perp^3}\frac{2\chi\chi'}{\varkappa}\ddot{h}(t_1)\mathrm{Ai}(y)\right]$$

$$M_s^\mu M_{s,\mu}^* \approx \frac{\omega^2}{4km^2\chi\chi'}\left(\frac{2\chi\chi'}{\varkappa}\right)^{\frac{4}{3}}\left[\left(\frac{\varkappa}{2\chi\chi'}\right)^{\frac{2}{3}}\mathrm{Ai}^2(y) - \left(y\mathrm{Ai}^2(y) + \mathrm{Ai}'^2(y)\right)\right]$$

$$\chi \approx \frac{\mathcal{E}(t_1)a_\perp\omega}{m^2}$$
$$\chi' \approx \frac{\mathcal{E}'(t_1)a_\perp\omega}{m^2}$$
$$\varkappa = \frac{ka_\perp\omega}{m^2}$$
$$y = \left(\frac{\varkappa}{2\chi\chi'}\right)^{\frac{2}{3}}\sigma$$

LCFA validity

- Compare what we neglected $\ddddot{f}(t_0)(\Delta t)^4$ with what we kept $\ddot{f}(t_0)(\Delta t)^2$ and $\dddot{f}(t_0)(\Delta t)^3$

$$\ddot{f}(t_0) \approx -a_\perp^2\sqrt{\sigma}\,\frac{\varkappa}{\chi\chi'}, \qquad \dddot{f}(t_0) \approx i a_\perp^3\,\frac{\varkappa}{\chi\chi'}$$

$$\ddddot{f}(t_0) \approx 3ia_\perp^3\,\frac{\varkappa}{\chi\chi'}\left[\frac{\boldsymbol{a}_\perp\dot{\boldsymbol{a}}_\perp}{a_\perp^2} - \frac{m a_\parallel(\mathcal{E}+\mathcal{E}')}{\mathcal{E}\mathcal{E}'}\right]$$

- Formation time $\ddot{f}(t_0)(\Delta t)^2 \sim 1,\ \dddot{f}(t_0)(\Delta t)^3 \sim 1$

$$\boxed{\Delta t \sim \frac{1}{a_\perp}\left(\frac{\chi\chi'}{\varkappa}\right)^{\frac{1}{3}}}$$

- Validity condition $\boxed{\left|\ddddot{f}(t_0)\right|(\Delta t)^4 \ll 1}$

$$\frac{1}{a_\perp}\left(\frac{\chi\chi'}{\varkappa}\right)^{1/3}\frac{\boldsymbol{a}_\perp\dot{\boldsymbol{a}}_\perp}{a_\perp^2} \ll 1$$

geometrical factor

$$\frac{a_\parallel}{a_\perp}\left(\frac{\chi\chi'}{\varkappa}\right)^{1/3}\frac{m\,(\mathcal{E}+\mathcal{E}')}{\mathcal{E}\mathcal{E}'} \ll 1$$

geometrical factor

$$\Delta t \sim \boxed{\frac{1}{a_\perp}\left(\frac{\chi\chi'}{\varkappa}\right)^{1/3} \ll 1}$$

locally constant

(formation time is smaller than laser period)

$$\boxed{\frac{m}{\mathcal{E}},\frac{m}{\mathcal{E}'} \ll \left(\frac{\varkappa}{\chi\chi'}\right)^{1/3}}$$

crossed

(particle is ultrarelativistic)

LC(C)FA validity and small parameters

Stationary point approximation provides two small parameters and one additional condition arises from using WKB approximation to solve Klein-Gordon equation

Locally constant: $\xi_1 = \frac{1}{a_\perp}\left(\frac{2\chi\chi'}{\varkappa}\right)^{1/3} \ll 1$

Crossed: $\xi_2[\xi_2'] = \frac{1}{\gamma}\left[\frac{1}{\gamma'}\right]\left(\frac{2\chi\chi'}{\varkappa}\right)^{\frac{1}{3}} \ll 1$

$$\boxed{\frac{\xi_2}{\xi_1} = \frac{a_\perp}{\gamma}}$$

WKB: $E \ll E_{cr}$

$$\gamma = \frac{\mathcal{E}(t_1)}{m}, \qquad \gamma' = \frac{\mathcal{E}'(t_1)}{m}$$

Corrections to LCFA

- Expand emission amplitude into series over $\xi_1, \xi_2, {\xi_2}'$

$$M_s^\mu = \frac{1}{2\pi\sqrt{2k}} \int_0^{2\pi} dt\, h^\mu(t) e^{f(t)} \quad \Rightarrow \quad M_s^\mu = \frac{1}{2\pi\sqrt{2k}} \int_0^{2\pi} dt\, h^\mu(t) e^{[f(t)-f_0(t)]} e^{f_0(t)}$$

expand up to desired order of ξ_1, ξ_2

$$\boxed{f_0(t) \approx f(t_0) + \frac{1}{2} \underset{-a_\perp^2\sqrt{\sigma}\frac{\varkappa}{\chi\chi'}}{\ddot{f}^{(0)}(t_0)} (\Delta t)^2 + \frac{1}{6} \underset{ia_\perp^3\frac{\varkappa}{\chi\chi'}}{\dddot{f}^{(0)}(t_0)} (\Delta t)^3}$$

- Squared emission amplitude

$$M^\mu M_\mu^* \approx \frac{\omega^2}{4km^2\chi\chi'} \left(\frac{2\chi\chi'}{\varkappa}\right)^{\frac{4}{3}} [\mathcal{M}_0 + \mathcal{M}_1 + \mathcal{M}_2 + \cdots]$$

- LCFA contribution

$$\mathcal{M}_0 = -[y\mathrm{Ai}^2(y) + \mathrm{Ai}'^2(y)] + \left(\frac{\varkappa}{2\chi\chi'}\right)^{\frac{2}{3}} \mathrm{Ai}^2(y)$$

- First order correction

$$\mathcal{M}_1 = -\underset{\text{small parameter}}{\xi_1} \frac{\tau}{\sqrt{1+\tau^2}} \underset{\text{geometrical factor}}{\frac{|\boldsymbol{a}_\perp \times \dot{\boldsymbol{a}}_\perp|}{3a_\perp^2}} \left[\Upsilon_1(y) + \left(\frac{\varkappa}{2\chi\chi'}\right)^{\frac{2}{3}} \Upsilon_2(y)\right]$$

$$\Upsilon_1(y) = y^{\frac{3}{2}}\left(2y^{\frac{3}{2}} + 5\right)\mathrm{Ai}^2 + 4y^{\frac{5}{2}}\mathrm{AiAi}' + 2\sqrt{y}\left(y^{\frac{3}{2}} + 2\right)\mathrm{Ai}'^2$$

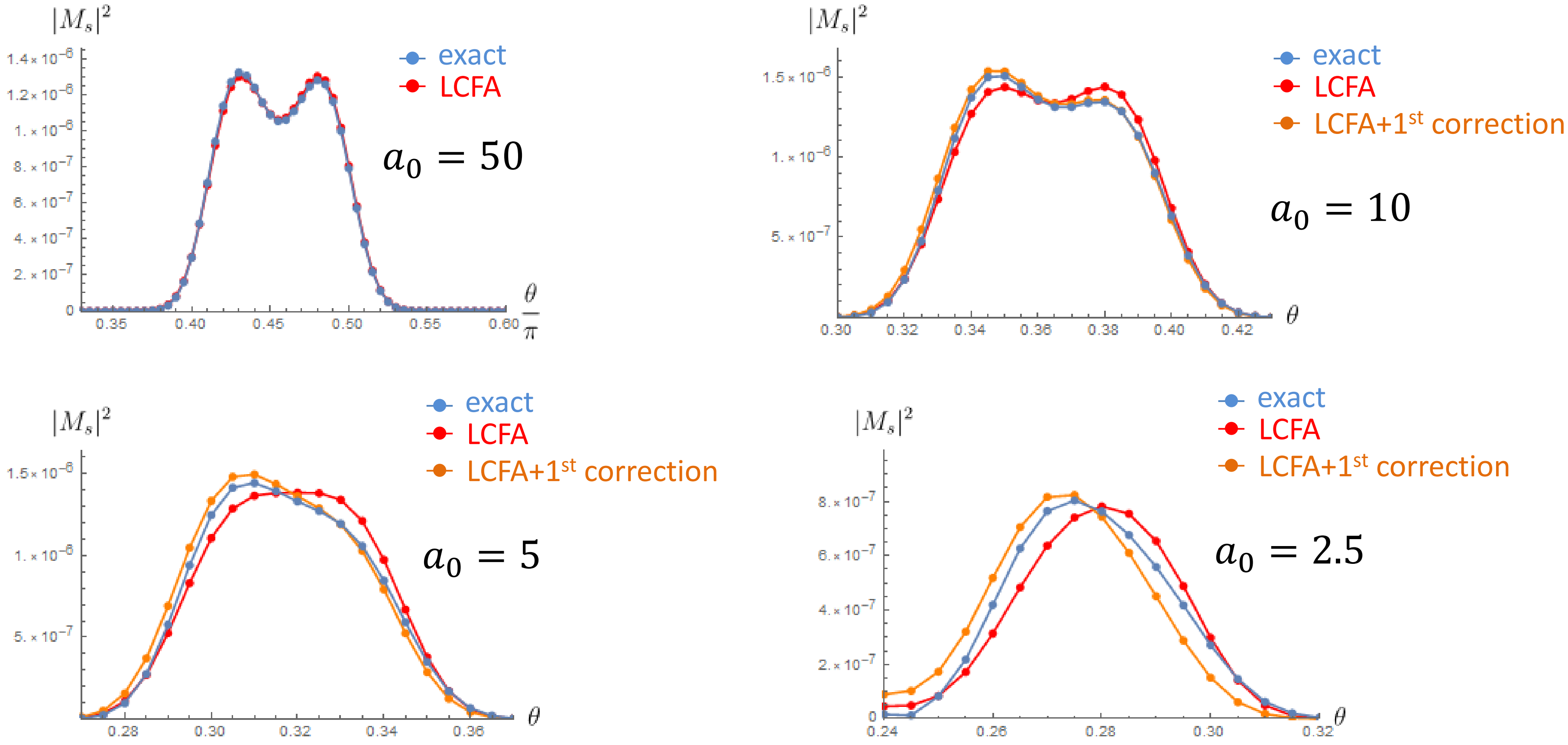
$$\Upsilon_2(y) = -2\left[\left(1 + y^{\frac{3}{2}}\right)\mathrm{Ai}^2 + \mathrm{yAiAi}'\right]$$

$$\tau = \frac{P_\perp}{m}, \qquad y = \left(\frac{2\chi\chi'}{\varkappa}\right)^{1/3} \sqrt{1+\tau^2}$$

$$\xi_1 = \frac{1}{a_\perp}\left(\frac{2\chi\chi'}{\varkappa}\right)^{1/3}$$

Numerical results

$eE(t) = m\omega a_0\{\cos t\,, \sin t\,, 0\}, \qquad \omega = 0.01m, \qquad p = 8m\{1,0,1\}, \qquad k = 5m\{\sin\theta\,, 0, \cos\theta\}$



For smaller a_0 (larger $\xi_{1,2}$) one needs next order corrections (to be implemented)

Also the first order correction is small if $\boldsymbol{p}$ is in (x, y) plane due to geometrical factor

Summary

- We calculated the probability of photon emission by a scalar particle in an arbitrary time-dependent strong electric field
- The result can be represented as a series expansion over small parameters ξ_1, ξ_2, ξ_2'
- Zeroth order term coincides with LCFA, and the corrections can be calculated up to (in principle) arbitrary order
- First order correction demonstrates good agreement with the exact calculation, second order correction will be implemented soon
- The approach is straightforwardly generalized for fermions

Corrections to LCFA

$$M^{\mu}M_{\mu}^{*} \approx \frac{\omega^2}{4km^2\chi\chi'}\left(\frac{2\chi\chi'}{\varkappa}\right)^{\frac{4}{3}}\left[\mathcal{M}_0 + \mathcal{M}_1 + \mathcal{M}_2 + \cdots\right]$$

$$\mathcal{M}_0 = -\left[y\Phi^2(y) + \Phi'^2(y)\right] + \left(\frac{\varkappa}{2\chi\chi'}\right)^{\frac{2}{3}}\Phi^2(y)$$

$$\mathcal{M}_1 = -\xi_1\frac{\tau\sqrt{y}}{\sqrt{1+\tau^2}}\frac{|\boldsymbol{a}_\perp \times \dot{\boldsymbol{a}}_\perp|}{3a_\perp^2}\left[\Upsilon_1(y) + \left(\frac{\varkappa}{2\chi\chi'}\right)^{\frac{2}{3}}\Upsilon_2(y)\right]$$

$$\xi_1 = \frac{1}{a_\perp}\left(\frac{2\chi\chi'}{\varkappa}\right)^{1/3}$$

$$\xi_2 = \left(\frac{2\chi\chi'}{\varkappa}\right)^{\frac{1}{3}}\frac{m}{\mathcal{E}}$$

$$\xi_2{}' = \left(\frac{2\chi\chi'}{\varkappa}\right)^{\frac{1}{3}}\frac{m}{\mathcal{E}'}$$

$$\mathcal{M}_2 = \xi_1^2 G_1(t_1)\left[\Upsilon_3(y) + \left(\frac{\varkappa}{2\chi\chi'}\right)^{\frac{2}{3}}\Upsilon_4(y)\right] + \xi_2^2 G_2(t_1)\left[\Upsilon_5(y) + \left(\frac{\varkappa}{2\chi\chi'}\right)^{\frac{2}{3}}\Upsilon_6(y)\right] +$$

$$+\xi_1\xi_2 G_3(t_1)\left[\Upsilon_7(y) + \left(\frac{\varkappa}{2\chi\chi'}\right)^{\frac{2}{3}}\Upsilon_8(y)\right] + \cdots$$