Chiral Spiral for finite number of flavors

Inhomogeneous phases of Four-Fermion theories on the lattice (handout version)

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with

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Four-Fermion theories (4FT)

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + i\mu\bar{\psi}\gamma_0\psi + (\bar{\psi}\Gamma_1\psi)(\bar{\psi}\Gamma_2\psi)$$

- Γ_i ... matrices
- μ ... chemical potential
- no bare mass term
- N_f flavors

Motivation & Goals

Why 4FT?

- **Low-energy effective theories for QCD**
  Asymptotic freedom, (spontaneous breaking of) chiral symmetry, . . .
- **Solid State Physics**
  Graphene, high-T_c superconductors, polymers, . . .
- . . .

In this talk

- Study thermodynamics of 4FT using Lattice Field Theory.
- Particular emphasis on inhomogeneous phases.
- 1+1 dimensions.

Gross-Neveu model

$$\mathcal{L} = i\bar{\psi}\left(\partial\!\!\!/ + \mu\gamma_0\right)\psi + \frac{g^2}{2N_f}(\bar{\psi}\psi)^2$$

or, equivalently

$$\mathcal{L} = i\bar{\psi}\left(\partial\!\!\!/ + \mu\gamma_0 + \sigma\right)\psi + \frac{N_f}{2g^2}\sigma^2$$

Ward identity

$$\langle\bar{\psi}\psi\rangle = \frac{iN_f}{g^2}\langle\sigma\rangle$$

discrete chiral symmetry

$$\psi \to \gamma_5\psi, \quad \bar{\psi} \to -\bar{\psi}\gamma_5, \quad \sigma \to -\sigma$$

*[1]GN, $N_f \to \infty$

σ is an auxiliary field that we have introduced for convenience. It is in general spacetime dependent and needs to be taken into account in the path integral.

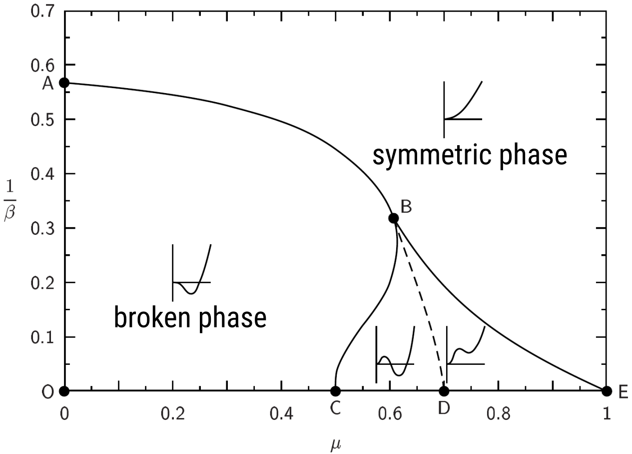
The GN model can be studied in the limit of $N_f \to \infty$. In this limit calculations simplify tremendously and mean field approaches become exact.

In the following we show a historic evolution of our knowledge about the phase structure of the 2D GN model, in chronological order, starting with large N_f results.

[1]Slides marked by an asterisk in the title were not included in the talk but serve as additional information.

$\overline{\text{GN}}$, $N_f \to \infty$

assuming $\sigma(x) = \sigma$ [Wolff, 1985]

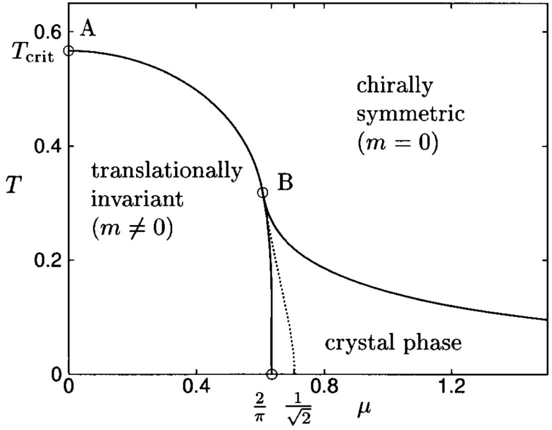


*GN, $N_f \to \infty$

The assumption that $\sigma(x)$ is a constant in space makes sense on theoretical grounds, given that the GN Lagrangian obeys translation symmetry. However it turns out that this assumtion does not give the true ground state.
Instead ...

$\overline{\text{GN}}$, $N_f \to \infty$

allowing for inhomogeneous $\sigma(x)$ [Thies, Urlichs, 2003]

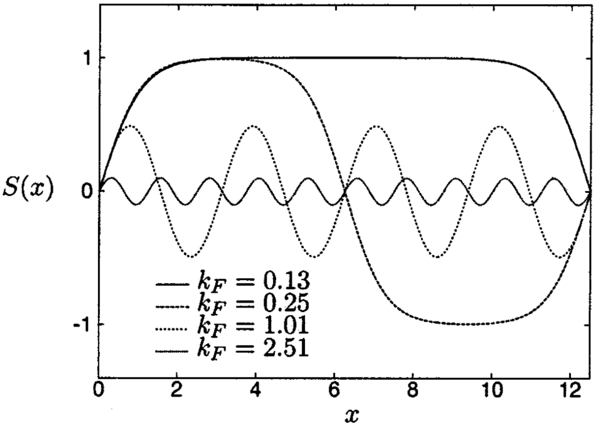


*GN, $N_f \to \infty$

The phase diagram on the previous slide now shows the true ground state. The ”translationally invariant” phase corresponds to the ”broken phase” on slide 5, while the ”chirally symmetric” phase corresponds to the ”symmetric phase”.
What is new here is the ”crystal phase”, which is now spatially inhomogeneous. We give the (now spatially dependent) chiral condensate on the next slide, showing that in addition to chiral symmetry now also translational symmetry is spontaneously broken!

$\overline{\mathrm{GN}}$, $N_f \to \infty$

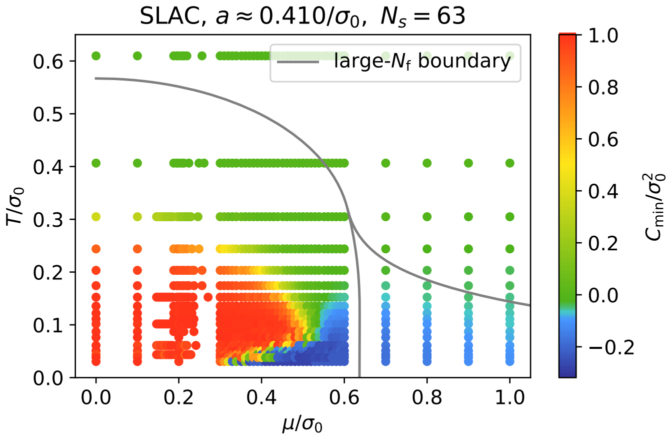
allowing for inhomogeneous $\sigma(x)$ [Thies, Urlichs, 2003]



*GN, finite N_f lattice results

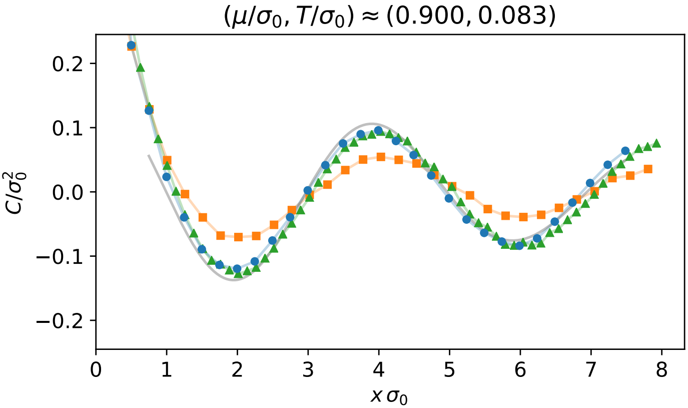
So far all results were obtained in mean field studies ($N_f \to \infty$). The question arises whether inhomogeneous phases survive for finite N_f. To answer that question one can use Lattice Field Theory models.
On the next two slides we show the phase diagram obtained on a spacetime lattice (the red, green and blue phases correspond to homogeneously broken, symmetric and inhomogeneously broken phases respectively) and a typical inhomogeneous configuration for low temperature and high chemical potential.

$\overline{\text{GN}}$, finite N_f lattice results



[Lenz et al., 2020]

$\overline{\text{GN}}$, finite N_f lattice results



[Lenz et al., 2020]

What do we learn?

- There are inhomogenous phases in the 2D GN model at finite N_f.
- They survive the continuum limit.
- Good agreement with $N_f \to \infty$ already at $N_f = 8$.
- ...

*Nambu-Jona-Lasinio (NJL) model

Let us now study a similar (but different in detail) model, going through the exact same steps as before.

Nambu-Jona-Lasinio (NJL) model

$$\mathcal{L} = i\bar{\psi}\left(\partial\!\!\!/ + \mu\gamma_0\right)\psi + \frac{g^2}{2N_f}\left((\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma_5\psi)^2\right)$$

or, equivalently

$$\mathcal{L} = i\bar{\psi}\left(\partial\!\!\!/ + \mu\gamma_0 + \sigma - i\pi\gamma_5\right)\psi + \frac{N_f}{2g^2}\left(\sigma^2 + \pi^2\right)$$

Ward identities

$$\langle\bar{\psi}\psi\rangle = \frac{iN_f}{g^2}\langle\sigma\rangle$$

$$\langle\bar{\psi}\gamma_5\psi\rangle = -\frac{N_f}{g^2}\langle\pi\rangle$$

continuous chiral symmetry

$$\psi \to e^{i\alpha\gamma_5}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_5}$$

$$\begin{pmatrix}\sigma \\ \pi\end{pmatrix} \to \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\sigma \\ \pi\end{pmatrix}$$

*NJL, $N_f \to \infty$

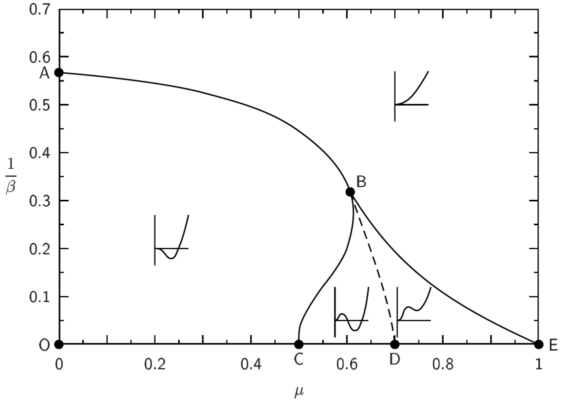
Notice that we now have 2 auxiliary fields, σ and π. The key difference is that chiral symmetry is now continous!

We proceed by performing the same steps as before, first investigating the $N_f \to \infty$ limit, assuming homogeneous σ and π. A simple argument proves that the result will be equivalent to the corresponding GN result, see the next slide.

On the next-to-next slide we will then see what happens if we allow for spatially dependent $\sigma(x)$ and $\pi(x)$.

$\overline{\text{NJL}}$, $N_f \to \infty$

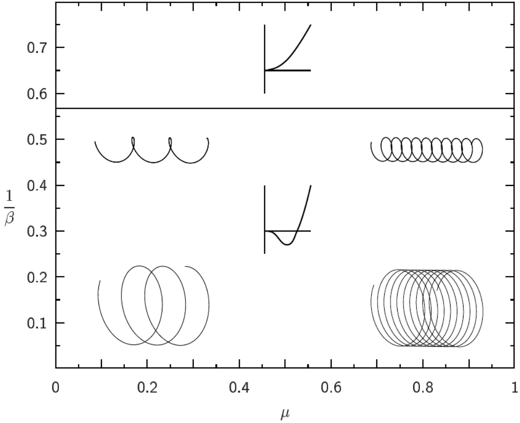
assuming $\sigma(x) = \sigma, \pi(x) = \pi$



$\Longrightarrow$ identical to GN??

$\overline{\text{NJL}}$, $N_f \to \infty$

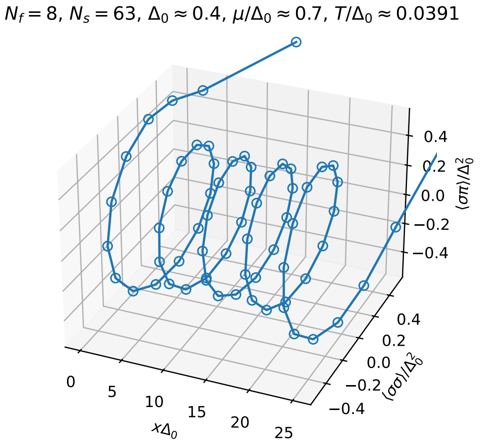
inhomogeneous $\sigma(x), \pi(x) \implies$ chiral spiral [Schön, Thies 2001]



*NJL, $N_f \to \infty$

We see that there are again inhomogenous phases, characterized by the chiral spiral (a 3D plot of σ and π as functions of x).
Now, what happens at finite N_f? We have studied this question on the lattice.

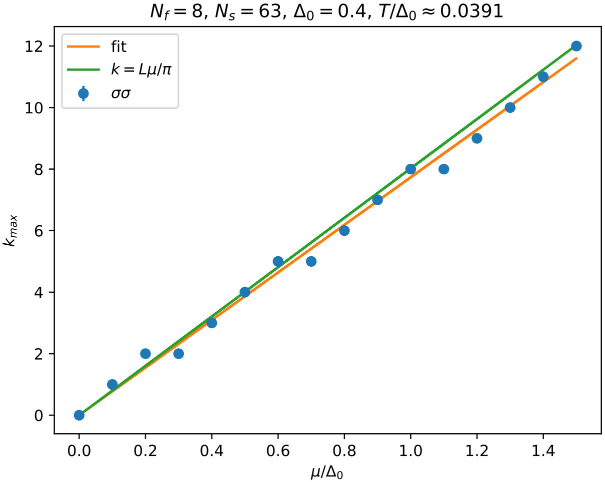
NJL, finite N_f lattice results



*NJL, finite N_f lattice results

This implies that the chiral spiral survives going to finite N_f.
We can now study the μ dependence of the wave number of the chiral spirals and will see very good agreement with the $N_f \to \infty$ results already at $N_f = 8$, as for the GN model.

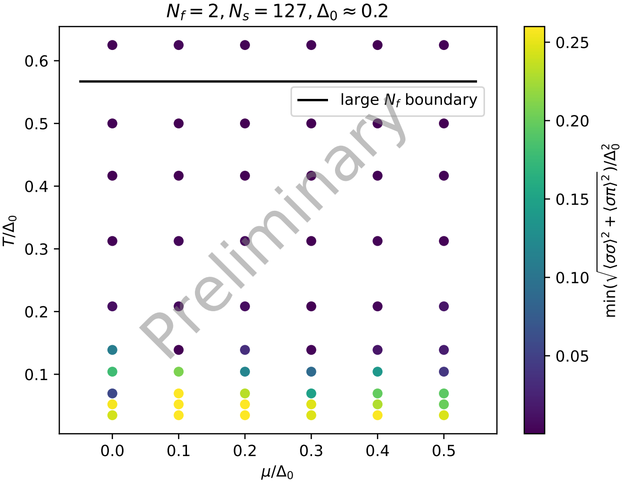
$\overline{\text{NJL}}$, finite N_f lattice results



*NJL, finite N_f lattice results

A preliminary (due to lack of statistics at the time of writing) phase diagram will look like the following, the yellow and green dots indicating the inhomogeneous phase and the dark dots indicating the symmetric phase. Since we work at finite N_f we expect the inhomogeneous phase to shrink since fluctuations tend to destroy the order.

$\overline{\text{NJL}}$, finite N_f lattice results



BUT...

Mermin, Wagner (1966), Coleman (1973)

There is no spontaneous breaking of continuous symmetries in two dimensions!

*BUT...

This is due to the non-existence of massless scalars in 2D.
There seems to be a conflict with our observations. However, we believe that this does, in fact, not pose a problem and on the following slide we give some alternative scenarios to spontaneous symmetry breaking that we could be observing instead.
It is highly non-trivial to determine which mechanism is at work exactly and we have not been able to do so yet.

Ways out

- Almost long-range order (BKT phase).

  [Berezinskiĭ 1970, 1971, Kosterlitz, Thouless , 1973, Witten 1978]

- Viscous fluid instead of crystal.

- (Decoupling Nambu-Goldstone boson.)

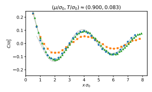
- . . .?

Summary & Outlook

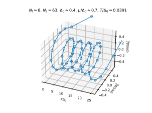
Summary

Inhomogeneous phases (in 2D) at $N_f < \infty$ in

- **GN model**



- **NJL model**



No conflict with CMW theorem

Outlook

- More statistics for phase diagram etc.
- Higher dimensions, more sophisticated models
- External fields, e.g. magnetic

Appendix

Observables

$$C_{\sigma\sigma}(x) = \frac{1}{N_t N_s} \sum_{t,y} \langle \sigma(t, y + x)\sigma(t, y) \rangle$$

$$C_{\sigma\pi}(x) = \frac{1}{N_t N_s} \sum_{t,y} \langle \sigma(t, y + x)\pi(t, y) \rangle$$

$$C_{\pi\sigma}(x) = \frac{1}{N_t N_s} \sum_{t,y} \langle \pi(t, y + x)\sigma(t, y) \rangle$$

$$C_{\pi\pi}(x) = \frac{1}{N_t N_s} \sum_{t,y} \langle \pi(t, y + x)\pi(t, y) \rangle$$

*Observables

These are the observables we actually measure in our lattice investigations. Attempting to simply measure $\langle \sigma \rangle$ or $\langle \pi \rangle$ would lead to destructive interference, destroying every order.

The correlators on the previous slide (only two of them are independent, we choose $C_{\sigma\sigma}$ and $C_{\sigma\pi}$) do not suffer from this problem and should reflect the inhomogeneity of $\langle \sigma \rangle$ and $\langle \pi \rangle$. Especially, if $\langle \sigma \rangle$ is a cosine and $\langle \pi \rangle$ is a sine function (as is the case in the $\mathbb{N}_f \to \infty$ limit), then $C_{\sigma\sigma}$ and $C_{\sigma\pi}$ will also be cosine and sine respectively.

On the next slide we give a typical example of $C_{\sigma\sigma}$ and $C_{\sigma\pi}$ and their Fourier transforms for a typical inhomogeneous configuration.

Correlators