Disc of dust: quasi-stationary routes to black holes in Einstein-Maxwell theory

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Physik-Combo, March 22-24, 2021

Outline

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Final product of a gravitational collapse

- Cosmic censorship hypothesis: no naked singularities exist in the universe [Penrose, 1969]
- No-hair theorem: black holes are completely characterized by their mass M, angular momentum J and electric charge Q [Hawking and Ellis, 1973; Wald, 1984]
- Is the final product of a gravitational collapse therefore always described by the Kerr-Newman solution (i.e. a black hole with mass, angular momentum and electric charge)?

Numerical and analytical approaches

- A *dynamical* collapse of rotating bodies can only be tackled with *numerical* methods
- Problem of analytical approaches: the shape of the surface of rotating fluid bodies is not known a priori (no spherical symmetry)
- Exceptions: stationary black holes and discs of dust (galaxies)
- Inverse scattering method: *analytical* tool (from soliton theory) that can be used to generate solutions to the axisymmetric, stationary vacuum Einstein-Maxwell equations $\rightarrow$ investigation of *quasi-stationary* collapse scenarios

[Neugebauer and R. Meinel, 2003]

Outline

Model

Electrically counterpoised dust:
Equilibrium configuration of charged dust (perfect fluid with vanishing pressure) where the gravitational binding energy is compensated by the electromagnetic field energy.

[Reinhard Meinel and Hütten, 2011]

Papapetrou-Majumdar class

Papapetrou-Majumdar class of static solutions to the Einstein-Maxwell equations:

$$\mathrm{d}s^2 = S^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) - S^{-2}\mathrm{d}t^2 \,. \tag{1}$$

Energy-momentum tensor:

$$T_{ik} = \rho u_i u_k + \frac{1}{4\pi}\left(F_{ij}F_k{}^j - \frac{1}{4}F^{mn}F_{mn}g_{ik}\right) , \tag{2}$$

with

$$u^i = \delta^i_4 S \,, \quad A_i = -\delta^4_i \phi \,, \quad \phi = -\epsilon\left(S^{-1} - 1\right) , \quad \epsilon = \pm 1 \,. \tag{3}$$

Einstein-Maxwell equations

For electrically counterpoised dust configurations with

$$J^i = \sigma u^i \,, \quad \sigma = \epsilon \rho \ \text{(charge density)}\,, \tag{4}$$

the Einstein-Maxwell equations

$$R_{ik} - \frac{1}{2} R\, g_{ik} = 8\pi T_{ik} \,, \quad F^{ik}{}_{;k} = 4\pi J^i \,, \tag{5}$$

reduce to

$$\Delta V = 4\pi\mu \,, \tag{6}$$

where

$$S = 1 - V \,, \quad \rho = \frac{\mu}{S^3} \,. \tag{7}$$

Solution of eq. (6) as Poisson integral:

$$V = -\int \frac{\mu(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{8}$$

Asymptotic behaviour:

$$r \equiv |\mathbf{r}| \to \infty: \quad V \to -\frac{M}{r}\,, \quad g_{44} = -S^{-2} \to -\left(1-\frac{2M}{r}\right)\,, \tag{9}$$

where

$$M = \int \mu(\mathbf{r})\,\mathrm{d}^3\mathbf{r} \tag{10}$$

is the gravitational mass.

Black hole limit

The electrically counterpoised dust configuration has finite extent:

$$\mu(\mathbf{r}) = f(\mathbf{r})\,, \tag{11}$$

with

$$f(\mathbf{r}) \equiv 0 \quad \text{for} \quad r > R\,. \tag{12}$$

Consider a one-parameter family of solutions:

$$\mu(\mathbf{r}) = \alpha^3 f(\alpha\mathbf{r})\,, \quad \alpha > 0\,. \tag{13}$$

This implies

$$\mu(\mathbf{r}) \equiv 0 \quad \text{for} \quad r > \frac{R}{\alpha}\,. \tag{14}$$

In the limit $\alpha \to \infty$ the distribution shrinks to $r = 0$ and

$$\mu(\mathbf{r}) = M\delta(\mathbf{r})\,. \tag{15}$$

As a consequence, for $r > 0$

$$V = -\frac{M}{r}\,, \quad S = 1 + \frac{M}{r}\,, \tag{16}$$

and thus

$$\mathrm{d}s^2 = \left(1 + \frac{M}{r}\right)^2 (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) - \left(1 + \frac{M}{r}\right)^{-2} \mathrm{d}t^2\,. \tag{17}$$

The limit leads to the formation of an extreme Reissner-Nordström black hole[1].

[1] Isotropic coordinate are used here $(r_S = r + M)$.

Overview: black holes

Black hole metrics:

	$J = 0$	$J \neq 0$
$Q = 0$	Schwarzschild	Kerr
$Q \neq 0$	Reissner-Nordström	Kerr-Newman

Black holes need to satisfy the following inequality:

$$Q^2 + \frac{J^2}{M^2} \leq M^2 \,. \tag{18}$$

Extreme case:

$$Q^2 + \frac{J^2}{M^2} = M^2 \,. \tag{19}$$

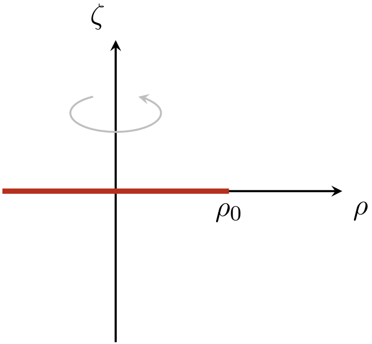
If the inequality is not fulfilled, there would be a naked singularity.

Outline

Model

We consider an infinitely thin disc of dust (pressureless perfect fluid) uniformly rotating around the axis of symmetry with constant angular velocity Ω.

[Reinhard Meinel, 1997; Neugebauer and R. Meinel, 2003; Reinhard Meinel, Ansorg, et al., 2008]



Metric and Symmetries

The metric can be written in terms of Weyl-Lewis-Papapetrou coordinates:

$$\mathrm{d}s^2 = e^{-2U}\left[e^{2k}\left(\mathrm{d}\rho^2 + \mathrm{d}\zeta^2\right) + \rho^2 \mathrm{d}\varphi^2\right] - e^{2U}\left(\mathrm{d}t + a\,\mathrm{d}\varphi\right)^2 \,, \quad (20)$$

where U, k and a depend only on ρ and ζ.
We assume axial symmetry and stationarity, represented by the Killing vectors

$$\eta^i = \delta^i_\varphi \,, \quad \xi^i = \delta^i_t \,, \quad (21)$$

and reflectional symmetry w.r.t. the plane $\zeta = 0$.

Boundary value problem

In case of axial symmetry and stationarity the vacuum Einstein equations reduce to the Ernst equation:

$$\Re f\, \nabla^2 f = (\nabla f)^2 \,, \tag{22}$$

where we introduced the complex Ernst potential

$$f := e^{2U} + \mathrm{i} b \,. \tag{23}$$

The function b is defined by

$$a_{,\rho} = \rho e^{-4U} b_{,\zeta} \,, \quad a_{,\zeta} = -\rho e^{-4U} b_{,\rho} \tag{24}$$

and k can be computed from U and b.

The Einstein equations (using reflectional symmetry) also provide us with the boundary condition:

$$f' = e^{2V_0} = \text{const.} \quad \text{for} \quad \zeta = 0,\ \rho \le \rho_0\,. \qquad (25)$$

The prime denotes the corotating frame with $\varphi' = \varphi - \Omega t$. Additionally, f has to be regular everywhere outside the disc and asymptotically flat:

$$f \to 1 \quad \text{as} \quad \rho^2 + \zeta^2 \to \infty\,. \qquad (26)$$

Equations (22), (25) and (26) form a boundary value problem of the Ernst equation for the rigidly rotating disc of dust.

Linear problem

There is a linear problem for the Ernst equation:

$$
\begin{aligned}
Y_{,z} &= \left\{ \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} + \lambda \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \right\} Y \,, \\
Y_{,\bar{z}} &= \left\{ \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{B} \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & \bar{A} \\ \bar{B} & 0 \end{pmatrix} \right\} Y \,,
\end{aligned}
\tag{27}
$$

where $Y(z, \bar{z}, \lambda)$ is a 2 x 2 matrix, $z = \rho + \mathrm{i}\zeta$, $\bar{z} = \rho - \mathrm{i}\zeta$ are complex coordinates and

$$
\lambda = \sqrt{\frac{K - \mathrm{i}\bar{z}}{K + \mathrm{i}z}} \,. \tag{28}
$$

K is the spectral parameter. $A(z, \bar{z})$ and $B(z, \bar{z})$ do not depend on λ.

If we introduce f via

$$A = \frac{f_{,z}}{f + \bar{f}}\,, \quad B = \frac{\bar{f}_{,z}}{f + \bar{f}}\,, \tag{29}$$

then the integrability condition of the linear problem

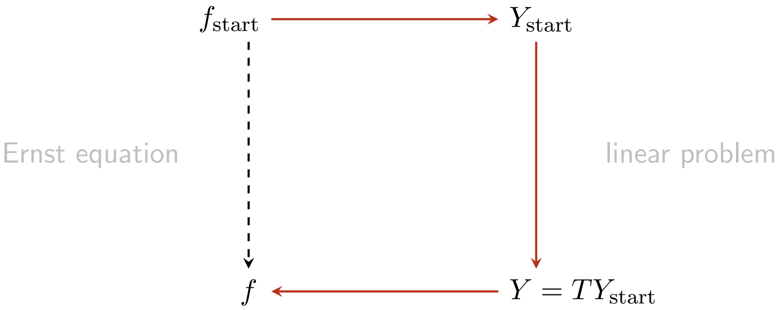
$$Y_{,z\bar{z}} = Y_{,\bar{z}z} \tag{30}$$

is the Ernst equation

$$\Re f \, \nabla^2 f = (\nabla f)^2\,. \tag{31}$$

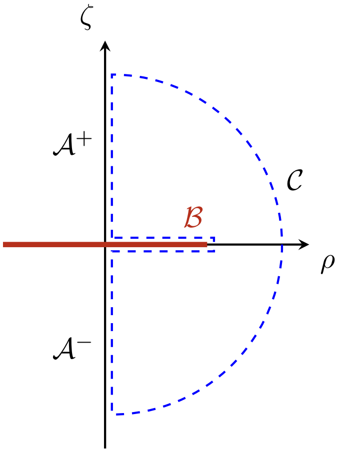
Inverse scattering method

General idea of the inverse scattering method for a given starting solution f_{start} of the Ernst equation: [S. Novikov, 1984]



T: Bäcklund transformation or Riemann-Hilbert problem

Integration of the linear problem



Integration path for the linear problem to obtain $Y(z, \bar{z}, \lambda)$ for $z, \bar{z} \in \mathcal{A}^{\pm}, \mathcal{B}, \mathcal{C}$:

1. $\mathcal{A}^+\mathcal{C}\mathcal{A}^-$
2. $\mathcal{B}$

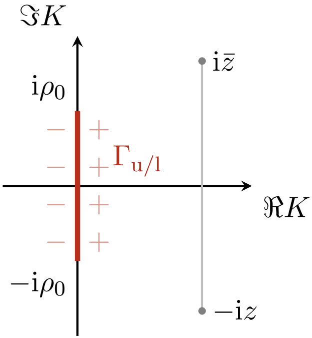
$\mathcal{A}^{\pm}$ denotes the symmetry axis, $\mathcal{B}$ the surface of the disc and $\mathcal{C}$ spatial infinity.

Riemann-Hilbert problem

The linear problem fails at the disc (no vacuum), therefore Y cannot be continuous at the contours $\Gamma_{u/l}: -\rho_0 \le \Im K \le \rho_0$.
Riemann-Hilbert problem for the disc:

$$\begin{aligned} K \in \Gamma_u: &\quad Y_- = Y_+ \mathcal{D}_u(K), \\ K \in \Gamma_l: &\quad Y_- = Y_+ \mathcal{D}_l(K), \\ K \notin \Gamma_{u/l}: &\quad Y \text{ analytic in } K. \end{aligned} \tag{32}$$

u/l denotes the upper/lower sheet of the two-sheeted Riemann K-surface.
($\lambda = \sqrt{\frac{K - i\bar{z}}{K + iz}}$ is double-valued.)



The solution

The solution for the disc of dust can be expressed in terms of hyperelliptic theta functions:

$$f = \frac{\vartheta(\alpha_0 u + \alpha_1 v - C_1, \beta_0 u + \beta_1 v - C_2; p, q, \alpha)}{\vartheta(\alpha_0 u + \alpha_1 v + C_1, \beta_0 u + \beta_1 v + C_2; p, q, \alpha)} e^{-(\gamma_0 u + \gamma_1 v + \mu w)}, \tag{33}$$

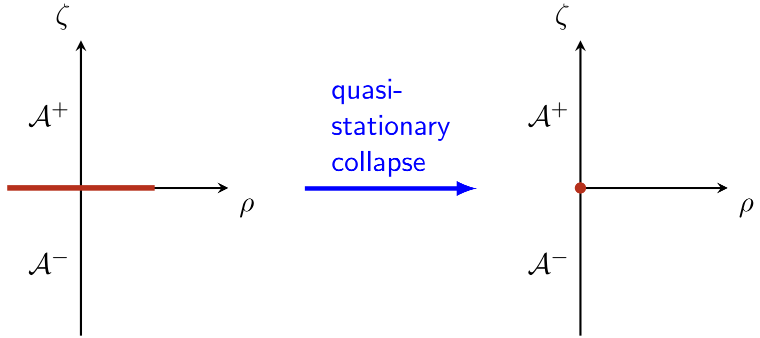
where $\mu := 2\Omega^2 \rho_0^2 e^{-2V_0}$ and the theta function is defined by

$$\vartheta(x, y; p, q, \alpha) := \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} p^{m^2} q^{n^2} e^{2mx+2ny+4mn\alpha}. \tag{34}$$

The definitions of u, v, w, the normalization parameters $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1$, the moduli p, q, α and the quantities C_1, C_2 can be found in Neugebauer and R. Meinel, 2003.

Black Hole Limit

The derived solution reveals that a quasi-stationary collapse of a rigidly rotating disc of dust (on the left) leads to an extreme Kerr black hole (on the right): [Reinhard Meinel, 1997]

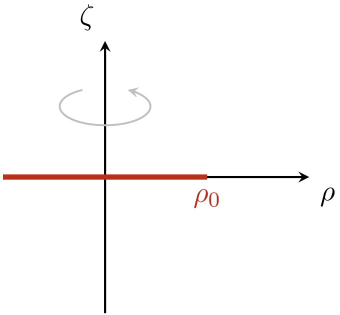


Outline

Model

We consider an infinitely thin disc of dust with constant specific charge $\epsilon \in [-1, 1]$ uniformly rotating around the axis of symmetry with constant angular velocity Ω. The relation between charge density ρ_{el} and mass density μ is given by $\rho_{\mathrm{el}} = \epsilon\mu$.

[Reinhard Meinel, 2012; Palenta and Reinhard Meinel, 2013]



Assumptions:

- axial symmetry
- stationarity
- reflectional symmetry w.r.t. the plane $\zeta = 0$

Boundary value problem

The Einstein-Maxwell (electro-)vacuum equations are equivalent to the Ernst equations due to axial symmetry and stationarity. Therefore, we can formulate the disc problem as a boundary value problem for the Ernst equations:

$$f\Delta\mathcal{E} = \left(\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi\right)\cdot\nabla\mathcal{E}\,, \qquad f\Delta\Phi = \left(\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi\right)\cdot\nabla\Phi\,,$$

$$\text{where} \quad f := \Re\,\mathcal{E} + |\Phi|^2\,, \quad \mathcal{E},\Phi \text{ complex Ernst potentials}\,. \tag{35}$$

Boundary conditions on the disc in the corotating frame:

$$(\sqrt{f'} + \epsilon\,\Re\,\Phi')_{,\rho} = 0,\ (\Re\,\Phi' + \epsilon\sqrt{f'})_{,\zeta} = 0,\ \Im\,\mathcal{E}' = 0,\ \Im\,\Phi' = 0. \tag{36}$$

Asymptotic flatness: $\mathcal{E} \to 1,\ \Phi \to 0$ as $\rho^2 + \zeta^2 \to \infty\,.$

Linear problem

The Ernst equations are the integrability condition of the following linear problem:

$$
\begin{aligned}
Y_{,z} &= \left\{ \begin{pmatrix} b_1 & 0 & c_1 \\ 0 & a_1 & 0 \\ d_1 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & b_1 & 0 \\ a_1 & 0 & -c_1 \\ 0 & d_1 & 0 \end{pmatrix} \right\} Y \,, \\
Y_{,\bar{z}} &= \left\{ \begin{pmatrix} b_2 & 0 & c_2 \\ 0 & a_2 & 0 \\ d_2 & 0 & 0 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & b_2 & 0 \\ a_2 & 0 & -c_2 \\ 0 & d_2 & 0 \end{pmatrix} \right\} Y \,,
\end{aligned}
\tag{37}
$$

with

$$
a_1 = \bar{b}_2 = \frac{\mathcal{E}_{,z} + 2\bar{\Phi}\,\Phi_{,z}}{2f} \,, \quad a_2 = \bar{b}_1 = \frac{\mathcal{E}_{,\bar{z}} + 2\bar{\Phi}\,\Phi_{,\bar{z}}}{2f} \,, \tag{38}
$$

$$
c_1 = f\bar{d}_2 = \Phi_{,z} \,, \quad c_2 = f\bar{d}_1 = \Phi_{,\bar{z}} \,. \tag{39}
$$

Next steps ...

- Applying the inverse scattering method to the linear problem (37)
- Performing an integration in the complex ρ-ζ-plane along a closed path $\mathcal{A}^{+}\mathcal{C}\mathcal{A}^{-}\mathcal{B}$
- **Construction of a Riemann-Hilbert problem (current research)**
- Solving the Riemann-Hilbert problem
- Black hole limit?

Outline

Summary and outlook

- Electrically counterpoised dust configurations admit a parametric transition to the extreme Reissner-Nordström black hole.
- A quasi-stationary collapse of a rigidly rotating disc of dust (without charge) leads to the formation of an extreme Kerr black hole.
- Our conjecture: For the rigidly rotating disc of dust with charge the final product of a quasi-stationary collapse is an extreme Kerr-Newman black hole and not a naked singularity.

[Breithaupt et al., 2015; Pynn et al., 2016]

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