Conformal Blocks with the Oscillator Construction

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March 22, 2021

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Oscillator Construction for the $\mathfrak{bms}_3$ Algebra

The $\mathfrak{bms}_3$ Algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_{\mathsf{L}}}{12} m(m^2-1)\delta_{m,-n}\,,$$
$$[L_m, M_n] = (m-n)M_{m+n} + \frac{c_{\mathsf{M}}}{12} m(m^2-1)\delta_{m,-n}\,,$$
$$[M_m, M_n] = 0$$

Associated Oscillator Generators:

$$l_0 = \Delta + \sum_{n=1}^{\infty} n\left(v_n^{(1)}\partial_{v_n^{(1)}} + v_n^{(2)}\partial_{v_n^{(2)}}\right),$$
$$l_k = \sum_{n=1}^{\infty} n\left(v_n^{(1)}\partial_{v_{k+n}^{(1)}} + v_n^{(2)}\partial_{v_{k+n}^{(2)}}\right) - \frac{1}{4}\sum_{n=1}^{k-1}\partial_{v_n^{(1)}}\partial_{v_{k-n}^{(2)}} + A_k\partial_{v_k^{(1)}} + B_k\partial_{v_k^{(2)}}\,,$$
$$l_{-k} = \sum_{n=1}^{\infty}(k+n)\left(v_{k+n}^{(1)}\partial_{v_n^{(1)}} + v_{k+n}^{(2)}\partial_{v_n^{(2)}}\right) - 4\sum_{n=1}^{k-1} n(k-n)v_n^{(1)}v_{k-n}^{(2)} + 4k\hat{B}_k v_k^{(1)} + 4k\hat{A}_k v_k^{(2)}\,,$$

$$m_0 = \xi + \sum_{n=1}^{\infty} n v_n^{(1)}\partial_{v_n^{(2)}}\,,$$
$$m_k = \sum_{n=1}^{\infty} n v_n^{(1)}\partial_{v_{k+n}^{(2)}} - \frac{1}{8}\sum_{n=1}^{k-1}\partial_{v_{k-n}^{(2)}}\partial_{v_n^{(2)}} + A_k\partial_{v_k^{(2)}}\,,$$
$$m_{-k} = \sum_{n=1}^{\infty}(k+n)v_{k+n}^{(1)}\partial_{v_n^{(2)}} - 2\sum_{n=1}^{k-1} n(k-n)v_{k-n}^{(1)}v_n^{(1)} + 4k\hat{A}_k v_k^{(1)},$$

with identifications

$$A_k = -\frac{i}{2}\sqrt{2\xi - \frac{c_{\mathsf{M}}}{12}} - k\sqrt{\frac{c_{\mathsf{M}}}{48}}\,, \qquad B_k = i\frac{c_{\mathsf{L}} - 2 - 24\Delta}{48\sqrt{2\xi - \frac{c_{\mathsf{M}}}{12}}} - k\frac{c_{\mathsf{L}} - 2}{48\sqrt{\frac{c_{\mathsf{M}}}{12}}}\,,$$
$$\hat{A}_k = \frac{i}{2}\sqrt{2\xi - \frac{c_{\mathsf{M}}}{12}} - k\sqrt{\frac{c_{\mathsf{M}}}{48}}\,, \qquad \hat{B}_k = -i\frac{c_{\mathsf{L}} - 2 - 24\Delta}{48\sqrt{2\xi - \frac{c_{\mathsf{M}}}{12}}} - k\frac{c_{\mathsf{L}} - 2}{48\sqrt{\frac{c_{\mathsf{M}}}{12}}}\,.$$

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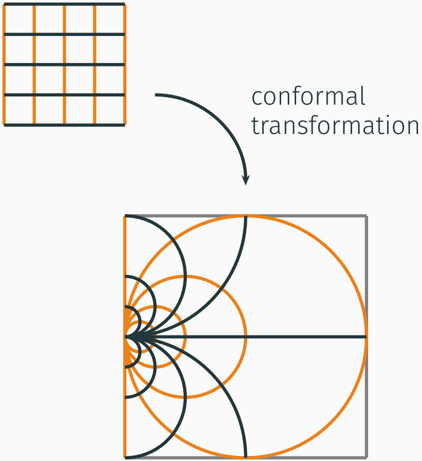
CFT Basics

Conformal Symmetry

diffeomorphisms $x^\mu \mapsto \tilde{x}^\mu(x)$ so that

$$d\tilde{s}^2 = \Omega^2(x) ds^2$$

- angles are preserved
- covariant transformations:
  - Poincaré transformations
  - scale transformations
  - special conformal transformations



CFT in 2 Dimensions: The Algebrae

The Virasoro Algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n-1)(n+1)\,\delta_{m+n,0}$$

- infinite algebra of infinitessimal local transformations with $n, m \in \{\mathbb{Z}\}$
- radial quantisation leads to central charge c

The "Global" Algebra or $\mathfrak{sl}(2,\mathbb{R})$:

$$[L_m, L_n] = (m-n)L_{m+n}$$

- finite algebra of finite global transformations with $n, m \in \{-1, 0, 1\}$
- not depending on the central charge c

CFT Basics: Fields

- Complexification of the two-dimensional plane: $(x^0, x^1) \to (z, \bar{z})$
  (but treat z and $\bar{z}$ independently!)
- Consider fields $\phi(z, \bar{z})$:
  - $\phi(z)$: *holomorphic*
  - $\phi(\bar{z})$: *anti-holomorphic*
- Primary field: obeys transformation rule under transformations $f(z)$ and $\bar{f}(\bar{z})$

$$\phi(z, \bar{z}) \mapsto \tilde{\phi}(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi\left(f(z), \bar{f}(\bar{z})\right)$$

- h and $\bar{h}$: conformal weights

CFT Basics: Two- and Three-Point Function

Using the Conformal Symmetry:

- Two-Point Function:

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}}{(z_1 - z_2)^{h_1+h_2}(\bar{z}_1 - \bar{z}_2)^{\bar{h}_1+\bar{h}_2}}$$

- Three-Point Function:

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle \sim \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{31}^{h_3+h_1-h_2}} \cdot \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{31}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}$$

where $z_{ij} = z_i - z_j$

CFT Basics: Two- and Three-Point Function

Using the Conformal Symmetry:

- Two-Point Function:

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{\delta_{h_1, h_2}\delta_{\bar{h}_1, \bar{h}_2}}{(z_1 - z_2)^{h_1+h_2}(\bar{z}_1 - \bar{z}_2)^{\bar{h}_1+\bar{h}_2}}$$

- Three-Point Function:

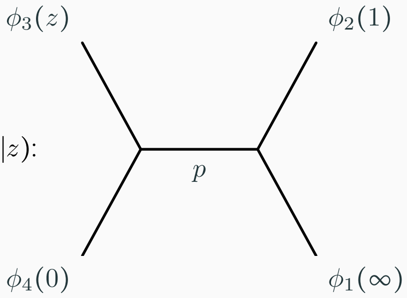
$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle \sim \frac{1}{z_{12}^{h_1+h_2-h_3}\, z_{23}^{h_2+h_3-h_1}\, z_{31}^{h_3+h_1-h_2}} \cdot \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\, \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}\, \bar{z}_{31}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}$$

where $z_{ij} = z_i - z_j$

CFT Basics: Four-Point Function

- more complicated to calculate
- Use Partial Wave Expansion:
  $\langle \phi_1(\infty,\infty)\phi_2(1,1)\phi_3(z,\bar z)\phi_4(0,0)\rangle = \sum_p C^p_{21} C^p_{34} A^{34}_{21}(p|z,\bar z)$
- Factorisation in Conformal Blocks: $A^{34}_{21}(p|z,\bar z) = \mathcal{F}^{34}_{21}(p|z)\,\bar{\mathcal{F}}^{34}_{21}(p|\bar z)$

Illustration of the Conformal Block $\mathcal{F}^{34}_{21}(p|z)$:



Calculating Conformal Blocks

Conformal Blocks: fully fixed by Conformal Symmetry

$\Rightarrow$ unfortunately not easy to calculate

Different Methods known:

- Recursion Relations (Zamolodchikov, 1984)
- Minimal Models ($c < 1$)
- Conformal Bootstrap
- Monodromy Method
- Oscillator Approach

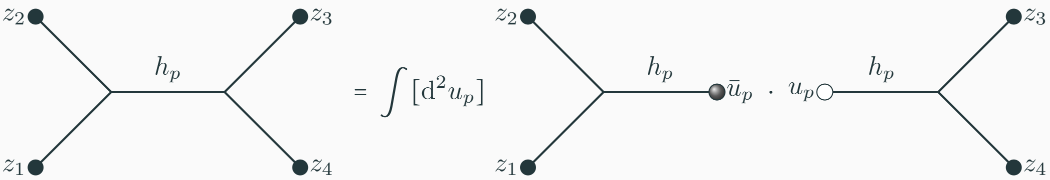
The Oscillator Approach for $\mathfrak{sl}(2, \mathbb{R})$

Introducing the Oscillator Variables

Different way to identify one specific Conformal Block:

Inserting a Projector (for a given representation): $P_{h_p} = \int [\mathrm{d}^2 u_p] \, |\bar{u}_p\rangle\langle u_p|$

$$\left\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)P_{h_p}\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\right\rangle = \int [\mathrm{d}^2 u_p] \, \langle 0|\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\bar{u}_p\rangle \, \langle u_p|\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)|0\rangle$$



$\Rightarrow$ calculate two second-level wavefunctions: $\chi_{h_p}(z_1, z_2, \bar{u}_p)$ and $\psi_{h_p}(z_3, z_4, u_p)$

Oscillator Basics I

The $\mathfrak{sl}(2,\mathbb{R})$ Algebra: $[\ell_m, \ell_n] = (m-n)\ell_{m+n}$ for $n, m \in \{-1, 0, 1\}$

- differential generators:

$$\ell_n^{(h)} = u^{(-n+1)}\partial_u + (1-n)hu^{-n}$$

Highest Weight Representation:

- built from primary state $|h\rangle$ with conformal weight h: $\ell_0 |h\rangle = h |h\rangle$
- h is the "highest weight" (lowest energy eigenvalue)
- lowering operators: $\ell_m |h\rangle = 0$ for $m > 0$
- raising operators create descendants $\ell_{-m} |h\rangle$ for $m > 0$
- Verma module $\mathcal{V}_h$: vector space spanned by primary state and its descendants

Oscillator Basics II

Goal: build unitary representation

- adjoint relation: $\ell_n^\dagger = \ell_{-n}$
- define Hermitian/inner (?) product:

$$(f,g) = \int_{|u|<1} [\mathrm{d}^2u]\overline{f(u)}g(u) = \frac{2h-1}{2\pi}\int_{|u|<1} \frac{\mathrm{d}^2u}{(1-u\bar{u})^{2-2h}}\overline{f(u)}g(u)$$

- general coherent state in Verma module: $\langle u| \equiv |\bar{u}\rangle^\dagger$

Useful orthogonality relations between:

- monomials: $(u^m, u^n) = \frac{n!}{(2h)_n}\delta_{m,n}$
- descendants: $((\ell_{-1})^m \cdot 1, (\ell_{-1})^n \cdot 1) = (2h)_n n! \delta_{m,n}$

Oscillator Basics III

- wavefunctions are Hermitian products:

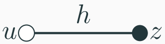
$$\psi(u) = \langle u|\psi\rangle$$

- relation between generators $L_n(z)$ and $\ell_n(u)$:

$$\langle u|L_n|\psi\rangle = \ell_n \psi(u)$$

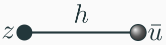
- First-level wavefunction:

$$\psi_h(z,u) = \langle u|\mathcal{O}_h(z)|0\rangle = (1 - zu)^{-2h}$$



- Dual first-level wavefunction:

$$\chi_h(z,\bar{u}) = \langle 0|\mathcal{O}_h(z)|\bar{u}\rangle = (z - \bar{u})^{-2h}$$



The Two-Point Function

Use wavefunctions to calculate two-point function:

$$\langle 0|\mathcal{O}_h(z_1)\mathcal{O}_h(z_2)|0\rangle = \int [\mathrm{d}^2 u]\, \langle 0|\mathcal{O}_h(z_1)|\bar{u}\rangle\, \langle u|\mathcal{O}_h(z_2)|0\rangle$$

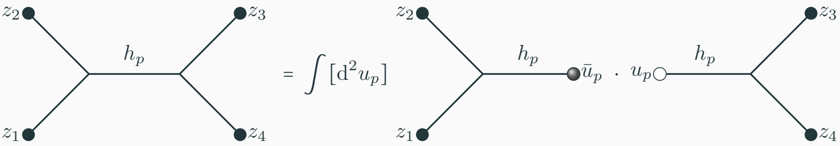
$$= \int [\mathrm{d}^2 u] (z_1 - \bar{u})^{-2h} (1 - z_2 u)^{-2h}$$

use series expansion in monomials and orthogonality relation $(u^m, u^n) = \frac{n!}{(2h)_n}\delta_{m,n}$:

$$= (z_1 - z_2)^{-2h}$$

⇒ known result for two-point function (modulo δ_{h_1,h_2})

The Four-Point Block I

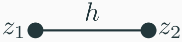


Need Second-Level Wave Functions (depending on oscillator variable):

$$\psi_{h_p}(z_1, z_2, u_p) = \frac{z_{12}^{h_p - h_2 - h_1}}{(1 - z_1 u_p)^{h_p - h_2 + h_1}(1 - z_2 u_p)^{h_p + h_2 - h_1}}$$

$$\chi_{h_p}(z_1, z_2, \bar{u}_p) = \frac{z_{12}^{h_p - h_2 - h_1}}{(\bar{u}_p - z_1)^{h_p - h_2 + h_1}(\bar{u}_p - z_2)^{h_p + h_2 - h_1}}$$

Not to confuse with the Two-Point Function:



The Four-Point Block II

$$\left\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)P_{h_p}\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\right\rangle$$
$$= \int [\mathrm{d}^2 u_p] \frac{z_{12}^{h_p-h_2-h_1}}{(\bar{u}_p - z_1)^{h_p-h_2+h_1}(\bar{u}_p - z_2)^{h_p+h_2-h_1}} \cdot \frac{z_{34}^{h_p-h_4-h_3}}{(1-z_3 u_p)^{h_p-h_4+h_3}(1-z_4 u_p)^{h_p+h_4-h_3}}$$

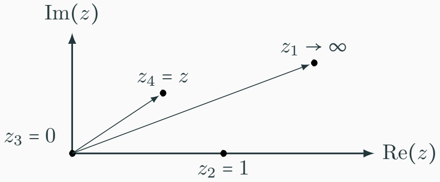
perform limit $z_1 \to \infty$

$$= z^{h_p-h_3-h_4} \int [\mathrm{d}^2 u_p](1-\bar{u}_p)^{h_1-h_2-h_p}(1-z u_p)^{h_3-h_4-h_p}$$

expand factors in monomials and use orthogonality relation

$$= z^{h_p-h_3-h_4} \sum_{k=0}^{\infty} \frac{(h_p-h_1+h_2)_k (h_p-h_3+h_4)_k}{(2h_p)_k k!} z^k$$
$$= z^{h_p-h_3-h_4} {}_2F_1(h_p-h_1+h_2, h_p-h_3+h_4, 2h_p; z)$$

use point configuration:

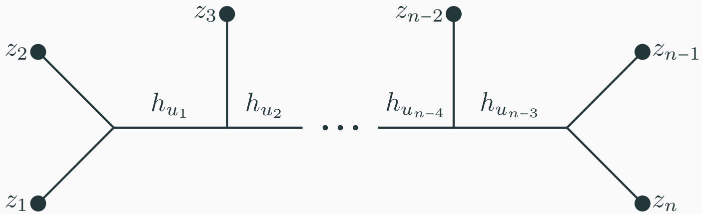


The n-Point Block in the Comb Channel

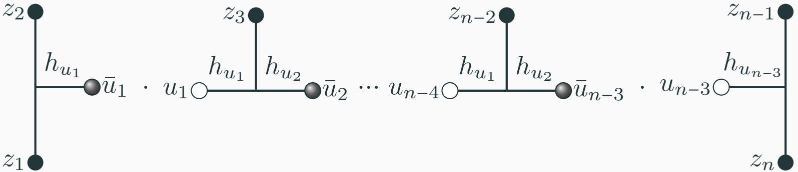
current work by group and mainly Tobias Hössel

$$\langle \mathcal{O}_{h_1}(z_1)\mathcal{O}_{h_2}(z_2)P_{h_{u_1}}\mathcal{O}_{h_3}(z_3)P_{h_{u_2}}\dots\mathcal{O}_{h_{n-2}}(z_{n-2})P_{h_{u_{n-3}}}\mathcal{O}_{h_{n-1}}(z_{n-1})\mathcal{O}_{h_n}(z_n)\rangle$$

$=$



$$= \int [\mathrm{d}^2 u_1] \cdots \int [\mathrm{d}^2 u_{n-3}]$$



Global Block vs. Virasoro Block

"Global" Algebra: $[L_m, L_n] = (m-n)L_{m+n}$ with $n, m \in \{-1, 0, 1\}$

Global Conformal Blocks $\mathcal{F}(h; z)$

- exist for $d \geq 2$
- dominated by hypergeometric functions ${}_2F_1$

Virasoro Conformal Blocks $\mathcal{V}(h, c; z)$

- exist only for $d = 2$
- infinite sum over global blocks
- more information
- more difficult to calculate

Virasoro Algebra: $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n-1)(n+1)\,\delta_{m+n,0}$ with $n, m \in \{\mathbb{Z}\}$

The Oscillator Approach for the Virasoro and the $\mathfrak{bms}_3$ Algebra

The Virasoro Oscillator Construction I

- not one oscillator variable u, but infinitely many u_n (Notation: $U = \{u_n\}$)
- Hermitian product given by:

$$(f(U), g(U)) = \int_{\mathbb{C}^\infty} \prod_{n=1}^{\infty} \mathrm{d}^2 u_n \frac{2n}{\pi} e^{-2n u_n \bar{u}_n} \overline{f(U)} g(U)$$

- Virasoro generators: (with $k > 0$ and $c = 1 + 24\mu^2$ and $h = \lambda^2 + \mu^2$)

$$\ell_0 = h + \sum_{n=1}^{\infty} n u_n \partial_{u_n} \,,$$

$$\ell_k = \sum_{n=1}^{\infty} n u_n \partial_{u_{n+k}} - \frac{1}{4} \sum_{n=1}^{k-1} \partial_{u_n} \partial_{u_{k-n}} + (\mu k + i\lambda) \partial_{u_k} \,,$$

$$\ell_{-k} = \sum_{n=1}^{\infty} (n+k) u_{n+k} \partial_{u_n} - \sum_{n=1}^{k-1} n(k-n) u_n u_{k-n} + 2k(\mu k - i\lambda) u_k \,,$$

The Virasoro Oscillator Construction II

- Wavefunction and dual wavefunction:

$$\psi_h(z,U) = \exp\left(2(\mu - i\lambda)\sum_{n=1}^{\infty} z^n u_n\right) \quad \text{and} \quad \chi_h(z,\bar{U}) = z^{-2h}\exp\left(2(\mu + i\lambda)\sum_{n=1}^{\infty} z^{-n}\bar{u}_n\right)$$

- Second-level wavefunctions: no general closed-form solution known
  - for specific values μ and h_i : result by Zamolodchikov (1985)
  - recursion relations
  - semiclassical limit ($c \to \infty$): saddle-point approximation
- Proof of Exponentiation (Beşken, Datta, Kraus, 2019):

$$\mathcal{V}(h_i, h_p, c; z) \approx \exp\left(-\frac{c}{6} f\left(\frac{h_i}{c}, \frac{h_p}{c}; z\right)\right)$$

- current work: (semiclassical) Virasoro n-point block

The $\mathfrak{bms}_3$ Oscillator Construction

- built from two copies of the Virasoro algebra and a non-relativistic İnönü-Wigner contraction
- assumed relation to asymptotically flat gravity via flat-space holography
- Lie brakets for generators:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_{\mathsf{L}}}{12} m(m^2-1)\delta_{m,-n}\,,$$
$$[L_m, M_n] = (m-n)M_{m+n} + \frac{c_{\mathsf{M}}}{12} m(m^2-1)\delta_{m,-n}\,,$$
$$[M_m, M_n] = 0$$

  L_n related to super-rotations, M_n to super-translations
- primary state $|\Delta, \xi\rangle$:

$$L_0 \,|\Delta, \xi\rangle = \Delta\,|\Delta, \xi\rangle \qquad \text{and} \qquad M_0\,|\Delta, \xi\rangle = \xi\,|\Delta, \xi\rangle$$

- two sets of oscillator generators l_n and m_n

Semiclassical $\mathfrak{bms}_3$ blocks

Semiclassical limit:

$$c_M \to \infty \qquad \text{with} \quad \frac{\Delta}{c_M}, \frac{\Delta_i}{c_M}, \frac{\xi}{c_M}, \frac{\xi_i}{c_M} \text{ and } \frac{c_L}{c_M} \text{fixed}$$

- Proof of Exponentiation for the conformal block
- perturbatively heavy conformal block
- heavy-light conformal block

Thank you for your attention!