# The Ritus-Narozhny conjecture: a tutorial

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ExHILP 2021, Jena via Zoom.

# Outline



Presentation will be pedagogical, not historical. Focus is on RN in wider context.

Details will be minimal.

# Invitation

# Invitation: a game of ~~thrones~~ two couplings

$a \leftrightarrow a_0,\ \varepsilon^2 \leftrightarrow \alpha$

- An observable $x(a, \varepsilon)$ obeys

$$\varepsilon^2 x^2 + x - a = 0$$

- Mainly interested in large $a$, while $\varepsilon$ is small.
- Furry expansion: treat $\varepsilon$ perturbatively, but treat $a$ exactly.

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  - NNLO: $x = a(1 - \varepsilon^2 a + 2\varepsilon^4 a^2)$

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Expansion not in powers of $\varepsilon$ . . .

. . . but in powers of $\varepsilon^2 a$: can be large!

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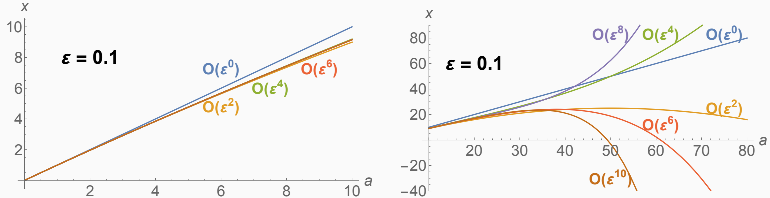
# Invitation: a game of two couplings

- $a$ large, $\varepsilon$ small: but the 'Furry expansion' breaks down.




- Resummation required. Shortcut: have exact solution.

$$\varepsilon^2 x^2 + x - a = 0 \implies x = \frac{-1+\sqrt{4a\varepsilon^2+1}}{2\varepsilon^2} \sim \sqrt{\frac{a}{\varepsilon^2}}$$

- Access to physics $a \gg 1$.
- Another lesson: non-perturbative physics.

$$x = \frac{-1-\sqrt{4a\varepsilon^2+1}}{2\varepsilon^2} \sim -\frac{1}{\varepsilon^2} - a + a^2\varepsilon^2 + \ldots$$

# Invitation . . . and, um, conclusions?

Furry expansion of $x(a,\varepsilon)$ for ($a \gg 1$ and) $\varepsilon \ll 1$:

$$\varepsilon^2 x^2 + x - a = 0$$

- Actual expansion parameter can depend on $a$: becomes large.

$$\varepsilon^2 \longrightarrow a\varepsilon^2$$

- Expansion requires resummation. Series may converge or not!
- Shortcut: exact solutions, if they exist. (Not often.)
- Physics at $a \gg 1$ different to perturbative predictions.

$$x \sim a \longrightarrow x \sim \sqrt{\frac{a}{\varepsilon^2}}$$

- Furry expansion may miss non-perturbative physics.

$$x \sim \frac{1}{\varepsilon^2}$$

# Classical physics

# An example in radiation reaction

$$m\ddot{x}_\mu = \underbrace{eF_{\mu\nu}\dot{x}^\nu}_{\text{big!}} + \underbrace{\frac{2}{3}\frac{e^2}{4\pi}(\dddot{x}_\mu\dot{x}_\nu - \dot{x}_\mu\dddot{x}_\nu)\dot{x}^\nu}_{\text{small...?}}$$

- Lorentz force, from strong field: treat exactly.
- Radiation-Reaction corrections: treat perturbatively.
- Classical Furry expansion. Look at an example.

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! $\mathcal{O}(e^{2n})$ classical from $\mathcal{O}(\alpha^n)$ QED.

! Loops contribute classically. Holstein & Donoghue PRL 93 (2004) 201602

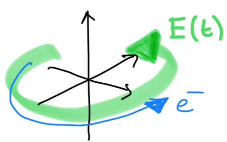
! No classical limit without loops. Higuchi PRD 69 (2004) 129903

Ilderton & Torgrimsson PLB 725 (2013) 481

# Rotating electric fields

- Rotating field $\mathbf{E}(t) = E_0(\cos\omega t, \sin\omega t, 0)$
  S.S. Bulanov et al PRL 105 (2010) 220407

$$a_0 = \frac{eE_0}{m\omega} \qquad \epsilon = \frac{2}{3}\frac{e^2}{4\pi}\frac{\omega}{m}$$




- Ask: stable electron orbit?

  $\implies$ condition on electron $\gamma$: solve in Furry expansion.

$$\gamma^2 - 1 \sim a_0^2\Big[\underset{\text{Lorentz}}{\overset{\uparrow}{1}} - \epsilon^2 a_0^6 + \mathcal{O}(\epsilon^4 a_0^{12})\Big]$$

- Corrections are a series in $\epsilon a_0^3$: can be large...

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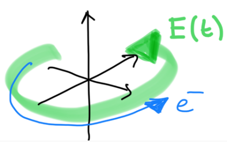


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Lorentz

- Corrections are a series in $\epsilon a_0^3$: can be large...

$\implies \gamma^2 < 1$, unphysical.

- Breakdown of the classical Furry expansion.

# “. . . second verse, same as the first . . .”

- Resummation needed at $\epsilon a_0^3 \gtrsim 1$.

- Shortcut! Exact solution: $a_0^2 = (\gamma^2 - 1)(1 + \epsilon^2\gamma^6)$

  S.S. Bulanov et al PRL 105 (2010) 220407

- Read off $a_0 \gg 1$ result:

$$\gamma^2 - 1 \sim \frac{a_0^2}{(\epsilon a_0^3)^{1/2}} \quad \text{for} \quad \epsilon a_0^3 \gg 1$$

- ‘Resummation’ fixes unphysical behaviour of pert. theory.

- $\epsilon \to \epsilon a_0^3$ & fractional powers at $a_0 \gg 1$, like example above.

- Don’t always have exact solutions . . .

# QED

# QED in the Furry picture

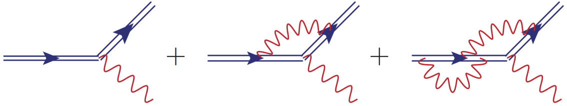
$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi - e\bar{\psi}\not{A}\psi$$

- Background $\mathcal{D}_\mu = \partial_\mu + ieA^{\text{bg}}_\mu$. Strong if $eA^{\text{bg}}/m \sim a_0 \gg 1$.

  Treat exactly.

- Generation/absorption of $e^\pm$, $\gamma$: controlled by $\alpha = e^2/(4\pi)$.

  Perturbation theory, as usual.



- Usually LO or NLO: hoping higher-orders $\propto \alpha^n$ small. . .
- Vital tool for theory, sims & experiment.

Bamber et al PRD 60 ('99) 092004,
Cole et al PRX 8 ('18) 011020, Poder et al PRX 8 ('18) 031004, LUXE, E320, ELI, SSL. . .

# "No alarms and..."

Ritus-Narozhny (RN) conjecture: Furry expansion breaks down.

Ritus Ann.Phys. 69 (1972) 555, Narozhny PRD 20 (1979) 1313, Narozhny PRD 21 (1980) 1176

- Based on (loop) calculations in a constant crossed field.
  Low frequency limit of a monochromatic plane wave

$$E^2 - B^2 = \mathbf{E}.\mathbf{B} = 0$$

- Only invariant: 'quantum nonlinearity parameter'

$$\chi = \gamma \frac{eE}{m^2}$$

Conjecture: loops$\sim$ powers of $\alpha\chi^{2/3}$, rather than $\alpha$.

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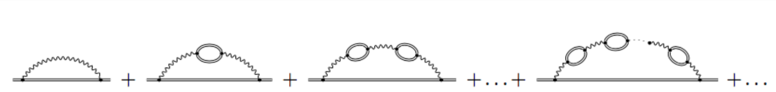
$\implies$ breakdown of (Furry) expansion when $\alpha\chi^{2/3} \gtrsim 1$.

$\implies$ cannot calculate in deeply quantum regime.

# Constant Crossed Fields: The State of The Art



Mironov et al PRD 102 (2020) 053005

1. $n$-loop polarisation operator insertion:

$$\frac{(n+1)\text{–loop}}{n\text{–loop}} \sim \alpha\chi^{2/3} \quad \text{rather than } \alpha \,, \quad \text{when} \quad \chi \gg 1$$

2. Resummed bubble chain correction: $\sim \chi^{-1/3}(\alpha\chi^{2/3})^2$

? Open questions.... are these the dominant corrections?

- Investigate! How do diagrams & processes behave at $\chi \gg 1$?

# "With ~~great~~ fractional power..."

Q. Are we talking about the same thing?

A. Breakdown of Furry expansion.

- Already seen/ will see: occurs classically & beyond LCFA.
- $\alpha \to \alpha a_0^n$.
- $\alpha\chi^{2/3}$ particular to (L)CFA.

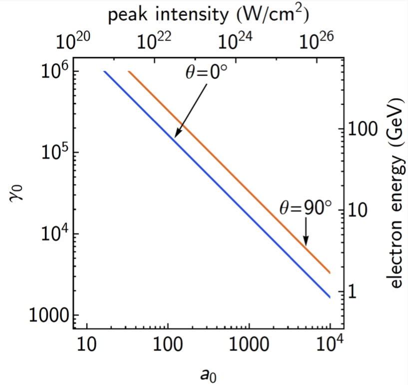
→ Different regimes, different observables, different parameters.

Podszus & DiPiazza, PRD 99 (2019) 076004, Ilderton PRD 99 (2019) 085002

Ekman et al PRD 102 (2020) 116005

# Experiment

# Don't be obtuse. . . no, wait. . .

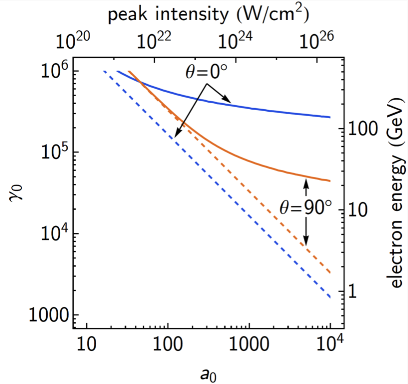



$\gamma_0$ = initial electron energy
$a_0$ = peak laser intensity
duration $= 50\lambda$

? Can we reach $\alpha\chi^{2/3} \sim 0.1$?
(10% loop corrections)

- $\chi \sim a_0\gamma_0(1 + \cos\theta)$.
  $\theta$ = angle to head-on.
- Head-on favoured. . .
  . . . neglecting energy loss!

# The right angle is a right angle.




$\gamma_0$ = initial electron energy
$a_0$ = peak laser intensity
duration $= 50\lambda$

- Include radiative losses

  Blackburn et al NJP21 (2019) 053040

- Oblique incidence favoured!

  $a_0 \gg 1$ tight focussing.

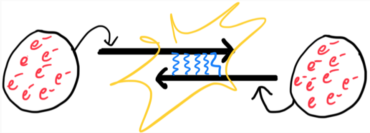
- Possible to reach $\alpha\chi^{2/3} \sim 0.1$:

  50GeV, $a_0 = 1000$.

# Beam-beam collisions

- Beam-beam collisions.

Del Gaudio et al, PRAB 22 (2019) 023402
Yakimenko et al PRL 122 (2019) 190404




- 125 GeV beams. v = 0.999999999992c
- 1 nC $(10^{10} e^{-})$ charge.

- One beams sees strong (summed) Coulomb field of other.

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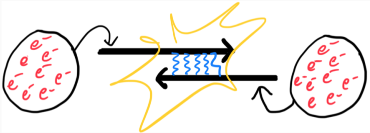



- 125 GeV beams. v = 0.999999999992c
- 1 nC $(10^{10}e^-)$ charge.

- One beams sees strong (summed) Coulomb field of other.
- Need high charge, otherwise high-energy regime.
- $\uparrow \chi$ by $\uparrow$ energy $\implies \log \chi$.

Podszus & DiPiazza, PRD 99 (2019) 076004
Ilderton PRD 99 (2019) 085002

- Highly-boosted beams with high charge density. . .

$\rightarrow$ Beyond-LCFA methods from hep. Adamo, Ilderton, MacLeod, *to appear*.

# Recent developments

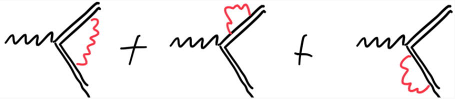
# “Step into the ~~vortex~~ vertex”

- Vertex correction:

Morozov Narozhny Ritus, Sov. Phys. JETP 53(1981)1103
DiPiazza & Lopez-Lopez, PRD 102 (2020) 076018

$$\Gamma^{\mu} \sim \alpha \chi^{2/3} \not{n} n^{\mu} + \ldots$$

- Gauge non-invariant but always comes in combination:



- Invariant. Leading term cancels against 1-loop self-energy.

- Conjectured to extend → all-orders.

Mironov & Fedotov, 2109.00634 [hep-th]

→ Dominance of bubble-chain contributions.

# All-in for inclusivity

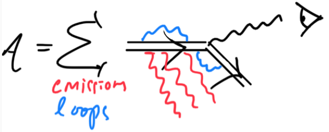
- Measurements: everything is inclusive.
- In vacuum: sum over soft (unobsevable) photons.

  Physically sensible & cures IR problems.
- Calculating inclusive observables: partial resummations.
- Resummation possible if approximations made.

# Out of sight, out of mind...

- In a laser: sum also over laser-degenerate emissions.

- Example: nonlinear Compton.




! All emissions, all loops can be resummed

in a restricted parameter region.

Edwards & Ilderton PRD 103 (2021) 016004

$$|\mathcal{A}_{\text{all orders}}|^2 \sim \exp\left(-\#\alpha a_0^2\right)|\mathcal{A}_{\text{tree}}|^2$$

- High powers of $\alpha a_0^2$ resum to exponentials.

# It all adds up

- $e^-$–plane wave collision: total radiated momentum $R_\mu$.

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Ilderton & Torgrimsson PLB 725 (2013) 481, de la Cruz et al JHEP 12 (2020) 076

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Ilderton & Torgrimsson PLB 725 (2013) 481, de la Cruz et al JHEP 12 (2020) 076

- Self-energies (powers of $\alpha\chi^{2/3}$) all multiply $p^{\text{in}}_\mu$.

- But…coeffcent of $p^{\text{in}}_\mu$ is $\sum_f \mathbb{P}(f) = 1$ by unitarity.

- Unitarity kills self-energy loops to all orders.

Heinzl Ilderton King PRL 127 (2021) 061601

- Can be important to look beyond subsets of diagrams.

Borinsky, Dunne, Meynig 2104.00593 [hep-th]

# “If you build it, he will ~~come~~ resum.”

- $e^-$ scattering on a plane wave.
- Expectation value of $e^-$ momentum. Inclusive!

$$\langle \hat{P} \rangle = \langle\, e^- \,| S^\dagger \hat{P} S |\, e^- \,\rangle$$

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$$\langle \hat{P} \rangle = \langle\, e^- \,| S^\dagger \hat{P} S |\, e^- \,\rangle = \sum_n \alpha^n M_n$$

- Strong field or long pulse limit:

  Recursion: $\mathcal{O}(\alpha^n)$ from $\mathcal{O}(\alpha)$ ‘building blocks’.


- QED version of ‘Müller matrices’.

Dinu & Torgrimsson PRD 102 (2020) 016018

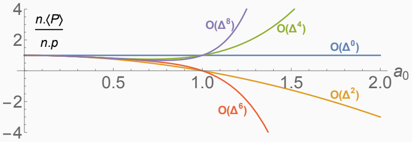
- Example: resummation in classical limit. . .

Torgrimsson PRL 127 (2021) 111602

# Breakdown and resummation

$$\frac{n\cdot\langle\hat{P}\rangle}{n\cdot p^{\text{in}}} = 1-\Delta+\Delta^2-\Delta^3+\dots \quad \text{where } \Delta\sim\alpha a_0^2$$

- Expansion in $\Delta\sim\alpha a_0^2$.
- Corrections become large & unphysical: $n\cdot p \overset{!}{<} 0$

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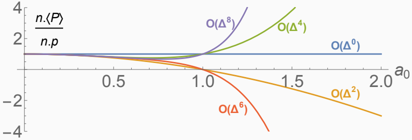


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DiPiazza LMP 83 (2008) 305

$$\frac{n\cdot\langle\hat{P}\rangle}{n\cdot p_{\text{in}}} \xrightarrow{\text{resum}} \frac{1}{1+\Delta} > 0$$

L.L. recovered from QED resummed to all orders in $\alpha$.

Torgrimsson PRL 127 (2021) 111602

# Conclusions

# Conclusions 1/2

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  - Beyond plane waves.
  - Classical and quantum.

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- **Formally**: the Furry expansion can and does break down.
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  - Beyond plane waves.
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- **Lesson:** Strong fields eventually require resummation.
  - Possible with some approximation/assumption/simplification.

    Heinzl Ilderton King PRL 127 (2021) 061601

    Torgrimsson PRL 127 (2021) 111602
  - Identifying relevant contributions essential.

    DiPiazza & Lopez-Lopez, PRD 102 (2020) 076018

    Mironov & Fedotov, 2109.00634 [hep-th]
  - Sometimes need everything.

# Conclusions 2/2

- **Experiment:** can we access the 'strongly coupled' regime?
  - Back-reaction / depletion of fields? Seipt et al PRL 118 (2017) 154803
  - 'Instability' of background to e.g. cascades?

    Fedotov et al PRL 105 (2010) 080402

- **Some topics not covered / an apology:**
  - Other experimental scenarios? DiPiazza et al PRL 124 (2020) 044801
  - All-Loop E-H Lagrangian for strong $B$-field.

    Karbstein, PRL 122 (2019) 211602
  - Reducible contributions. Gies & Karbstein, JHEP 03 (2017) 108

    Karbstein, 2109.04823
  - Other resummations! Podszus & DiPazza, PRD 104 (2021) 016014

    Ekman, Heinzl, Ilderton PRD 104 (2021) 036002

    Torgrimsson 2105.02220