# Spacetime singularities and cosmic censorship III

Håkan Andréasson

University of Gothenburg / Chalmers University of Technology

Mitteldeutsche Physik Combo
Jena-Leipzig, March 2021

# Einstein-Vlasov in the New York Times 1991

The New York Times

## Computer Defies Einstein's Theory

By JOHN NOBLE WILFORD
Published: March 10, 1991

A supercomputer at Cornell University, simulating a tremendous gravitational collapse in the universe, has startled and confounded astrophysicists by producing results that should not be possible according to Einstein's general theory of relativity.

(Reconsidered 2019 by W. East who finds numerical support in favour of weak CC.)

# Criticism raised by Alan Rendall '92

- Shapiro and Teukolsky first made simulations for the Vlasov-Poisson (VP) system. They then generalized this code to the relativistic case.
- In their VP code they took data close to dust (since the dust solution is known) and found that the kinetic energy and the potential energy diverge as the singularity was approached.
- Shapiro and Teukolsky considered this as support for the reliability of their numerical code. Dust and Vlasov matter, however, behave very differently in some situations.
- Pfaffelmoser and Lions/Perthame showed in the early '90s that global existence holds for the Vlasov-Poisson system. In particular, the kinetic energy and the potential energy do not blow up.

# Dust from Vlasov

Dust is a pressureless fluid which can be approximated by Vlasov matter.

The Vlasov equation is linear in $f$ and distributional solutions make sense. One class of distributional solutions is given by

$$f(x^\gamma, p^a) = -u_0|g|^{-1/2}\rho(x^\gamma)\delta(p^a - u^a),$$

where $\rho \geq 0$ and $u^a(x^\gamma)$ is a mapping from spacetime into the mass shell and $u_0$ is given by $u^a$ from the mass shell relation.

Solutions of the EV system where the phase space density $f$ has this form are in one-to-one correspondence with dust solutions of the Einstein equations with density $\rho$ and four-velocity $u^\alpha$.

Dust may thus be considered as a *singular* case of matter described by the Vlasov equation.

# The Euler-Poisson system and the Vlasov-Possion system

The pressureless Euler-Poisson system reads

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0,\\
&\partial_t u + (u \cdot \partial_x) u = -\partial_x U(t,x),\\
&\Delta U = 4\pi\rho, \quad \lim_{|x|\to\infty} U(t,x) = 0.
\end{aligned}$$

The Vlasov-Possion system reads

$$\begin{aligned}
&\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0,\\
&\Delta U = 4\pi\rho, \quad \lim_{|x|\to\infty} U(t,x) = 0,\\
&\rho(t,x) = \int f(t,x,v)\, dv.
\end{aligned}$$

# A collapsing ball of dust

Let

$$\rho(t,x) := \frac{3}{4\pi}\frac{1}{r^3(t)}\mathbf{1}_{B_{r(t)}(0)},$$

where $r(t)$ solves

$$\ddot{r} = -\frac{1}{r^2},\ r(0) = 1,\ \dot{r} = 0.$$

Also, let

$$u(t,x) = \frac{\dot{r}(t)}{r(t)}x,$$

then $(\rho, u, U)$ is a solution of the Euler-Poisson system above (where $U$ is determined via the Poisson equation).

This solution describes a ball of dust, initially at rest, which collapses under its own gravitational field to a point in finite time, since it can be shown that $\lim_{t\to T} r(t) = 0$ for some $T > 0$.

If we swop matter model, from dust to Vlasov, and consider the Vlasov-Poisson system instead, then the global existence results for the Vlasov-Poisson system guarantee that no singularity will form.

The global existence results say nothing about the behaviour for such solutions, only that they will not break down. In a work by Rein and Taegert (2016) this question is investigated more carefully.

**Theorem**

*For any constants* $C1, C2 > 0$ *there exists a smooth, spherically symmetric solution* $f$ *of the Vlasov-Poisson system such that initially*

$$\|\rho(0)\|_\infty < C_1,$$

*but at some time* $t > 0$

$$\|\rho(t)\|_\infty > C_2.$$

# Dust collapse in general relativity

The homogeneous Oppenheimer-Snyder collapse from 1939, and the numerous investigations about inhomogeneous dust collapse, have had an immense impact in general relativity.

In particular there is a huge literature on dust collapse in cases where naked singularities form.

1. arXiv:2103.07190 [pdf, other] gr-qc

**Causal Structure of Singularity in non-spherical Gravitational Collapse**

**Authors:** Dipanjan Dey, Pankaj S. Joshi, Karim Mosani, Vitalii Vertogradov

**Abstract**: We investigate here the final state of gravitational collapse of a non-spherical and non-marginally bound dust cloud as modeled by the Szekeres spacetime. We show that a directionally globally naked singularity can be formed in this case near the collapsing cloud boundary, and not at its geometric center as is typically the case for a spherical gravitational collapse. This is a strong curvature na... ▽ More

**Submitted** 15 March, 2021; **v1** submitted 12 March, 2021; **originally announced** March 2021.

**Comments:** 5 pages, 2 figures

# A review of the theory of dust collapse

In the literature on dust collapse it is standard to use co-moving coordinates where the metric is written as follows:

$$ds^2 = -dt^2 + e^{2\lambda(t,r)}dr^2 + R^2(t,r)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \qquad (1)$$

Asymptotic flatness means that the metric quantities $\lambda$ and $R$ satisfy the boundary conditions

$$\lim_{r\to\infty} \lambda(t,r) = 0, \quad \lim_{r\to\infty} \frac{R(t,r)}{r} = 1.$$

A regular center requires that

$$\lim_{r\to 0} \frac{re^{\lambda(t,r)}}{R(t,r)} = 1.$$

# The Einstein-Euler system

The Einstein-dust system consists of the Euler equations

$$u^\alpha \nabla_\alpha u_\beta := u^\alpha \left( \partial_\alpha u_\beta - \Gamma^\gamma_{\alpha\beta} u_\gamma \right) = 0, \tag{2}$$

$$u^\alpha \nabla_\alpha \rho + \rho \nabla_\alpha u^\alpha = 0 \tag{3}$$

coupled to the Einstein equations with energy momentum tensor

$$T_{\alpha\beta} = \rho u_\alpha u_\beta. \tag{4}$$

We use comoving coordinates and require that the four velocity field initially is given by

$$u^{\alpha}_{|t=0} = (1,0,0,0), \;\; u_{\alpha|t=0} = (-1,0,0,0).$$

Since $\Gamma^0_{0\beta} = 0$ for a metric of the form (1),

$$u^{\alpha} = (1,0,0,0), \;\; u_{\alpha} = (-1,0,0,0)$$

solves (2). The energy momentum tensor takes the form

$$T_{00} = \rho, \quad T_{\alpha\beta} = 0 \text{ for } (\alpha,\beta) \neq (0,0),$$

and the continuity equation (3) becomes

$$\partial_t \rho + \left( \partial_t \lambda + 2\frac{\partial_t R}{R} \right) \rho = 0.$$

We also write down the field equations again for the case at hand:

$$\begin{aligned}
2R\partial_t R\partial_t\lambda + 2Re^{-2\lambda}\partial_r R\partial_r\lambda + (\partial_t R)^2 - 2Re^{-2\lambda}\partial_r^2 R & \\
- e^{-2\lambda}(\partial_r R)^2 + 1 &= 8\pi R^2\rho, \\
\partial_r R\partial_t\lambda - \partial_r\partial_t R &= 0, \\
2R\partial_t^2 R + (\partial_t R)^2 - e^{-2\lambda}(\partial_r R)^2 + 1 &= 0, \\
R(\partial_t\lambda)^2 + \partial_t R\partial_t\lambda + R\partial_t^2\lambda + \partial_t^2 R - e^{-2\lambda}\partial_r R\partial_r\lambda & \\
- e^{-2\lambda}\partial_r^2 R &= 0.
\end{aligned}$$

As initial condition we prescribe

$$\rho(0,r) = \mathring{\rho}(r),\ R(0,r) = \mathring{R}(r),\ \partial_t R(0,r) = \mathring{v}(r). \qquad (5)$$

# Solution of the Einstein-dust system

Given initial data as prescribed in (5) we define

$$\mathring{m}(r) := 4\pi \int_0^r \mathring{R}^2(s)\mathring{R}'(s)\mathring{\rho}(s)\, ds,$$

and

$$W^{-2}(r) := (\mathring{v}(r))^2 + 1 - \frac{2\mathring{m}(r)}{\mathring{R}(r)} \quad (\geq 0).$$

Let $R = R(t,r)$ solve *the master equation*

$$\partial_t R(t,r) = -\sqrt{\frac{2\mathring{m}(r)}{R(t,r)} + W^{-2}(r) - 1},$$

then $\lambda$ and $\rho$ are given by

$$e^{\lambda(t,r)} = W(r)\,\partial_r R(t,r),$$

$$\rho(t,r) = \frac{\mathring{R}^2(r)\mathring{R}'(r)}{R^2(t,r)\partial_r R(t,r)}\mathring{\rho}(r).$$

# Solving the master equation

The master equation reads

$$\partial_t R(t,r) = -\sqrt{\frac{2\mathring{m}(r)}{R(t,r)} + f(r)}, \qquad (6)$$

where $f(r) = (W(r))^{-2} - 1$.

Using separation of variables the following function turns up:

$$G(y) := \begin{cases} \frac{\arcsin\sqrt{y}}{y^{3/2}} - \frac{\sqrt{1-y}}{y} & , \quad 0 < y \leq 1, \\ \frac{2}{3} & , \quad y = 0, \\ -\frac{\operatorname{arcsinh}\sqrt{-y}}{(-y)^{3/2}} - \frac{\sqrt{1-y}}{y} & , \quad -\infty < y < 0. \end{cases}$$

If we define

$$t_0(r) := \frac{\mathring{R}(r)^{3/2} G\left(-\frac{\mathring{R}(r) f(r)}{2\mathring{m}(r)}\right)}{\sqrt{2\mathring{m}(r)}}$$

then for $0 \leq t \leq t_0(r),\ r > 0$ the implicit relation

$$t_0(r) - t = \frac{R^{3/2} G\left(-\frac{R f(r)}{2\mathring{m}(r)}\right)}{\sqrt{2\mathring{m}(r)}}$$

defines the desired solution to (6).

Since $\lim_{t \to t_0(r)} R(t, r) = 0,\ t_0(r)$ is the coordinate time at which the solution blows up at $r$ which is the time by which the dust particle which started out initially at $r$ has reached the center.

# Trapped surfaces

The general condition for a trapped surface in the given coordinates is that along all radial null geodesics

$$\frac{d}{d\tau}R(t,r) = \frac{dt}{d\tau}\left(\partial_t R + \partial_r R\frac{dr}{dt}\right) < 0,$$

which together with

$$\frac{dr}{dt} = \pm e^{-\lambda}$$

and the fact that $\partial_t R \leq 0$ means that

$$\partial_t R(t,r) < -e^{-\lambda(t,r)}\partial_r R(t,r). \tag{7}$$

For the dust case this is equivalent to

$$\sqrt{\frac{2\mathring{m}(r)}{R(t,r)} + f(r)} > W^{-1}(r)$$

which in turn can be written as

$$R(t,r) < 2\mathring{m}(r). \tag{8}$$

If we take initial data which contain no trapped surface and observe that as

$$\lim_{t \to t_0(r)} R(t,r) = 0,$$

it follows that there is a unique time $t_H(r) \in ]0, t_0(r)[$ such that $R(t_H(r), r) = 2\mathring{m}(r)$, which is given by the relation

$$t_0(r) - t_H(r) = 2\mathring{m}(r) G(-f(r)) \tag{9}$$

which follows from (7) and characterizes the time when a trapped surface forms at $r$.

Hence we have derived expressions for both the blow up time $t_0(r)$ and the time when a trapped surface forms $t_H(r)$.

# The Oppenheimer-Snyder solution

A special case is the Oppenheimer-Snyder solution for which $W(r) \equiv 1$ and

$$\mathring{\rho} = c\,1_{[0,1]}.$$

Then for $r \leq 1$

$$R(t,r) = \left(1 - \sqrt{6\pi c}\,t\right)^{2/3} r =: \gamma(t)r,$$

$$\rho(t,r) = \frac{c}{\gamma^3(t)},$$

and

$$e^{\lambda(t,r)} = \partial_r R(t,r).$$

Moreover,

$$t_0(r) = \frac{1}{\sqrt{6\pi c}}, \quad t_H(r) = \frac{1}{\sqrt{6\pi c}} - \frac{16\pi c}{9} r^3.$$

# Remark on stability and instability

A general feature of the dust solutions is that they blow up in finite time *independently* of the amplitude $c$; if the amplitude is taken very small, the blow up occurs later but still after a finite time.

*Hence, the Einstein-Dust system might be said to be unstable.*

How does this relate to the stability results for the EV system? For small initial data global existence has been shown for the Einstein-Vlasov system:

- Rein and Rendall '92 in spherical symmetry.
- Lindblad and Taylor '17 in the general case
- Fajman, Joudioux and Smulevici '17 in the general case

# Approximate dust with Vlasov matter

Roughly, to approximate dust we choose

$$\mathring{f}(x, v) = h_\epsilon(v)\mathring{\rho}(x),$$

where $h_\epsilon$ is approximating a Dirac delta function, and $\mathring{\rho}(x) = c1_{[0,1]}$.

Hence we have two parameters, $\epsilon$ and $c$. If we fix $c$ and let $\epsilon \to 0$ then $\mathring{f} \to \infty$, whereas if we fix $\epsilon$ and let $c \to 0$ then $\mathring{f} \to 0$.

In the former case the above stability results do not apply whereas in the latter they do.

This simple observation shows a fundamental difference between dust and Vlasov matter!

# The Einstein-Vlasov system in co-moving coordinates

In kinetic theory the movement of the particles is complex and one cannot speak about co-moving coordinates literally. However, in order to compare solutions for dust and for the EV system it is convenient to use the same type of coordinates.

(In fact local existence fails at $r = 0$ in co-moving coordinates. However, we can show local existence using other coordinates where dust collapse also can be described in a natural way.)

# Approximate dust solutions by EV solutions

It is possible to approximate dust solutions arbitrary well by a solution of the Einstein-Vlasov system on any time interval $[0, T]$, where $T < t_0$.

*Remark:* The solution of the Einstein-Vlasov system is regular also at the boundary of the collapsing body.

The following result is in preparation together with Gerhard Rein.

**Theorem**

*Consider a collapsing dust solution with blow up time $t_0$. Given $\delta > 0$ and $T < t_0$, there exists initial data for the Einstein-Vlasov system such that the corresponding solutions are "$\delta$-close" to the dust solution for all $t \in [0, T]$.*

# Naked singularities for dust

In the Oppenheimer-Snyder collapse the density is homogeneous within the matter; $\rho(t, r) = \rho(t)$. It is well-known that inhomogeneous data can be prescribed which lead to naked singularities for dust.

Numerically this was first studied by Eardley and Smarr in 1979 and then a rigorous proof was given by Christodoulou 1984:

- Time functions in numerical relativity: Marginally bound dust collapse, *Phys. Rev.* D 19, 2239 (1979).
- Violation of Cosmic Censorship in the Gravitational Collapse of a Dust Cloud, *Commun. Math. Phys.* 93, 171-195 (1984).

It is of great interest to evolve initial data for the Einstein-Vlasov system having the property that analogous data for dust form naked singularities.

# Naked singularity data

We consider initial data specified by Eardley and Smarr.

For $0 \leq r \leq 1$ let

$$\mathring{\rho}(r) = \frac{1}{6\pi}(\mathring{t} + \zeta r^p)^{-1}(\mathring{t} + \zeta(1 + 2p/3)r^p)^{-1} 1_{[0,1]}(r),$$

where $\zeta, p, \mathring{t} \in [0, \infty[$ are parameters. It follows that for $0 \leq r \leq 1$

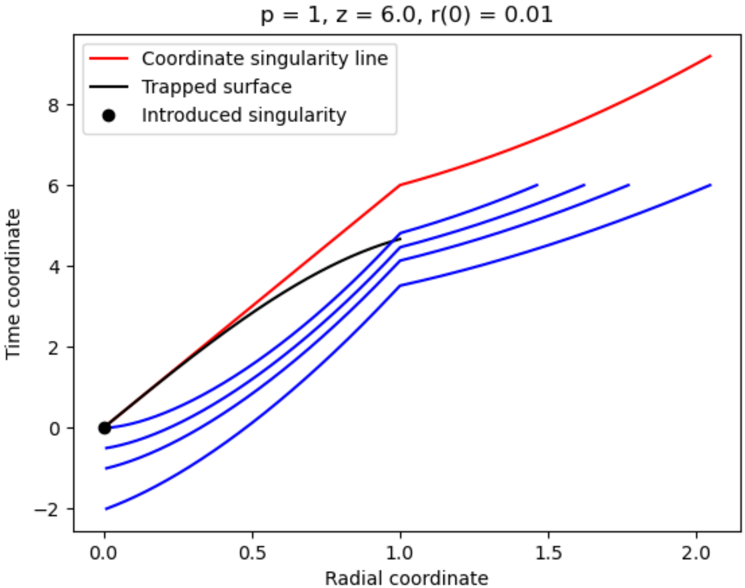
$$\mathring{R}(r) = \left(\frac{9}{2}\right)^{1/3} r\,(\mathring{t} + \zeta r^p)^{2/3}.$$

For $0 \leq r \leq 1$ the blow up time is given by

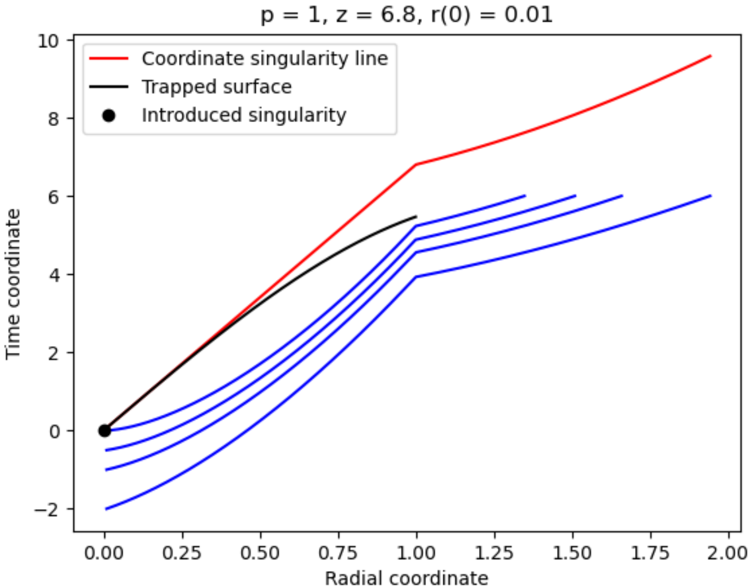
$$t_0(r) = \mathring{t} + \zeta r^p,$$

and the time $t_H(r)$ when a trapped surface forms is given by

$$t_H(r) = \mathring{t} + \zeta r^p - \frac{4}{3} r^3.$$


p = 1, z = 6.0, r(0) = 0.01
Coordinate singularity line
Trapped surface
Introduced singularity
Time coordinate
Radial coordinate
8
6
4
2
0
−2
0.0
0.5
1.0
1.5
2.0

# Naked singularity ($\zeta > \zeta_{crit} \approx 6.3$)



# Homogeneous solutions

It turns out to be possible to construct spatially homogeneous solutions of the EV system in co-moving coordinates. Of course, these violate the boundary conditions at infinity, and the idea is that they are cut-off along some suitable curve.

The homogeneous solutions have $j = 0$ but the radial pressure $p$ and the tangential pressure $p_T$ are non-zero.

These solutions are useful to understand the mechanism which makes dust and Vlasov behave very differently close to the blow up.

# The energy density

The energy density takes the following form in the case of Vlasov matter:

$$\rho(t,r) = \frac{\pi}{R^2}\int_{-\infty}^{\infty}\int_0^{\infty}\sqrt{1+w^2+\frac{L}{R^2}}f(t,r,w,L)\,dL\,dw.$$

Initially we start from initial data which are dust-like in the sense that the support of the momenta $w$ and $L$ is very small.

The closer to the blow up time we get the larger the momenta will become; only if a particle has zero momentum initially it will continue to be zero.

Hence, the kinetic energy of the particles will eventually give an essential contribution to the energy density.

# Main mechanism

The density $\rho^{EV}$ of the homogeneous Einstein-Vlasov solution turns out to take the form

$$\rho^{EV}(t) = \frac{1}{\tilde{\gamma}^3(t)} \int_0^1 \sqrt{1 + \frac{\epsilon\eta}{\tilde{\gamma}^2(t)}}\, G(\eta)\, d\eta,$$

where $G$ is such that $\rho^{EV} = \rho^D$ when $\epsilon = 0$. Here $\epsilon << 1$ measures how closely a solution approximates dust.

Recall that the density for a homogeneous dust solution evolves as

$$\rho^D(t) = \frac{\mathring{\rho}^D}{\gamma^3(t)}, \text{ where } \mathring{\rho}^D = \text{constant.} \;\; (\gamma(t) \approx \tilde{\gamma}(t))$$

Clearly the character of $\rho^D$ and $\rho^{EV}$ becomes quite different when $\tilde{\gamma}^2(t) \sim \epsilon$, which eventually will be the case since $\tilde{\gamma}(t) \to 0$ as $t \to t_0$. In fact $\rho^{EV} >> \rho^D$ when $\tilde{\gamma}(t) \sim 0$.

Thank You Jena and Leipzig!