Spacetime singularities and cosmic censorship I

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Outline

- Lecture I: A historical review of singularities in general relativity; the maximally extended Schwarzschild spacetime; the singularity theorem by Penrose (Nobel prize 2020).
- Lecture II: Cosmic censorship; a historical perspective; an initial value formulation; the Einstein-Vlasov system; rigorous black hole formation in the case of Vlasov matter.
- Lecture III: A review of dust collapse; naked singularities in the inhomogeneous case; approximate dust with Vlasov matter with the hope of eliminating the naked singularities.

Dark stars versus black holes

In Newtonian gravity the escape velocity v_e for a particle of mass m, on a surface of a spherically symmetric object with radius r and mass M, is given by

$$\frac{GMm}{r} = \frac{mv_e^2}{2}.$$

This equation follows from energy conservation.

If the radius r is small, then v_e has to be large. In particular, if

$$r := r_S = \frac{2GM}{c^2} \quad (\sim 1\,cm \text{ for our planet}),$$

then the escape velocity $v_e = c$, i.e. the speed of light.

Hence if $r < r_S$ then not even light can escape to infinity. Such a star would be dark. This was discovered by John Michell already in 1783 and later Laplace included the concept in one of his books.

Dark stars versus black holes

As we will see, r_S is the Schwarzschild radius which is obtained by solving the Einstein equations! So, are dark stars black holes? No they are not. No information carrying signals can be transmitted to infinity from a black hole. For a dark star the situation is different.

For a dark star information can in fact be carried to infinity by a chain process. Indeed, if a photon is emitted at the surface of the star at radius r_0, it can reach radius r_1 determined by

$$\frac{-GM}{r_1} = \frac{-GM}{r_0} + \frac{c^2}{2} \text{ or equivelently } \frac{1}{r_1} = \frac{1}{r_0} - \frac{1}{r_S}.$$

If $r_0 < r_S$ the photon does not reach infinity. However, another photon can transfer the information from r_1 to a radius r_2 and after n steps $(n \geq r_S/r_0)$ information has been transmitted to infinity.

General relativity

The classical gravitational theory by Newton cannot easily be modified to capture the theory of special relativity.

In November 1915, Einstein gave a talk at the Prussian Academy of Science where he presented his General theory of relativity, which is a theory for gravity.

- Spacetime is a four dimensional manifold M equipped with a Lorentzian metric $g_{ab},\ a, b = 0, 1, 2, 3$.
- The metric g_{ab} is determined through the Einstein field equations (below $c = G = 1$)

$$G_{ab} := R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi T_{ab}$$

In this talk we take the cosmological constant $\Lambda = 0$.

The left hand side of Einstein's equations

The Einstein equations *look* easy: $G_{ab} = 8\pi T_{ab}$

However, the left hand side of the equations (*"made of marble"*) written out in local coordinates is very complicated since

$$G_{ab} := R_{ab} - \frac{1}{2} g_{ab} R$$

where $R_{ab} = R^{d}_{adb}$, and $R = R^{a}_{a}$, and

$$R^{d}_{abc} = \frac{\partial}{\partial x^b}\Gamma^{d}_{ac} - \frac{\partial}{\partial x^a}\Gamma^{d}_{bc} + \Gamma^{e}_{ac}\Gamma^{d}_{be} - \Gamma^{e}_{bc}\Gamma^{d}_{ae},$$

where

$$\Gamma^{c}_{ab} = \frac{1}{2} g^{cd} \left(\frac{\partial g_{bd}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^d} \right).$$

The right hand side of $G_{ab} = 8\pi T_{ab}$

The right hand side (*"made of wood"*) is the energy momentum tensor of the matter. We have to make a choice of matter model:

- fluid matter models (dust, Euler, Navier-Stokes)
- kinetic matter models (Vlasov or Boltzmann matter)
- elastic matter models
- field theoretical models (a scalar field, a Yang-Mills field,...)

Some models are used by astrophysicists and some are used due to mathematical convenience.

In addition to the system $G_{ab} = 8\pi T_{ab}$, equations for the evolution of the matter must often be included. Later we will look in detail about this point for the Einstein-Vlasov system.

The Schwarzschild solution (1916)

Consider a spherically symmetric object with mass M and radius R. For $r > R$ there is vacuum and the Einstein equations can be solved uniquely and explicitly by the Schwarzschild solution:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (1)$$

Here and below we use geometrical units so that $c = G = 1$, otherwise $2M/r \to 2GM/c^2 r$.

The object could have a time-dependent matter distribution but the external spacetime is anyhow static* and given by (1). This follows from *Birkhoff's theorem*.

A consequence is that a spherical mass distribution cannot emit gravitational waves. An axially symmetric mass distribution can on the other hand emit gravitational waves.

Black holes were disputed

The Schwarzschild solution is a vacuum solution. What happens when matter is included?

Schwarzschild studied a static star of uniform density (an incompressible fluid) in 1916 and concluded that $2M/R < 8/9 < 1$ so that the "singularity" is avoided.

Einstein studied a static star described by a compressible fluid and concluded: "The Schwarzschild singularity does not appear for the reason that matter cannot be concentrated arbitrarily" (Ann. Math. 1939).

Oppenheimer-Snyder studied spherically symmetric gravitational collapse of dust (time dependent problem) in 1939 and concluded that a singular spacetime is obtained and that an event horizon forms.

Kruskal-Szekeres coordinates

The Schwarzschild solution is singular for $r = 2M$ and $r = 0$. The latter is a true singularity in the sense that curvature invariants blow up but this is not the case for $r = 2M$. Spacetime is perfectly well behaved there. It is merely a coordinate singularity.

It took almost 50 years to understand this issue. In 1960 Kruskal and Szekeres independently came up with a construction.

The Kruskal-Szekeres coordinates are defined by replacing t and r in the Schwarzschild coordinates by a new timelike coordinate T and a new spacelike coordinate X by

$$T = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right)$$
$$X = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right),$$

for $r > 2M$ and $-\infty < t < \infty$. The transformation for the region $r < 2M$ is similar.

The maximally extended Schwarzschild metric

It follows that the (maximally extended) Schwarzschild metric takes the form

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}(-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where the range of T and X is given by

$$-\infty < X < \infty$$
$$-\infty < T^2 - X^2 < 1,$$

and where $r > 0$ is implicitly given as the unique solution to the equation

$$T^2 - X^2 = \big(1 - \frac{r}{2M}\big) e^{r/2M}.$$

The event horizon, given by $r = 2M$ in Schwarzschild coordinates, is here given by $T^2 - X^2 = 0$ where the metric is non-singular and well defined.

There are two singularities at $r = 0$, corresponding to $T^2 - X^2 = 1$, one for positive T and one for negative T. The former is the singularity of a black hole and the latter is the singularity of a white hole.

The maximally extended solution is typically divided into four regions where regions I and III are the exterior and the parallel exterior regions respectively, and regions I and IV are the interior black hole region and interior white hole region.

The four regions of the maximally extended solution

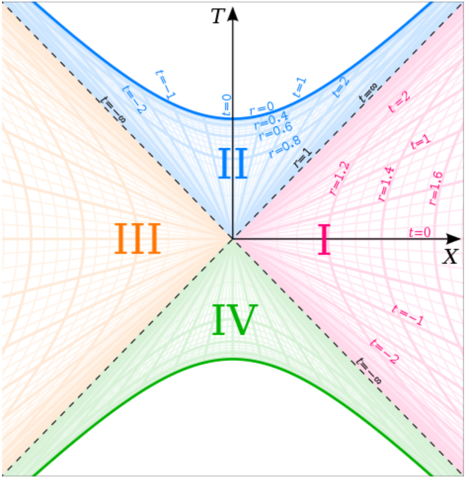


Figure: Here $2M = 1$. The event horizons are the dotted 45° lines and the singularities are the hyperbolas at the top and bottom. (Picture by Dr Greg)

Penrose-Carter diagrams

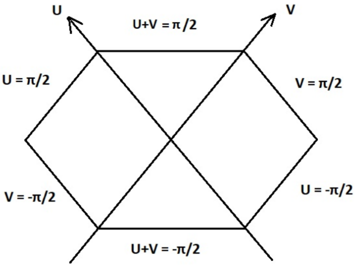
Often so called light-cone coordinates are used instead, i.e.

$$\tilde{U} = T - X$$
$$\tilde{V} = T + X,$$

and moreover it is common to compactify spacetime by the transformation

$$U = \arctan \tilde{U}$$
$$V = \arctan \tilde{V}.$$

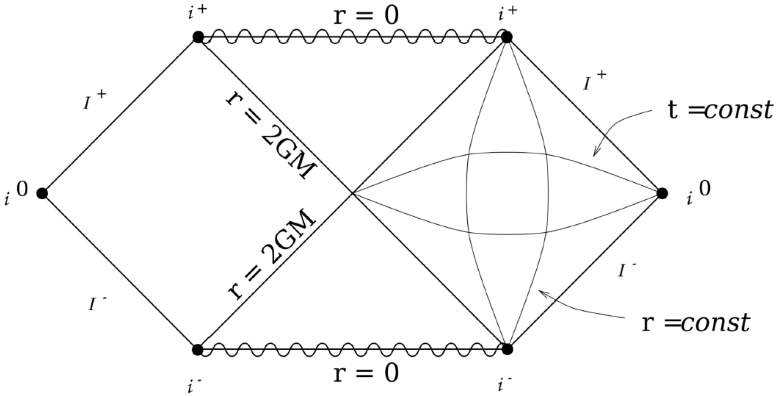
This results in so called Penrose-Carter diagrams which are frequently used to depict different spacetimes.



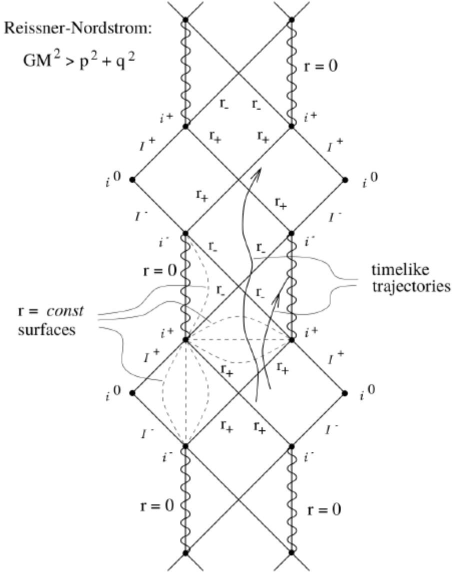
The boundaries have names:

- $U = \pi/2$ and $V = \pi/2$ are called future null infinity, denoted by $\mathcal{I}^+$ ("scri plus"). Analogous for $\mathcal{I}^-$ ("scri minus").
- Spacelike infinity is denoted by i^0 and are the points where $\mathcal{I}^+$ and $\mathcal{I}^-$ meet.
- Future timelike infinity is denoted by i^+ and are the points $(U, V) = (0, \pi/2)$ and $(U, V) = (\pi/2, 0)$. Analogous for i^-.

Penrose-Carter diagram for the Schwarzschild spacetime



Penrose diagram for the Reissner-Nordström spacetime



The Einstein-Rosen bridge/wormhole

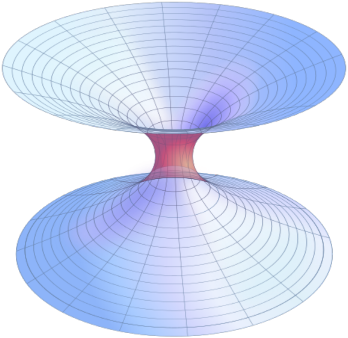


Figure: Embedding diagram for the spacelike surface $t = 0$. (One dimension supressed.)

The Schwarzschild spacetime is static for *external observers* and it is called an eternal black hole spacetime. The picture above indicates that one should be able to travel between the two exterior regions (universes).

Wormholes and exotic matter

However, the black hole region II is dynamic (since $t \to t + \Delta t$ is a spacelike motion and not a timelike motion). When the spacelike surface $t = 0$ is pushed forward in time it enters region II and its geometry changes. A careful analysis shows that no particle can travel between region I and III.

In the literature there are many attempts to obtain a stable wormhole solution. However, these attempts require "exotic matter".

A general comment about this is that if you are allowed to have "any" matter then you can find any solution by simply choose a metric and define the energy momentum tensor by the Einstein equations.

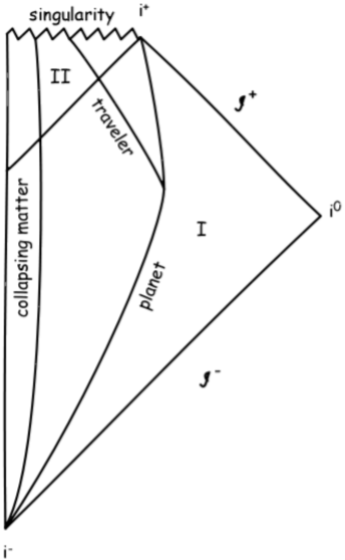
What is realistic in these diagrams?

The wormhole is an example of a property which is likely not realistic in nature.

We believe that black holes have *formed* from a non-singular mass distribution rather than being eternal as the Schwarzschild spacetime.

If we use a Penrose diagram to depict the gravitational collapse of a star then the regions III and IV in fact disappear and only regions I and II remain.

Penrose diagram of gravitational collapse of a star



Nobel prize 2020

What is realistic in the Schwarzschild solution and in the Oppenheimer-Snyder dust collapse?

Could the singularity and the event horizon be a result of the assumption of (perfect) spherical symmetry?

The Nobel prize in physics 2020 was awarded to Roger Penrose for his celebrated singularity theorem from 1965 which (partially) answers this question.

Nobel prize motivation: "For the discovery that black hole formation is a robust prediction of the general theory of relativity".

The Penrose singularity theorem

Penrose shows in his singularity theorem that any spacetime (spherically symmetric or not) which satisfies some rather general conditions contains a *singularity*.

Definition: A spacetime is singular if it possesses at least one incomplete geodesic.

A "not satisfying example": A spacetime with a conical singularity is singular although the Riemann curvature tensor vanishes everywhere away from the singularity.

Penrose wanted to understand the *nature* of the singularity. Is it a black hole or not, i.e., is the singularity clothed by an event horizon and is curvature blowing up at the singularity? This wish led him to propose the cosmic censorship conjecture (1969) that we will discuss later.

Conditions in the singularity theorem

There are several versions of singularity theorems with various assumptions. The Penrose singularity theorem from 1965 requires:

- A condition on the curvature: $R_{ab}k^a k^b \geq 0$ for all null geodesics k^a.
- A causality condition: spacetime is *globally hyperbolic*.
- (Most importantly:) That spacetime contains a trapped surface T.

The first condition follows from the strong energy condition for matter.

A spacetime M is globally hyperbolic if it contains a spacelike surface Σ such that $D(\Sigma) = M$, i.e., Σ is a Cauchy surface. (PDE connection.)

A trapped surface T is a two dimensional spacelike surface with the property that the *expansion* of both sets ("ingoing" and "outgoing") of future directed null geodesics orthogonal to T is everywhere negative.

Formation of a trapped surface

If the assumption about a trapped surface should make sense we have to ask if it is possible to start from initial data without a trapped surface and show that a trapped surface forms in the evolution. Let me mention a few situations where this is the case.

- It is known for dust (1939) and for the spherically symmetric Einstein-Vlasov system (2010). We will get back to this later.
- Demetrios Christodoulou showed formation of a trapped surface for the vacuum Einstein equations in a groundbreaking work (600 pages) from 2009.

Concerning the latter work let me cite Piotr Chrusciel in his Mathematical Reviews: "In spite of the above reservations, most likely originating in this reviewer's ignorance, this is a visionary proof, of terrifying complexity, which opens new avenues in our understanding of mathematics of the Einstein equations."

Christodoulou was awarded the Shaw prize for his result.

A trrapped surface implies a black hole - sometimes

If your Einstein-matter system satisfies some conditions then a black hole will result if there is a trapped surface.

In spherical symmetry the conditions are satisfied for the Einstein-Vlasov system (Dafermos-Rendall 2005). This is the case for the spherically symmetric Einstein-Vlasov system.

However, this is not generally true; if the singularity is not a black hole singularity (i.e., clothed by an event horizon) it is called a *naked singularity*. This leads us to the weak cosmic censorship conjecture.