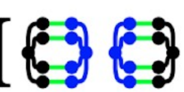
Spin-foam models of Lorentzian quantum space-time

Emerging geometry at the semi-classical limit

$$Z(\Delta^*) = \sum_{j\to\mathcal{F}}\sum_{\iota\to\mathcal{E}}\int \mathrm{d}\chi_i\Delta_{\chi_i}\left[\prod_{f\in\mathcal{F}} j_f^2(\gamma^2+1)\right]\left[\prod_{e\in\mathcal{E}}\right.$$



$$\left.\right]\left[\prod_{v\in\mathcal{V}}\right.$$

$$\left.\right]$$

Spin-foam models of Lorentzian quantum space-time

Emerging geometry at the semi-classical limit

How to quantize gravity

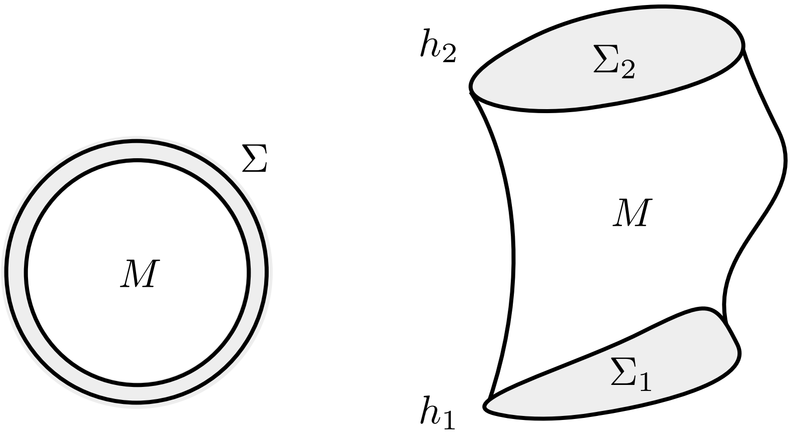
- Quantization of gravity via "sum over histories", reminiscent of transition amplitudes.

- Would like to compute

Note the i

$$Z(h) = \int_{h=g|_{\partial M}} \mathcal{D}g \; e^{iS_{\mathrm{GR}}}$$

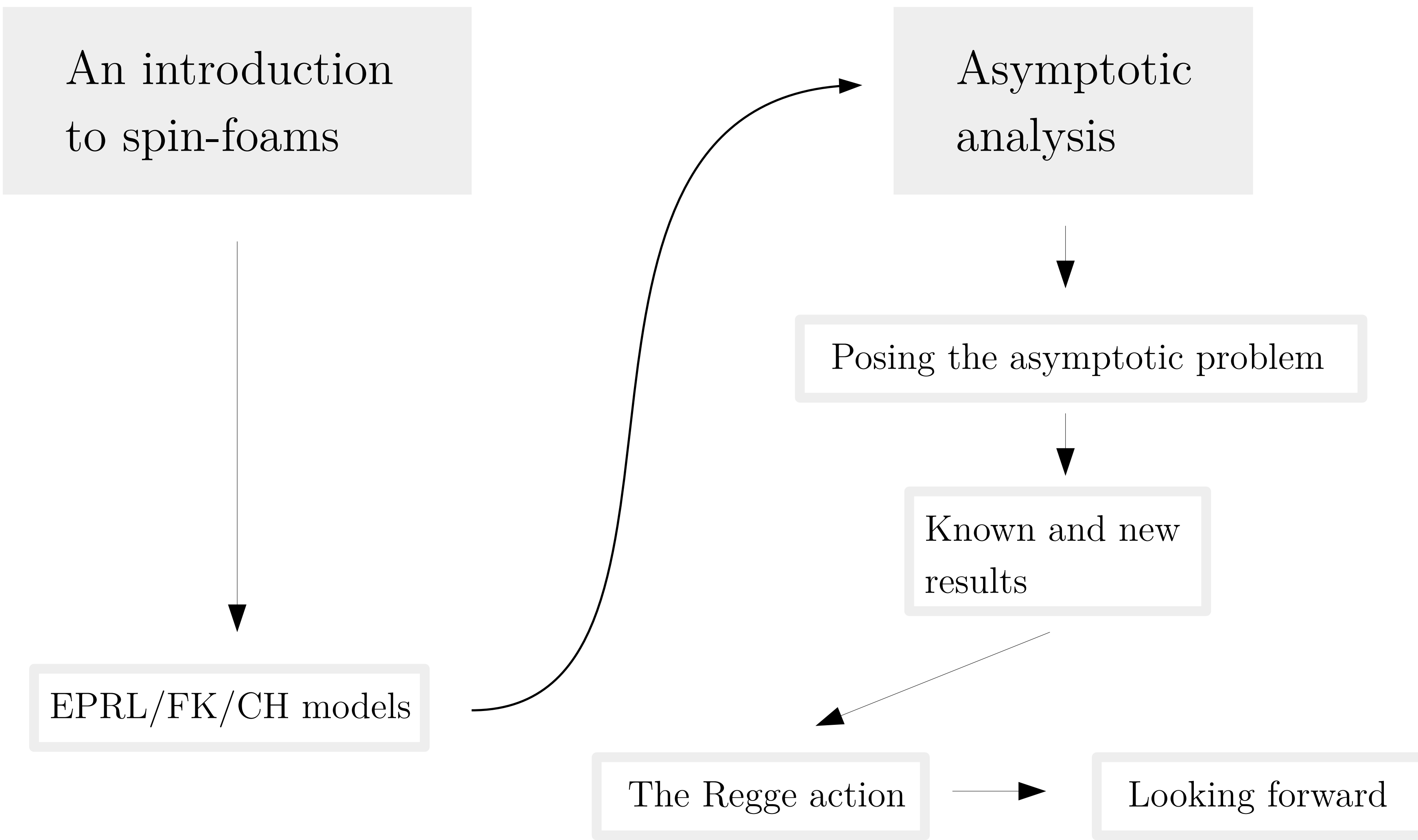
but this is technically difficult.



Lorentzian space-times

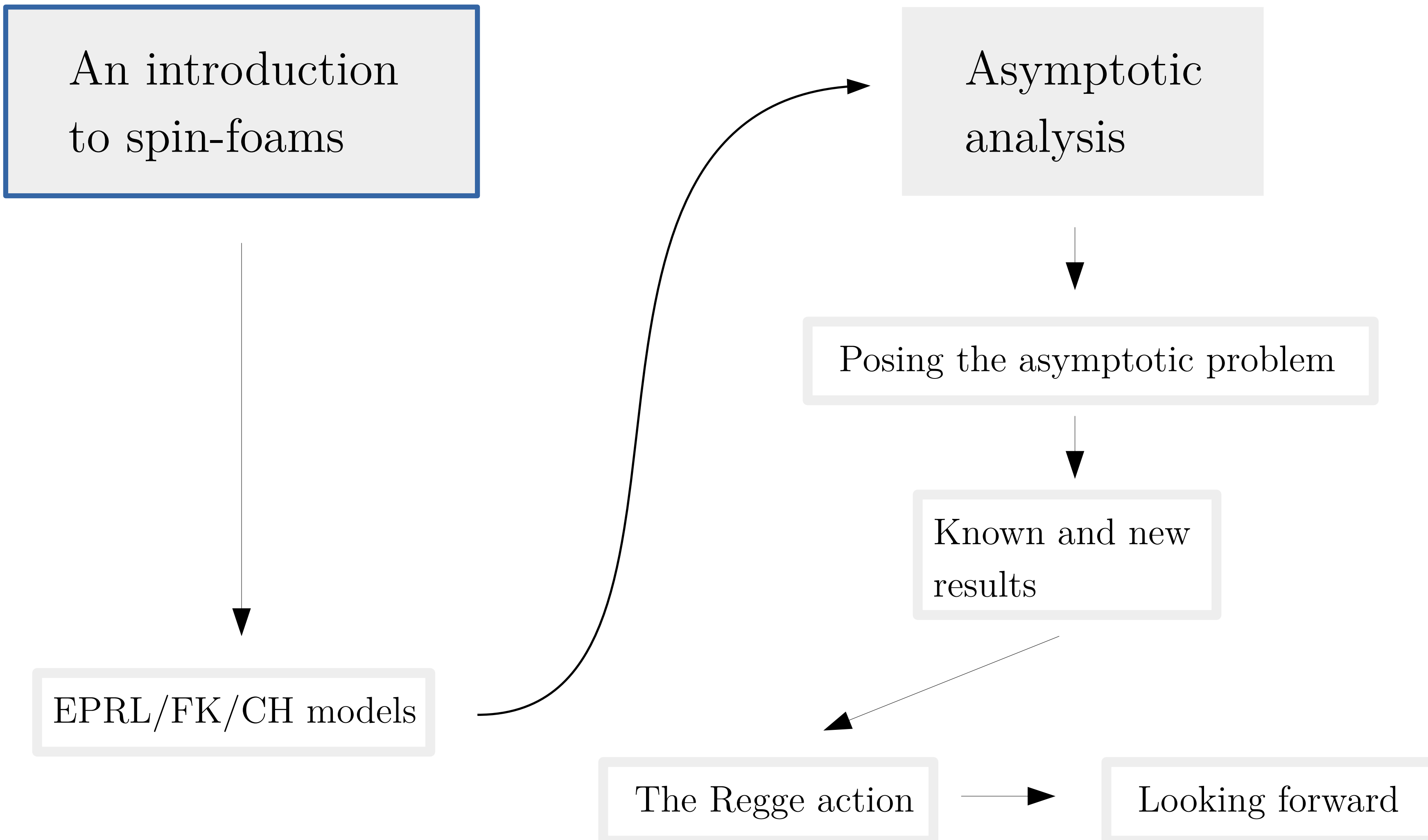
A possible way forward:
the Spin-foam framework

The road-map



An introduction to spin-foams
EPRL/FK/CH models
Asymptotic analysis
Posing the asymptotic problem
Known and new results
The Regge action
Looking forward

The road-map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EPRL/FK/CH spin foam model

- GR = BF + constraints

$$(g = \eta^{IJ} \theta_I \otimes \theta_J)$$

$$S_T = \int_M F^{IJ} \wedge \left(\star + \frac{1}{\gamma} \right) \theta_I \wedge \theta_J$$

"Simplicity constraints"

$$\int_M \mathrm{Tr}\; B \wedge F \,, \quad B = (\star + \gamma^{-1})\, \theta \wedge \theta$$

- The general idea: (Engle et al. '07; Freidel, Krasnov '07; Conrady, Hnibyda '10)

Discretize BF → Quantize → Apply constraints at quantum level

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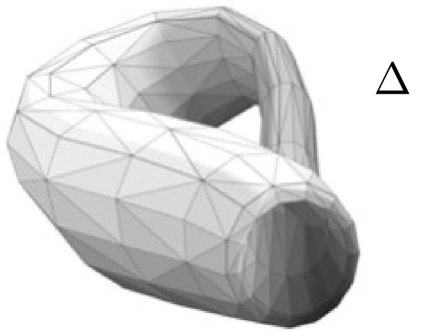
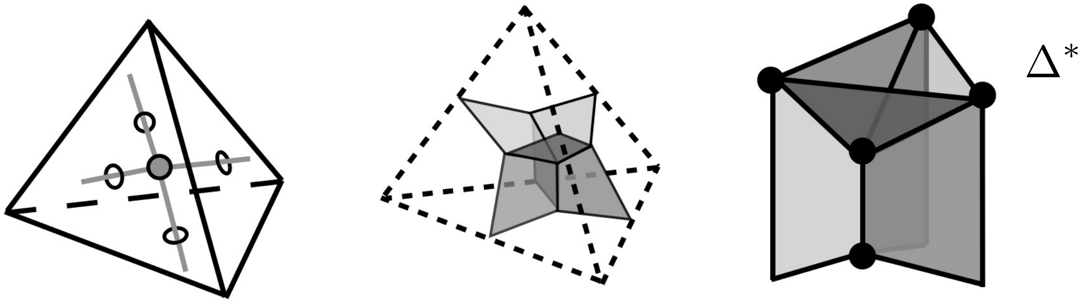
Cellular decomposition of M



Dual 2d cell complex



EPRL/FK/CH spin foam model

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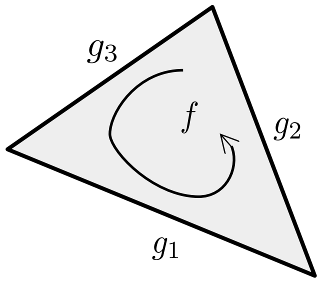

$$Z(M) \sim \int \mathcal{D}A\mathcal{D}B\; e^{i \int_M \mathrm{Tr}\; B \wedge F} \sim \int \mathcal{D}A\; \delta(F)$$

$$Z(\Delta^*) = \int_{\mathrm{SL}(2,\mathbb{C})} \prod_{e \in \mathcal{E}} \mathrm{d}g_e \prod_{f \in \mathcal{F}} \delta\left(\prod_{e \in \partial f} g_e\right)$$

EPRL/FK/CH spin foam model

- GR = BF + constraints

$$S_T = \int_M F^{IJ} \wedge \left(\star + \frac{1}{\gamma}\right) \theta_I \wedge \theta_J$$

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Discretize BF → **Quantize** → Apply constraints at quantum level

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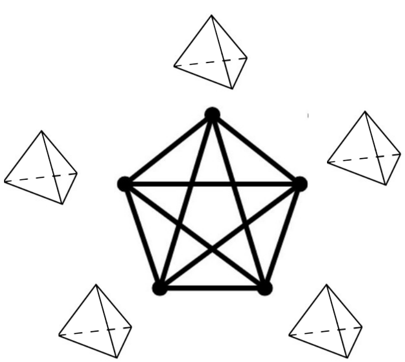
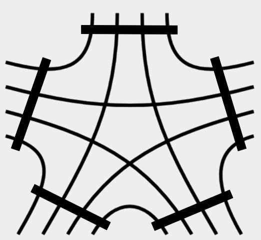
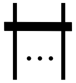


$$Z(\Delta^*) = \sum_{\{\rho\} \to \{f\}} \prod_{f \in \mathcal{F}} \dim(\rho) \prod_{v \in \mathcal{V}}$$



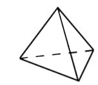
SL(2,C) unitary irreducible reps.

$\Big| = \rho(g)$

 $= \int \mathrm{d}g\, \rho_1(g) \ldots \rho_n(g)$

- The general idea:

Discretize BF → Quantize → **Apply constraints at quantum level**

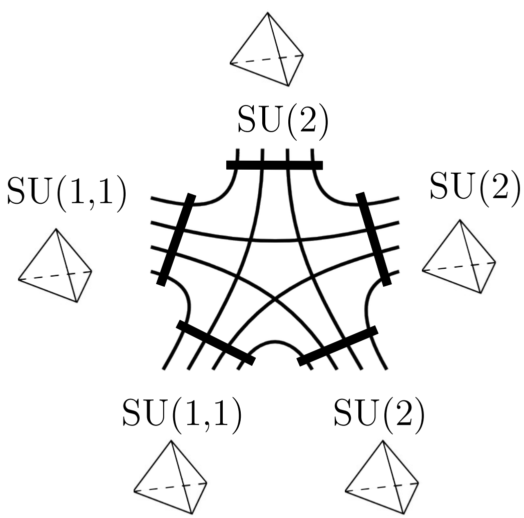
- Discretized constraints require choice of causal character of each ;
- Completeness relations from SU(2) and SU(1,1) unitary irreps.

$$\mathbb{1}_\rho = \sum_{j\,m} |j\,m\rangle \langle j\,m|$$

SU(2), space-like

$$\mathbb{1}_\rho = \sum_{k\,m} |k\,m\rangle \langle k\,m| + \sum_{m} \int \mathrm{d}s\ |s\,m\rangle \langle s\,m|$$

SU(1,1), time-like



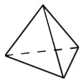
- Constraints single out a set of admissible representations $\rho^{(p,n)}$ depending on causal character

$$n = f(p)$$

EPRL/FK/CH spin foam model

- The general idea:

Discretize BF → Quantize → Apply constraints at quantum level

- Discretized constraints require choice of causal character of each ;
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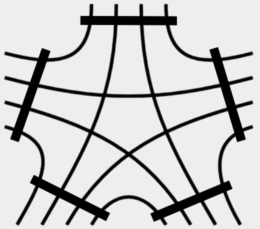
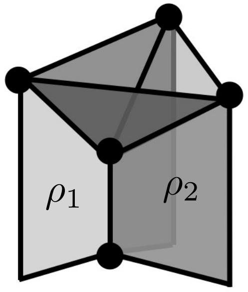
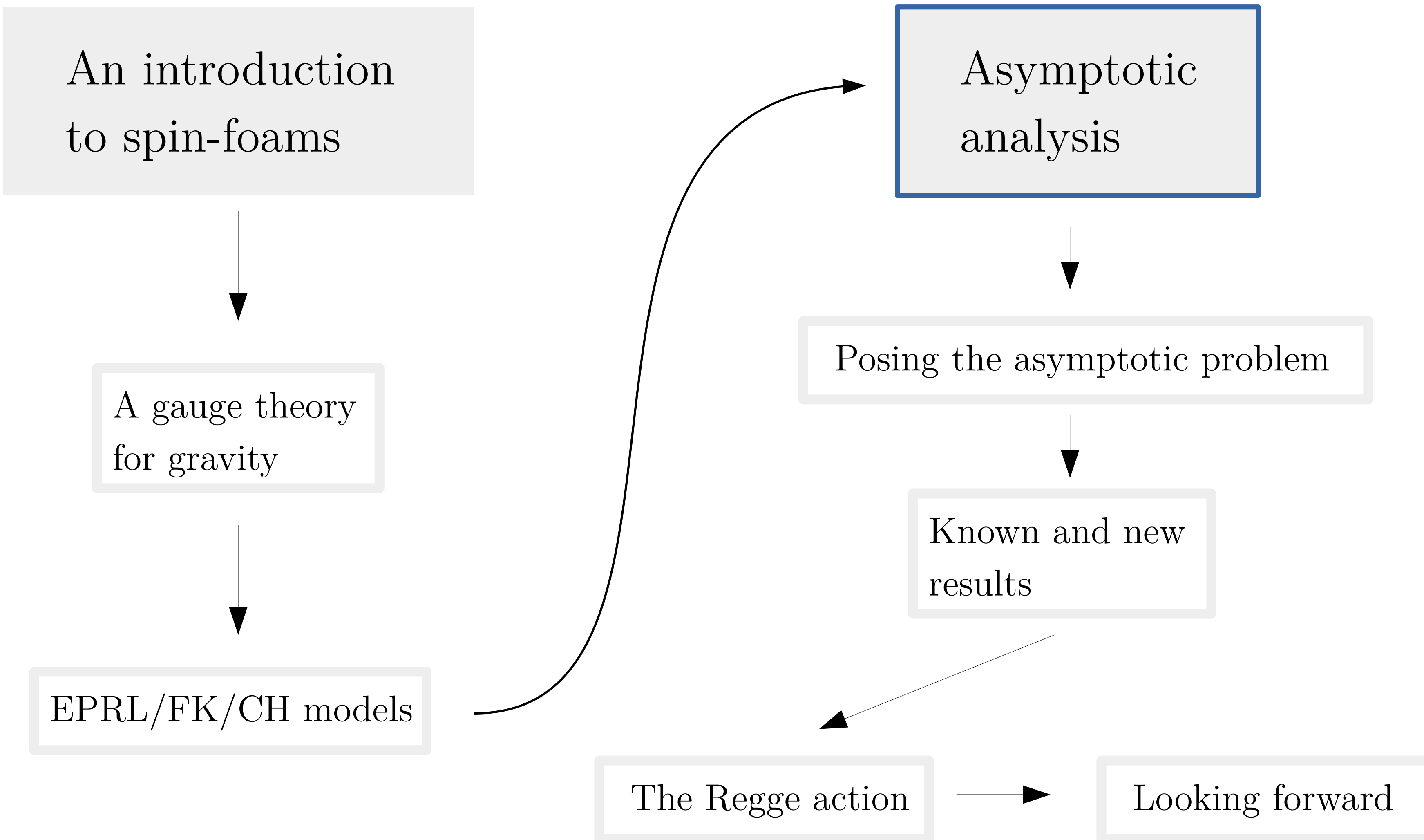
SU(1,1), time-like

$$Z(\Delta^*) = \sum_{\{\rho\}\to\{f\}} \prod_{f\in\mathcal{F}} \dim(\rho) \prod_{v\in\mathcal{V}}$$ 

Restrict to admissible representations



The road-map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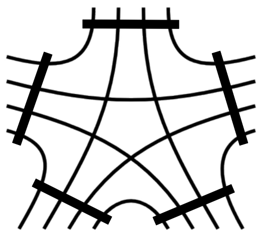
Posing the asymptotic problem

- States of the theory live in spaces of unitary irreps. of SU(2) and SU(1,1), labeled by spins $\{|j\,m\rangle\,,|k,m\rangle\,,|s\,m\rangle\}$;

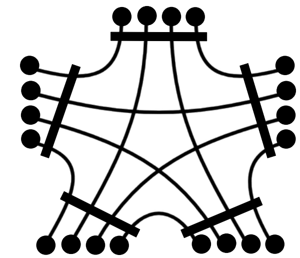
Vertex amplitude

$\prod_{v\in\mathcal{V}}$ 

 $=\rho(g)$

$$\mathbb{1}_\rho=\sum_{j\,m}|j\,m\rangle\langle j\,m|$$

$$\mathbb{1}_\rho=\sum_{k\,m}|k\,m\rangle\langle k\,m|+\sum_{m}\int \mathrm{d}s\;|s\,m\rangle\langle s\,m|$$

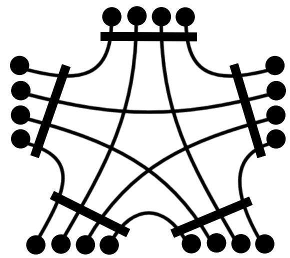


 $=\langle j\,m|\,\rho(g)\,|j'\,m'\rangle$

- What happens at large j, k and s?

- Focus on vertex amplitude, take uniform scaling $\{j,\,k,\,s\}\to\{\Lambda j,\,\Lambda k,\,\Lambda s\}$, and rewrite:

Fix boundary states

 $=\int_{\mathrm{SL}(2,\mathbb{C})}\left[\prod_i \mathrm{d}g_i\right]f(\{g_i\})e^{i\Lambda S(\{g_i\})}$

Look at critical points of action

Known and new results

- Asymptotic analysis for entirely space-like foams done in (Barret et al. '09);

$$S|_{\text{critical}} = S_{\text{Regge}}$$

"cosine problem"

$$A \sim e^{iS_{\text{Regge}}} + e^{-iS_{\text{Regge}}}$$

- Inclusion of time-like tetrahedra, but only space-like triangle interfaces in (Kamiński et al. '17);
- Extension to time-like triangles by (Liu, Han '18);

Unanswered questions in time-like analyses

- Minkowski's theorem and rigidity for Lorentzian polyhedra?
- Extension to non-simplicial cells?
- Really need to assume at least one time-like triangle in each time-like tetrahedron? (Liu, Han '18)
- Explicit asymptotic formula?

- Asymptotic analysis for entirely space-like foams done in (Barret et al. '09);

$$S|_{\text{critical}} = S_{\text{Regge}}$$

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Unanswered questions in time-like analyses

- Minkowski's theorem and rigidity for Lorentzian polyhedra? ✓, **holds**
- Extension to non-simplicial cells? ✓
- Really need to assume at least one time-like triangle in each time-like tetrahedron? ✓, **no!**
- Explicit asymptotic formula? **wip**

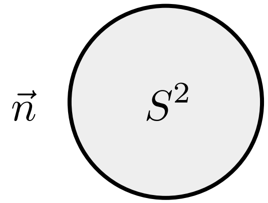
+

No "cosine problem" if 4-cell includes both time-like and space-like cells.

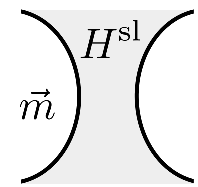
(J.D.S., Steinhaus, soon)

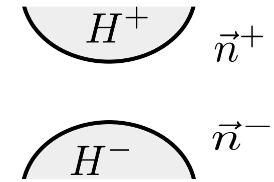
Critical points of the action

- Boundary states can be characterized by a spin and by a vector in S^2, $H^{\pm}$, H^{sl}.



$$\{|j\,\vec{n}\rangle\,,|s\,\vec{m}\rangle\,,|k,\vec{n}^{\tau}\rangle\}$$





- After **a lot** of massaging,

$$\sum_f j_f \vec{n}_f = 0$$

Space-like 3-cells

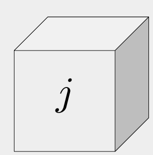
$$\sum_{\mathrm{s.l.}\,f} k_f \vec{n}_f^{\tau} + \sum_{\mathrm{t.l.}\,f} s_f \vec{m}_f = 0\,.$$

Time-like 3-cells

- **Minkowski's theorem**: there exists a unique polyhedron satisfying any given closure condition

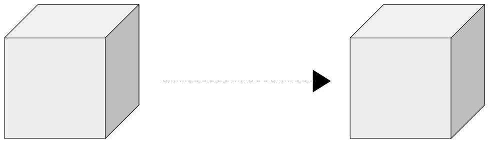
$$\sum_f A_f \vec{n}_f = 0 \qquad \text{(Aleksandrov 1950)}$$

with A_f the areas of faces and $\{n_f\}$ a set of l.i. normals to the faces.

Can reconstruct  with spin areas

Critical points of the action

- One more set of critical point equations, prescribing **gluing rules for faces of polyhedra**,



following combinatorics of cellular decomposition;

- One obtains (under well-behaved data) a 4-dimensional polytope with curvature at the boundary;

- The asymptotic action generally becomes the Regge action associated with the polytope,

$$S \sim \sum_f \theta_f A_f \quad \text{(discrete curvature)}$$

with θ_f the Euclidian/Lorentzian dihedral angle between normals of polyhedra sharing a face.

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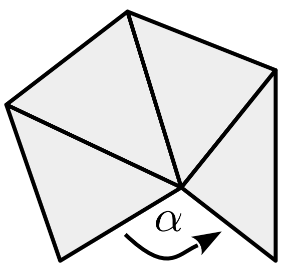
(discrete curvature)

with θ_f the Euclidian/Lorentzian dihedral angle between normals of polyhedra sharing a face.

$$\mathrm{d}s^2 = \mathrm{d}r^2 + (1 - \frac{\alpha}{2\pi})\mathrm{d}\phi^2 - \mathrm{d}t^2 + \mathrm{d}z^2$$

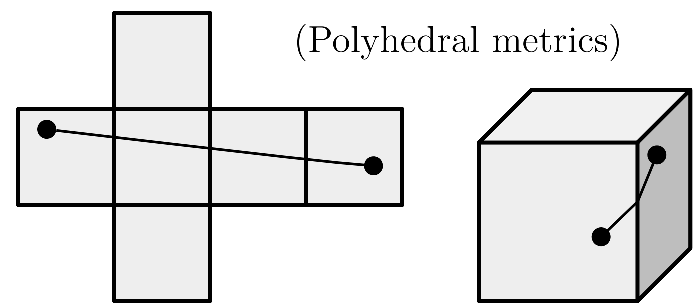


$$\frac{\sqrt{-g}}{2} R = \frac{\alpha \delta(r)}{2\pi}$$

(Sorkin 1974)

$$\frac{1}{2} \int \sqrt{-g} R \, \mathrm{d}r \mathrm{d}\phi \mathrm{d}z \mathrm{d}t = \alpha A$$

(Polyhedral metrics)



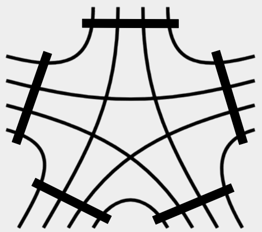
(Aleksandrov 1950)

Synthesis

- The spin-foam frameworks provides a **well-defined notion of a sum-over-histories** for gravity;

- The model is structured around **combinatorics** and **algebraic data** – geometry is *diluted;*

$$Z(\Delta^*) = \sum_{\{\rho\}\to\{f\}} \prod_{f\in\mathcal{F}} \dim(\rho) \prod_{v\in\mathcal{V}}$$



- Triangulated gravity is found in the large-spin regime, where one recovers **geometry**;

Many open questions: inclusion of matter, cellular-decomposition dependence, continuum limit,...

Looking forward

- States associated to time-like 2-cells are generalized eigenstates of a non-compact operator, admitting complex eigenvalues;

  - Possible inclusion of causal ordering between time-like cells?
  - Simpler parametrization of the model?

- Numerical analysis of renormalization flow using hypercuboids;

  - Can one find a phase transition signaling lattice independence?

- Inclusion of matter fields at the foam level;

- Operational meaning of the framework;

- ...?

Thank you for your attention!