Towards exact FRG flows of a UV-interacting scalar field theory

Bootstrapping the Wetterich Equation

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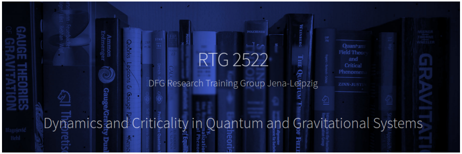


Outline

Quantum Field Theory

Computation Schemes

- Perturbation Theory
- Lattice Simulations
- RG Techniques
- ...

$\Rightarrow$ In general, no exact results and no rigorous error estimates
$\Rightarrow$ Quantum Field Theory is hard

Functional Renormalisation Group

A cutoff dependent functional differential equation

$$\partial_k \Gamma_{k,\Lambda}(\phi) = \frac{1}{2} \mathrm{Tr}_\Lambda \left[(\partial_k R_{k,\Lambda}) \left(\Gamma^{(2)}_{k,\Lambda}\Big|_\phi + R_{k,\Lambda} \right)^{-1} \right] \tag{1}$$

- Λ is a cutoff
- Derivation can be made fully rigorous
  - $\Longrightarrow$ typically on a lattice
  - $\Longrightarrow$ $\Gamma_{k,\Lambda}$ on *function* space possible (WIP)
- Limit $\Lambda \to \infty$ problematic

Functional Renormalisation Group

The Λ-free Wetterich Equation

$$\partial_k \Gamma_k(\phi) = \frac{1}{2} \operatorname{Tr}\left[(\partial_k R_k) \left(\Gamma_k^{(2)}\Big|_\phi + R_k \right)^{-1} \right] \tag{2}$$

Common Treatments

- Local Potential Approximation
- Vertex Expansion
- ...

Assumptions of the Vertex Expansion

Analyticity of Γ_k

$$\Gamma_k(\phi) = \sum_{n=0}^{\infty} \frac{\mathrm{D}^n|_0\, \Gamma_k(\phi^{\otimes n})}{n!} \tag{3}$$

Analyticity of $\phi \mapsto \left(\Gamma_k^{(2)}\Big|_\phi + R_k \right)^{-1}$

$$\left(\Gamma_k^{(2)}\Big|_\phi + R_k \right)^{-1} = \sum_{n=0}^{\infty} \frac{\text{diagrams with n external legs}\,(\phi^{\otimes n})}{n!} \tag{4}$$

The Flow Equations

1PI n-Point Functions

$$\Gamma_k^{(n)}\left(p_1, \ldots, p_n\right) = \kappa_n\left(p_1, \ldots, p_{n-1}\right)\delta\left(p_1 + \ldots + p_n\right) \tag{5}$$

Flow Equation for κ_n

$$\partial_k \kappa_n = \frac{1}{2\left(2\pi\right)^d} \sum_{c \in \mathcal{C}(n)} \left(-1\right)^{\#c} \frac{n!}{c!} \bar{\lambda}_c \tag{6}$$

- $\mathcal{C}\left(n\right)$ - partitions of n including permutations
- $\bar{\lambda}_c$ - integral over symmetrized one-loop diagram indexed by c

Structure of the Flow Equations

$\partial_k \kappa_n = \dots$

- Captures full momentum dependence
- $\bar{\lambda}_c$ depends non-linearly on $\kappa_1, \dots, \kappa_{n+2}, R_k$
- Infinite tower of flow equations coupling all κ_n

Open Questions?

- Can we construct full solutions for all κ_n?
- Which boundary conditions are appropriate?
- Which sets of κ_n correspond to some Γ_k?

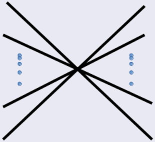
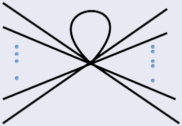
A Bootstrap Method

The Observation

- $\partial_k \kappa_n$ depends *linearly* on κ_{n+2}:

$$\partial_k \kappa_n \left(p_1, \ldots, p_{n-1}\right) = -\frac{1}{2\left(2\pi\right)^d} \int_{\mathbb{R}^d} \frac{\partial_k R_k\left(q\right)}{\left[\kappa_2\left(q\right) + R_k\left(q\right)\right]^2} \kappa_{n+2}\left(p_1, \ldots, p_{n-1}, q, -q\right) \, \mathrm{d}q + \ldots \tag{7}$$

The Linear Part of the Flow Equation

∂_k  $=$  $+ \ldots$

The Linear Structure

Linear Operators I_n

$$(I_n f)\left(p_1, \ldots, p_{n-1}\right) = \int_{\mathbb{R}^d} \frac{\partial_k R_k\left(q\right)}{\left[\kappa_2\left(q\right) + R_k\left(q\right)\right]^2} f(p_1, \ldots, p_{n-1}, q, -q)\, \mathrm{d}q \tag{8}$$

- preserves relevant symmetries
- has a linear right inverse ρ_n such that $I_n \circ \rho_n = \mathrm{id}$
- ρ_n can be constructed to also preserve relevant symmetries

Constructing a Solution

Setting

$$\kappa_{n+2} = \rho_n \left[-2\,(2\pi)^d\,\partial_k \kappa_n + \sum_{c \in \mathcal{C}(n) \setminus \{(n)\}} (-1)^{\#c}\,\frac{n!}{c!}\,\bar{\lambda}_c \right] \tag{9}$$

solves the flow equation!
Note, that the right-hand side depends only on $\kappa_1, \ldots, \kappa_{n+1}, R_k$.

Constructing a Solution (continued)

An Iterative Procedure

1. For some $N \in \mathbb{N}$ find $\kappa_1, ..., \kappa_{N+1}$ satisfying the flow equations for all $n \in \mathbb{N}_{<N}$
2. Find a right inverse ρ_N of I_N
3. Construct κ_{N+2} as on the last slide
4. Increase N by 1 and go back to step 2

$\Rightarrow$ Obtain all κ_n and their full momentum dependences!

A ϕ^4-Like Boundary Condition

A UV-Interacting Limit

- $\lim_{k\to\infty} \kappa_4 = \frac{\lambda}{|m|^{d-4}}$
- $\lim_{k\to\infty} \kappa_2(p) = \|p\|^2 + m^2$
- For all $n \in \mathbb{N} \setminus \{2,4\} : \lim_{k\to\infty} \kappa_n = 0$

Important!

- Physical boundary conditions should diverge in $k \to \infty$ limit

  $\Leftrightarrow$ divergence of coupling constants upon removal of cutoffs

Problematic k-scaling of ρ_n

k-Scaling

- Most commonly $R_k \sim k^2$, hence $\rho_n \sim k^{3-d}$

$\implies$ Great for $k \to \infty$ whenever $d > 3$

$\implies$ Bad for $k \to 0$ whenever $d > 3$

$\implies$ Need for strong control over divergences

The Exponential Regulator

$$R_k\left(q; k\right) = \frac{\left\|q\right\|^2}{\exp\left[\frac{\left\|q\right\|^2}{k^2}\right] - 1} \tag{10}$$

The ϕ^4-Like Ansatz

Make the ansatz (note the strong regularity for $k \to 0$)

$$\kappa_4\left(p, q, r; k\right) = \frac{\lambda}{|m|^{d-4}} \exp\left[- \frac{\|p\|^d + \|q\|^d + \|r\|^d + \|p+q+r\|^d + |m|^d}{k\,|m|^{d-1}} \right] \tag{11}$$

for all $p, q, r \in \mathbb{R}^d$, $k > 0$ and $m \neq 0$. Then,

$$\lim_{k\to\infty} \kappa_4 = \frac{\lambda}{|m|^{d-4}} \qquad \text{and} \qquad \lim_{k\to 0} \kappa_4 = 0 \tag{12}$$

Finding $\kappa_1, \kappa_2, \kappa_3$

The Odd Correlators

- $\kappa_1 = 0$
- $\kappa_3 = 0$

$\Rightarrow$ By linearity of ρ_n all odd correlators vanish

The Case for κ_2

- Flow equation $\partial_k \kappa_2 = \ldots$

  $\Leftarrow$ Can be solved iteratively for $0 \leq \lambda$ not too large

  $\Rightarrow \lim_{k \to \infty} \kappa_2(p) = \|p\|^2 + m^2$

Summary

A Full Solution

- We have $\kappa_1, \kappa_2, \kappa_3$ and κ_4

  $\Rightarrow$ satisfy boundary conditions
- All higher correlators may be constructed through the ρ_n

  $\Rightarrow$ A full solution to the flow equations

  $\Rightarrow$ Works (at least) in $d > 2$

  $\Rightarrow$ All correlators are finite

All proofs may be found in [Phys. Rev. D 103, 025002].

Physical Outlook

Next Steps

- Relation to beta functions of coupling constants

  $\Rightarrow$ Fixed Points?
- Generalisation to fermions
- Generalisation to multiple fields
- What are the kernels of the I_n

  $\Rightarrow$ How much choice is there for ρ_n?
- What are physical ansatzes? (divergences)
- Knowledge of ρ_n enables rigorous error estimates

Mathematical Outlook

Next Steps

- Derive FRGE on function space from first principles (almost done)

  $\Rightarrow$ Appears to require auxillary IR cutoff ϵ

  $\Leftarrow$ Can ϵ be traded for k?

  $\Rightarrow$ Reveal "half" of the divergence structure of couplings
- Take the $\Lambda \to \infty$ limit

  $\Rightarrow$ Reveal the other half (Landau poles, triviality, ...)
- Generalise (multiple fields, fermions)
- Gauge theories (worth 1M USD)

Symmetries of κ_n

For all $\sigma \in \mathrm{Sym}_{n-1}$ and $p_1, \dots, p_{n-1} \in \mathbb{R}^d$

$$\kappa_n\left(p_1, \dots, p_{n-1}\right) = \kappa_n\left(-\left[p_1 + \dots + p_{n-1}\right], p_2, \dots, p_{n-1}\right) \quad (13)$$

$$\kappa_n\left(p_{\sigma(1)}, \dots, p_{\sigma(n-1)}\right) = \kappa_n\left(p_1, \dots, p_{n-1}\right) \quad (14)$$

Isomorphic to Sym_n invariance under action

Let $f\colon \left(\mathbb{R}^d\right)^{n-1} \to \mathbb{R}$, $\sigma \in \mathrm{Sym}_n$, $p_1, \dots, p_{n-1} \in \mathbb{R}^d$.

$$(\sigma f)\left(p_1, \dots, p_{n-1}\right) = f\left(p_{\sigma(1)}, \dots, p_{\sigma(n-1)}\right) , \quad (15)$$

where $p_n := -\left[p_1 + \dots + p_{n-1}\right]$.

Definition of $\bar{\lambda}_n$

For all $p_1, \dots, p_{n-1} \in \mathbb{R}^d$,

$$\bar{\lambda}_c\left(p_1, \dots, p_{n-1}\right) = \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} \lambda_c\left(p_{\sigma(1)}, \dots, p_{\sigma(n-1)}\right) \tag{16}$$

where $p_n := -\left[p_1 + \dots + p_{n-1}\right]$.

Definition of λ_n

For all $c \in \bigcup_{n\in\mathbb{N}} \mathcal{C}(n)$ and all $p_1^1, \dots, p_{c_1}^1, p_1^2, \dots, p_{c_{\#c}-1}^{\#c} \in \mathbb{R}^d$,

$$\lambda_c\left(p_1^1, \dots, p_{c_1}^1, \dots, p_{c_{\#c}-1}^{\#c}\right) = \int_{\mathbb{R}^d} \frac{(\partial_k R_k)(q)}{\left[\kappa_2(q) + R_k(q)\right]^2} \kappa_{2+c_{\#c}}\left(p_1^{\#c}, \dots, p_{c_{\#c}-1}^{\#c}, -\sum_{a=1}^{\#c}\sum_{b=1}^{c_{\#c}-1} p_b^a, q\right) \prod_{l=1}^{\#c-1} \frac{\kappa_{2+c_l}\left(p_1^l, \dots, p_{c_l}^l, q - \sum_{a=1}^{l}\sum_{b=1}^{c_l} p_b^a\right)}{(\kappa_2 + R_k)\left(q - \sum_{a=1}^{l}\sum_{b=1}^{c_l} p_b^a\right)} \mathrm{d}q \tag{17}$$

The Right Inverses ρ_n

For all $n \in \mathbb{N}$ and $p_1, \ldots, p_{n+1} \in \mathbb{R}^d$,

$$
\begin{aligned}
(\rho_n g)(p_1, \ldots, p_{n+1}) = & \sum_{J \subseteq \{0, \ldots, n+1\}} \sum_{l=0}^{\left\lfloor \frac{n-1-\#J}{2} \right\rfloor} \frac{\alpha^n_{\#J,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-\#J-l}} \\
& \times \int_{(\mathbb{R}^d)^{n-1-\#J-l}} g(p_J, -s_1, s_1, \ldots, -s_l, s_l, t_1, \ldots, t_{n-1-\#J-2l}) \\
& \times K(s_1) \ldots K(s_l) K(t_1) \ldots K(t_{n-1-\#J-2l}) \, \mathrm{d}s_{\ldots} \mathrm{d}t_{\ldots} .
\end{aligned}
\tag{18}
$$

The Right Inverses ρ_n (continued)

... with

$$\alpha^n_{a,b} = \frac{(-1)^{n-1-a-b}}{n} 2^{n-1-a-2b} \binom{n-1-a-b}{b}, \qquad (19)$$

and

$$K = \frac{\partial_k R_k}{[\kappa_2 + R_k]^2}. \qquad (20)$$