On the quantum Rényi relative entropies and their use

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Motivation

- **Entropy** and **relative entropy** are at the core of information theory
- **Rényi entropy** and **Rényi relative entropy** are fruitful generalizations of this concept, having many applications
- When generalizing to the quantum case, there is not a unique way to generalize Rényi relative entropy
- In this talk, I'll review three quantum generalizations of Rényi relative entropy and their information-processing properties
- I'll also discuss their applications in quantum information theory

Entropy

- One of the most famous formulas in all of science [Sha48]:

$$H(p_X) := -\sum_x p_X(x) \log_2 p_X(x)$$

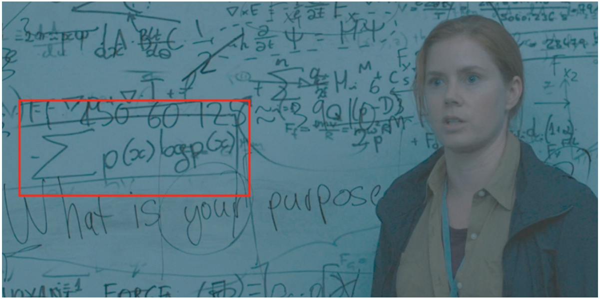


Figure: Screen capture from the movie *Arrival* (2016)

- Equal to the **optimal limit** of classical data compression [Sha48]

Relative entropy

- **Relative entropy** [KL51] generalizes entropy
- Let $p_X \in \mathcal{P}(\mathcal{X})$ be a probability distribution, and let q_X be a measure. Then

$$D(p_X \| q_X) := \sum_x p_X(x) \log_2 \left(\frac{p_X(x)}{q_X(x)} \right)$$

- **Distinguishability measure** with interpretation in hypothesis testing

Special cases of relative entropy

- **Entropy** is a special case of relative entropy:

$$H(p_X) = -D(p_X \| \mathbf{1}),$$

where $\mathbf{1}$ is the vector of all ones.

- **Mutual information** is a special case of relative entropy:

$$I(X;Y) \coloneqq D(p_{XY} \| p_X \otimes p_Y) = \inf_{q_Y \in \mathcal{P}(\mathcal{Y})} D(p_{XY} \| p_X \otimes q_Y)$$

Data-processing inequality

- Let $N_{Y|X}$ be a classical channel and define

$$q_Y := N_{Y|X}(q_X)$$

where

$$q_Y(y) = \sum_{x \in \mathcal{X}} N_{Y|X}(y|x) q_X(x).$$

- Then the **data-processing inequality** for relative entropy is

$$D(p_X \| q_X) \geq D(N_{Y|X}(p_X) \| N_{Y|X}(q_X))$$

Rényi relative entropy

Rényi relative entropy is a generalization of relative entropy with parameter $\alpha \in (0,1) \cup (1,\infty)$ [Rén61]:

$$D_\alpha(p_X \| q_X) := \frac{1}{\alpha - 1} \log_2 \left(\sum_{x \in \mathcal{X}} p_X(x)^\alpha q_X(x)^{1-\alpha} \right)$$

Special cases of Rényi relative entropy

- **Rényi entropy** [Rén61] is a special case:

$$H_\alpha(p_X) := \frac{1}{1-\alpha} \log_2 \left(\sum_{x \in \mathcal{X}} p_X(x)^\alpha \right) = -D_\alpha(p_X \| \mathbf{1})$$

- **Rényi mutual information** [Csi95] defined as

$$I_\alpha(X;Y) := \inf_{q_Y \in \mathcal{P}(\mathcal{Y})} D_\alpha(p_{XY} \| p_X \otimes q_Y)$$

Properties of Rényi relative entropy

- Converges to relative entropy:

$$D(p_X \| q_X) = \lim_{\alpha \to 1} D_\alpha(p_X \| q_X)$$

- **Data-processing inequality** for all $\alpha \in (0,1) \cup (1,\infty)$:

$$D_\alpha(p_X \| q_X) \geq D_\alpha(N_{Y|X}(p_X) \| N_{Y|X}(q_X))$$

- **Additivity**: for probability distributions p_{X_1} and p_{X_2}, measures q_{X_1} and q_{X_2}, and for all $\alpha \in (0,1) \cup (1,\infty)$:

$$D_\alpha(p_{X_1} \otimes p_{X_2} \| q_{X_1} \otimes q_{X_2}) = D_\alpha(p_{X_1} \| q_{X_1}) + D_\alpha(p_{X_2} \| q_{X_2})$$

- **Ordering**: For $\alpha > \beta > 0$

$$D_\alpha(p_X \| q_X) \geq D_\beta(p_X \| q_X)$$

Quantum states

- A quantum state ρ_A is a positive semi-definite, unit trace operator (i.e., Hermitian matrix with all eigenvalues non-negative and summing to one)
- Subscript notation indicates ρ_A is a state of a quantum system A
- Also called **density operator** or **density matrix**
- $\mathcal{D}(\mathcal{H}_A)$ denotes set of density operators acting on a Hilbert space $\mathcal{H}_A$
- Classical probability distributions are a special case in which density operator is **diagonal**

Quantum channels

- A **quantum channel** $\mathcal{N}_{A\to B}$ is a completely positive, trace-preserving map, which evolves system A to system B

- Complete positivity and trace preservation are equivalent to Choi state

$$\Phi^{\mathcal{N}}_{RB} := (\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(\Phi_{RA})$$

being positive semi-definite and $\mathrm{Tr}_B[\Phi^{\mathcal{N}}_{RB}] = \frac{1}{d_R} I_R$, where $R \simeq A$ and maximally entangled state Φ_{RA} is defined as

$$\Phi_{RA} := \frac{1}{d_R} \sum_{i,j} |i\rangle\langle j|_R \otimes |i\rangle\langle j|_A,$$

- A classical channel is a special case in which the Choi state is a diagonal density operator.

Quantum relative entropy

- **Quantum relative entropy** of a state ρ and a positive semi-definite operator σ is defined as [Ume62]

$$D(\rho\|\sigma) := \operatorname{Tr}[\rho(\log_2 \rho - \log_2 \sigma)]$$

- Standard definition with operational meaning [HP91, ON00]

Special cases of quantum relative entropy

- **Quantum entropy** [vN27] is a special case of relative entropy:

$$H(\rho) := -\operatorname{Tr}[\rho \log_2 \rho] = -D(\rho \| I)$$

- **Quantum mutual information** [Str65] is a special case of relative entropy:

$$I(A;B)_\rho := D(\rho_{AB} \| \rho_A \otimes \rho_B) = \inf_{\sigma_B \in \mathcal{D}(\mathcal{H}_B)} D(\rho_{AB} \| \rho_A \otimes \sigma_B)$$

Data-processing inequality for quantum relative entropy

Let ρ be a state, σ a positive semi-definite operator, and $\mathcal{N}$ a quantum channel. Then [Lin75]

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

Petz–Rényi relative entropy

- There are at least two meaningful ways to generalize the classical Rényi relative entropy to the quantum case. Let us begin with the Petz–Rényi relative entropy.

- **Petz–Rényi relative entropy** [Pet86] defined for $\alpha \in (0,1) \cup (1,\infty)$:

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log_2 \operatorname{Tr}[\rho^\alpha \sigma^{1-\alpha}]$$

Properties of Petz–Rényi relative entropy

- Converges to quantum relative entropy in limit $\alpha \to 1$:

$$D(\rho\|\sigma) = \lim_{\alpha\to 1} D_\alpha(\rho\|\sigma)$$

- **Data-processing inequality** [Pet86] for all $\alpha \in (0,1) \cup (1,2]$:

$$D_\alpha(\rho\|\sigma) \geq D_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

- **Additivity**: For states ρ_1 and ρ_2, and positive semi-definite operators σ_1 and σ_2:

$$D_\alpha(\rho_1 \otimes \rho_2\|\sigma_1 \otimes \sigma_2) = D_\alpha(\rho_1\|\sigma_1) + D_\alpha(\rho_2\|\sigma_2)$$

- **Ordering** [MH11]: For $\alpha > \beta > 0$

$$D_\alpha(\rho\|\sigma) \geq D_\beta(\rho\|\sigma)$$

Sandwiched Rényi relative entropy

- Different quantum generalization of classical Rényi relative entropy:
- **Sandwiched Rényi relative entropy** [MLDS+13, WWY14] defined for all $\alpha \in (0,1) \cup (1,\infty)$:

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log_2 \operatorname{Tr}[(\sigma^{(1-\alpha)/2\alpha} \rho \sigma^{(1-\alpha)/2\alpha})^\alpha]$$

Properties of sandwiched Rényi relative entropy

- Converges to quantum relative entropy in limit $\alpha \to 1$ [MLDS$^+$13, WWY14]:

$$D(\rho\|\sigma) = \lim_{\alpha\to 1} \widetilde{D}_\alpha(\rho\|\sigma)$$

- **Data-processing inequality** for all $\alpha \in [1/2, 1) \cup (1, \infty)$ [FL13]:

$$\widetilde{D}_\alpha(\rho\|\sigma) \geq \widetilde{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

- **Additivity**: For states ρ_1 and ρ_2, and positive semi-definite operators σ_1 and σ_2:

$$\widetilde{D}_\alpha(\rho_1 \otimes \rho_2\|\sigma_1 \otimes \sigma_2) = \widetilde{D}_\alpha(\rho_1\|\sigma_1) + \widetilde{D}_\alpha(\rho_2\|\sigma_2)$$

- **Ordering** [MLDS$^+$13]: For $\alpha > \beta > 0$,

$$\widetilde{D}_\alpha(\rho\|\sigma) \geq \widetilde{D}_\beta(\rho\|\sigma)$$

Quantum hypothesis testing [Hel67, Hel69, Hol72, Hol73]

- Quantum system prepared in the state ρ or σ and objective is to figure out which one was prepared
- Make a quantum measurement $\{\Lambda, I - \Lambda\}$ to figure out which was prepared
- Assign outcome Λ to "guess ρ", and the outcome $I - \Lambda$ to "guess σ"
- Probability of committing a **Type I error** ("false alarm"):

$$\mathrm{Tr}[(I - \Lambda)\rho]$$

- Probability of committing a **Type II error** ("missed detection"):

$$\mathrm{Tr}[\Lambda\sigma]$$

Hypothesis testing relative entropy [BD10, BD11, WR12]

- Minimize Type II error probability subject to a constraint on Type I error probability:

$$D_H^{\varepsilon}(\rho\|\sigma) := -\log_2 \inf_{\Lambda\geq 0}\{\mathrm{Tr}[\Lambda\sigma] : \mathrm{Tr}[(I-\Lambda)\rho] \leq \varepsilon,\, \Lambda \leq I\}$$

- More generally can define this quantity when σ is positive semi-definite

- Obeys **data-processing inequality** (operational argument):

$$D_H^{\varepsilon}(\rho\|\sigma) \geq D_H^{\varepsilon}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

Quantum Stein's lemma [HP91, ON00]

- **Optimal achievable rate** for hypothesis testing:

$$E(\rho,\sigma) \coloneqq \inf_{\varepsilon\in(0,1)} \liminf_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n}\|\sigma^{\otimes n})$$

- **Optimal strong converse rate** for hypothesis testing:

$$\widetilde{E}(\rho,\sigma) \coloneqq \sup_{\varepsilon\in(0,1)} \limsup_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n}\|\sigma^{\otimes n})$$

- Always have $E(\rho,\sigma) \le \widetilde{E}(\rho,\sigma)$
- **Quantum Stein's lemma**:

$$E(\rho,\sigma) = \widetilde{E}(\rho,\sigma) = D(\rho\|\sigma)$$

Relating quantum hypothesis testing and relative entropy

- Let ρ be a state and σ a positive semi-definite operator

- **Lower bound** [Hay07, AMV12, QWW18]: For $\varepsilon \in (0,1]$, and $\alpha \in (0,1)$:

$$D_H^{\varepsilon}(\rho\|\sigma) \geq D_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{\varepsilon}\right)$$

- **Upper bound** [CMW16]: For $\varepsilon \in [0,1)$, and $\alpha \in (1,\infty)$:

$$D_H^{\varepsilon}(\rho\|\sigma) \leq \widetilde{D}_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right)$$

Application of lower bound

- Apply lower bound and additivity to find for all $\varepsilon \in (0,1]$ and $\alpha \in (0,1)$ that

$$\frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \| \sigma^{\otimes n}) \geq \frac{1}{n} D_\alpha(\rho^{\otimes n} \| \sigma^{\otimes n}) + \frac{\alpha}{n(\alpha - 1)} \log_2\left(\frac{1}{\varepsilon}\right)$$
$$= D_\alpha(\rho \| \sigma) + \frac{\alpha}{n(\alpha - 1)} \log_2\left(\frac{1}{\varepsilon}\right)$$

- Take $n \to \infty$ limit to find for all $\alpha \in (0,1)$ that

$$\liminf_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \| \sigma^{\otimes n}) \geq D_\alpha(\rho \| \sigma)$$

- Since lower bound holds for all $\alpha \in (0,1)$, conclude that

$$\liminf_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \| \sigma^{\otimes n}) \geq \sup_{\alpha \in (0,1)} D_\alpha(\rho \| \sigma) = D(\rho \| \sigma)$$

Application of upper bound

- Apply upper bound and additivity to find for all $\varepsilon \in [0,1)$ and $\alpha \in (1,\infty)$ that

$$\frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \| \sigma^{\otimes n}) \leq \frac{1}{n} \widetilde{D}_{\alpha}(\rho^{\otimes n} \| \sigma^{\otimes n}) + \frac{\alpha}{n(\alpha-1)} \log_2\left(\frac{1}{1-\varepsilon}\right)$$
$$= \widetilde{D}_{\alpha}(\rho \| \sigma) + \frac{\alpha}{n(\alpha-1)} \log_2\left(\frac{1}{1-\varepsilon}\right)$$

- Take $n \to \infty$ limit to find for all $\alpha \in (1,\infty)$ that

$$\limsup_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \| \sigma^{\otimes n}) \leq \widetilde{D}_{\alpha}(\rho \| \sigma)$$

- Since upper bound holds for all $\alpha \in (1,\infty)$, conclude that

$$\limsup_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \| \sigma^{\otimes n}) \leq \inf_{\alpha \in (1,\infty)} \widetilde{D}_{\alpha}(\rho \| \sigma) = D(\rho \| \sigma)$$

Quantum Stein's lemma

- Combining lower and upper bound gives **quantum Stein's lemma**:

$$E(\rho,\sigma) = \widetilde{E}(\rho,\sigma) = D(\rho\|\sigma)$$

- Actually something slightly stronger: for all $\varepsilon \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n}\|\sigma^{\otimes n}) = D(\rho\|\sigma)$$

- **Operational interpretation** of quantum relative entropy

Generalized divergence

- **Generalized divergence** $\boldsymbol{D}(\rho\|\sigma)$ [PV10, SW12] is a function that satisfies data processing; i.e., for every state ρ, positive semi-definite operator σ, and channel $\mathcal{N}$, the following inequality holds

$$\boldsymbol{D}(\rho\|\sigma) \geq \boldsymbol{D}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

- Examples include relative entropy, Petz– and sandwiched Rényi relative entropies (for certain α values), hypothesis testing relative entropy, etc.

Information measures from generalized divergence

- **Generalized mutual information** of a bipartite state ρ_{AB}:

$$\boldsymbol{I}(A;B)_\rho := \inf_{\sigma_B \in \mathcal{D}(\mathcal{H}_B)} \boldsymbol{D}(\rho_{AB} \| \rho_A \otimes \sigma_B)$$

- **Alt. generalized mutual information** of a bipartite state ρ_{AB}:

$$\overline{\boldsymbol{I}}(A;B)_\rho := \boldsymbol{D}(\rho_{AB} \| \rho_A \otimes \rho_B)$$

Entanglement measures from generalized divergence

- **Generalized divergence of entanglement** of a bipartite state ρ_{AB} [VP98, Das18]:

$$\boldsymbol{E}_R(A;B) := \inf_{\sigma_{AB} \in \text{SEP}(A:B)} \boldsymbol{D}(\rho_{AB} \| \sigma_{AB})$$

- **Generalized Rains divergence** [TWW17] of a bipartite state ρ_{AB}:

$$\boldsymbol{R}(A;B) := \inf_{\sigma_{AB} \in \text{PPT}'(A:B)} \boldsymbol{D}(\rho_{AB} \| \sigma_{AB})$$

$$\text{where } \text{PPT}'(A:B) := \{\sigma_{AB} : \sigma_{AB} \geq 0, \| T_B(\sigma_{AB}) \|_1 \leq 1\}$$

$$T_B(\cdot) := \sum_{i,j} |i\rangle\langle j|_B (\cdot) |i\rangle\langle j|_B$$

- These are **entanglement measures** [HHHH09] because they do not increase under the action of local operations and classical communication on ρ_{AB}

Channel information measures

- Let $\mathcal{N}_{A\to B}$ be a quantum channel.
- We can define channel measures by optimizing state measures [Wil17].
- For example, generalized mutual information of a channel defined by

$$\boldsymbol{I}(\mathcal{N}) := \sup_{\psi_{RA}} \boldsymbol{I}(R;B)_\omega,$$

  where $\omega_{RB} := \mathcal{N}_{A\to B}(\psi_{RA})$
- More generally, if $\boldsymbol{S}(A;B)_\rho$ is one of the state measures given previously, then channel measure is

$$\boldsymbol{S}(\mathcal{N}) := \sup_{\psi_{RA}} \boldsymbol{S}(R;B)_\omega.$$

Entanglement-assisted classical communication [BSST99]

- Suppose Alice and Bob are connected by a quantum channel $\mathcal{N}_{A\to B}$.
- An $(|M|, \varepsilon)$ entanglement-assisted classical comm. code consists of an shared state $\Psi_{A'B'}$, and encoding channel $\mathcal{E}_{M'A'\to A}$, and a decoding measurement channel $\mathcal{D}_{BB'\to\hat{M}}$ such that

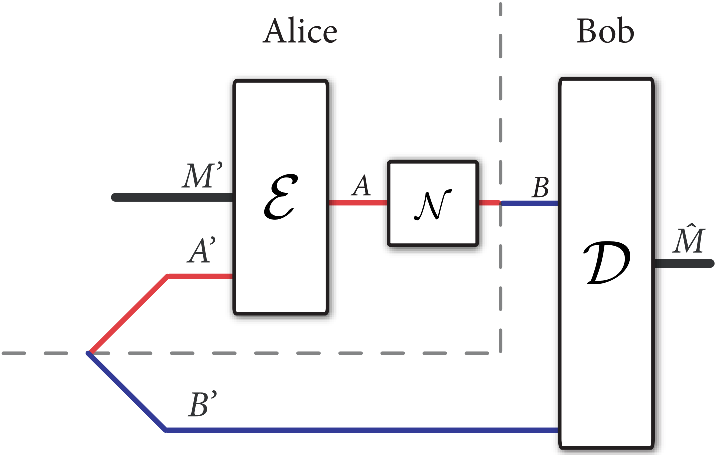
$$\frac{1}{2}\left\|\overline{\Phi}_{M\hat{M}} - (\mathcal{D}_{BB'\to\hat{M}} \circ \mathcal{N}_{A\to B} \circ \mathcal{E}_{M'A'\to A})(\overline{\Phi}_{MM'} \otimes \Psi_{A'B'})\right\|_1 \leq \varepsilon,$$

  where

$$\overline{\Phi}_{M\hat{M}} := \frac{1}{\dim(\mathcal{H}_M)} \sum_m |m\rangle\langle m|_M \otimes |m\rangle\langle m|_{\hat{M}},$$

- $|M|$ = number of messages.
- Note that $\overline{\Phi}_{M\hat{M}}$ represents a classical state, and the goal is for the coding scheme to **preserve the classical correlations** in this state.

Schematic of an entanglement-assisted code



Entanglement-assisted classical capacity

- One-shot entanglement-assisted classical capacity [DH13]:

$$C_{\mathrm{EA}}^{\varepsilon}(\mathcal{N}) := \sup_{\Psi_{A'B'},\mathcal{E},\mathcal{D},M} \{\log_2 |M| : \exists\,(|M|,\varepsilon) \text{ EA code for } \mathcal{N}\}$$

- Entanglement-assisted capacity:

$$C_{\mathrm{EA}}(\mathcal{N}) := \inf_{\varepsilon\in(0,1)} \liminf_{n\to\infty} \frac{1}{n} C_{\mathrm{EA}}^{\varepsilon}(\mathcal{N}^{\otimes n})$$

- Strong converse entanglement-assisted capacity:

$$\widetilde{C}_{\mathrm{EA}}(\mathcal{N}) := \sup_{\varepsilon\in(0,1)} \limsup_{n\to\infty} \frac{1}{n} C_{\mathrm{EA}}^{\varepsilon}(\mathcal{N}^{\otimes n})$$

- Always have $C_{\mathrm{EA}}(\mathcal{N}) \leq \widetilde{C}_{\mathrm{EA}}(\mathcal{N})$

Bounds on one-shot EA capacity

- Using methods called **position-based coding** [AJW19] and **sequential decoding** [GLM12, Sen11, OMW19], we find a lower bound on one-shot EA capacity, holding for $\eta \in (0, \varepsilon)$:

$$\bar{I}_H^{\varepsilon-\eta}(\mathcal{N}) - \log_2\left(\frac{4\varepsilon}{\eta^2}\right) \leq C_{\text{EA}}^{\varepsilon}(\mathcal{N})$$

- By relating EA communication task to hypothesis testing, we obtain an upper bound on one-shot EA capacity [MW14]:

$$C_{\text{EA}}^{\varepsilon}(\mathcal{N}) \leq I_H^{\varepsilon}(\mathcal{N})$$

Lower bound on EA capacity

- Lower bound on one-shot EA capacity implies lower bound on EA capacity
- Now pick $\eta = \varepsilon/2$ and, for $\alpha \in (0,1)$, apply lower bound for hypothesis testing relative entropy from before:

$$
\begin{aligned}
\frac{1}{n} C_{\text{EA}}^{\varepsilon}(\mathcal{N}^{\otimes n}) &\geq \frac{1}{n} \bar{I}_{H}^{\frac{\varepsilon}{2}}(\mathcal{N}^{\otimes n}) - \frac{1}{n} \log_2 \left(\frac{16}{\varepsilon} \right) \\
&\geq \frac{1}{n} \bar{I}_{\alpha}(\mathcal{N}^{\otimes n}) - \frac{\alpha}{n(1-\alpha)} \log_2 \left(\frac{2}{\varepsilon} \right) - \frac{1}{n} \log_2 \left(\frac{16}{\varepsilon} \right) \\
&\geq \bar{I}_{\alpha}(\mathcal{N}) - \frac{\alpha}{n(1-\alpha)} \log_2 \left(\frac{2}{\varepsilon} \right) - \frac{1}{n} \log_2 \left(\frac{16}{\varepsilon} \right)
\end{aligned}
$$

Lower bound on EA capacity (ctd.)

- Take $n \to \infty$ limit to find for all $\alpha \in (0,1)$ that

$$\liminf_{n\to\infty} \frac{1}{n} C_{\text{EA}}^{\varepsilon}(\mathcal{N}^{\otimes n}) \geq \bar{I}_\alpha(\mathcal{N})$$

- Since it holds for all $\alpha \in (0,1)$, we conclude that

$$\liminf_{n\to\infty} \frac{1}{n} C_{\text{EA}}^{\varepsilon}(\mathcal{N}^{\otimes n}) \geq \sup_{\alpha\in(0,1)} \bar{I}_\alpha(\mathcal{N}) = I(\mathcal{N})$$

- and thus

$$C_{\text{EA}}(\mathcal{N}) \geq I(\mathcal{N})$$

Upper bound on EA capacity

- Upper bound on one-shot EA capacity implies upper bound on EA capacity
- Apply upper bound for hypothesis testing relative entropy from before for $\alpha \in (1, \infty)$ and **additivity** of sandwiched Rényi channel mutual information [DJKR06, GW15]:

$$\begin{aligned}\frac{1}{n} C^{\varepsilon}_{\mathrm{EA}}(\mathcal{N}^{\otimes n}) &\leq \frac{1}{n} I^{\varepsilon}_{H}(\mathcal{N}^{\otimes n}) \\ &\leq \frac{1}{n} \widetilde{I}_{\alpha}(\mathcal{N}^{\otimes n}) + \frac{\alpha}{n(\alpha - 1)} \log_2\left(\frac{1}{1-\varepsilon}\right) \\ &= \widetilde{I}_{\alpha}(\mathcal{N}) + \frac{\alpha}{n(\alpha - 1)} \log_2\left(\frac{1}{1-\varepsilon}\right)\end{aligned}$$

Upper bound on EA capacity (ctd.)

- Take $n \to \infty$ limit to find for all $\alpha \in (1, \infty)$ that

$$\limsup_{n\to\infty} \frac{1}{n} C_{\text{EA}}^{\varepsilon}(\mathcal{N}^{\otimes n}) \leq \widetilde{I}_{\alpha}(\mathcal{N})$$

- Since it holds for all $\alpha \in (1, \infty)$, we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} C_{\text{EA}}^{\varepsilon}(\mathcal{N}^{\otimes n}) \leq \inf_{\alpha\in(1,\infty)} \widetilde{I}_{\alpha}(\mathcal{N}) = I(\mathcal{N})$$

- and thus

$$\widetilde{C}_{\text{EA}}(\mathcal{N}) \leq I(\mathcal{N})$$

Entanglement-assisted capacity theorem

- Combining lower and upper bounds:

$$C_{\mathsf{EA}}(\mathcal{N}) = \widetilde{C}_{\mathsf{EA}}(\mathcal{N}) = I(\mathcal{N})$$

- **Operational meaning** for mutual information of a quantum channel as entanglement-assisted classical capacity [BSST02, BCR11, BDH^{+}14]

Geometric Rényi relative entropy

- **Geometric Rényi relative entropy** [PR98, Mat13] is a generalization of classical Rényi relative entropy that is useful for bounding feedback-assisted capacities [FF19].

- For ρ a state, σ a positive semi-definite operator, and $\alpha \in (0,1) \cup (1,\infty)$, **geometric Rényi relative entropy** defined as

$$\widehat{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log_2 \operatorname{Tr}[\sigma(\sigma^{-1/2}\rho\sigma^{-1/2})^\alpha]$$

Namesake for geometric Rényi relative entropy

- Called geometric Rényi relative entropy because it can be written in terms of **weighted operator geometric mean** [LL01]:

$$G_\alpha(\sigma, \rho) := \sigma^{1/2}(\sigma^{-1/2}\rho\sigma^{-1/2})^\alpha\sigma^{1/2}$$

- so that

$$\widehat{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log_2 \mathrm{Tr}[G_\alpha(\sigma, \rho)]$$

Properties of geometric Rényi relative entropy

- Converges to **Belavkin–Staszewski relative entropy** [BS82] in limit $\alpha \to 1$:

$$\widehat{D}(\rho\|\sigma) = \lim_{\alpha\to 1} \widehat{D}_\alpha(\rho\|\sigma)$$

where $\widehat{D}(\rho\|\sigma) := \text{Tr}[\rho \log_2 \rho^{1/2}\sigma^{-1}\rho^{1/2}]$

- **Data-processing inequality** [PR98, Mat13] for $\alpha \in (0,1) \cup (1,2]$:

$$\widehat{D}_\alpha(\rho\|\sigma) \geq \widehat{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

- **Additivity**: For states ρ_1 and ρ_2, and positive semi-definite operators σ_1 and σ_2:

$$\widehat{D}_\alpha(\rho_1 \otimes \rho_2\|\sigma_1 \otimes \sigma_2) = \widehat{D}_\alpha(\rho_1\|\sigma_1) + \widehat{D}_\alpha(\rho_2\|\sigma_2)$$

- **Ordering** [KW20]: For $\alpha > \beta > 0$

$$\widehat{D}_\alpha(\rho\|\sigma) \geq \widehat{D}_\beta(\rho\|\sigma)$$

Geometric Rényi channel divergence [LKDW18, FF19, KW20]

Geometric Rényi channel divergence: Distinguishability measure for quantum channel $\mathcal{N}_{A\to B}$ and a completely positive map $\mathcal{M}_{A\to B}$, for $\alpha \in (0,1) \cup (1,2]$:

$$\widehat{D}_\alpha(\mathcal{N}\|\mathcal{M}) := \sup_{\psi_{RA}} \widehat{D}(\mathcal{N}_{A\to B}(\psi_{RA})\|\mathcal{M}_{A\to B}(\psi_{RA}))$$

Subadditivity of geometric Rényi channel divergence

- Key property: **subadditivity** with respect to serial composition [FF19].

- For channels $\mathcal{N}^1_{A\to B}$ and $\mathcal{N}^2_{B\to C}$, completely positive maps $\mathcal{M}^1_{A\to B}$ and $\mathcal{M}^2_{B\to C}$, and $\alpha \in (0,1) \cup (1,2]$:

$$\widehat{D}_\alpha(\mathcal{N}^2 \circ \mathcal{N}^1 \| \mathcal{M}^2 \circ \mathcal{M}^1) \leq \widehat{D}_\alpha(\mathcal{N}^1 \| \mathcal{M}^1) + \widehat{D}_\alpha(\mathcal{N}^2 \| \mathcal{M}^2)$$

Geometric Rains entanglement [BDWW19, FF19]

- Let $\mathcal{N}_{AB\to A'B'}$ be a bipartite quantum channel.

- **Geometric Rains entanglement** of $\mathcal{N}_{AB\to A'B'}$ defined as

$$\widehat{R}_\alpha(\mathcal{N}) := \inf_{\mathcal{M}:E_N(\mathcal{M})\leq 0} \widehat{D}_\alpha(\mathcal{N}\|\mathcal{M})$$

  where **logarithmic negativity** of $\mathcal{M}_{AB\to A'B'}$ defined as

$$E_N(\mathcal{M}) := \log_2 \| T'_B \circ \mathcal{M}_{AB\to A'B'} \circ T_B\|_\diamond$$

- **Diamond norm** of a Hermiticity-preserving map $\mathcal{P}_{C\to D}$ defined as

$$\|\mathcal{P}\|_\diamond := \sup_{\psi_{RC}} \|\mathcal{P}_{C\to D}(\psi_{RC})\|_1$$

  where ψ_{RC} is a pure state with $R \simeq C$

Special cases of geometric Rains entanglement

- Can be evaluated for a bipartite state $\rho_{A'B'}$, which is a bipartite channel with AB inputs trivial
- Can be evaluated for a point-to-point channel $\mathcal{N}_{A\to B'}$, which is a bipartite channel with input B and output A' trivial

Subadditivity of geometric Rains entanglement [FF19]

- Important property of logarithmic negativity: **subadditivity** with respect to serial composition.
- For completely positive maps $\mathcal{M}^1_{AB\to A'B'}$ and $\mathcal{M}^2_{A'B'\to A''B''}$:

$$E_N(\mathcal{M}^2 \circ \mathcal{M}^1) \leq E_N(\mathcal{M}^1) + E_N(\mathcal{M}^2)$$

- This and subadditivity of geometric Rényi channel divergence imply subadditivity for geometric Rényi entanglement: For bipartite channels $\mathcal{N}^1_{AB\to A'B'}$ and $\mathcal{N}^2_{A'B'\to A''B''}$ and $\alpha \in (0,1) \cup (1,2]$:

$$\widehat{R}_\alpha(\mathcal{N}^2 \circ \mathcal{N}^1) \leq \widehat{R}_\alpha(\mathcal{N}^1) + \widehat{R}_\alpha(\mathcal{N}^2)$$

LOCC channel

- In the theory of entanglement and quantum communication, one often assumes that Alice and Bob can communicate classical data for free.
- Paradigm is local op.'s and classical comm. (LOCC) [BDSW96].
- A **one-way LOCC channel** from Alice to Bob consists of Alice performing a quantum instrument, sending classical outcome to Bob, who performs a quantum channel conditioned on the classical data.
- An **LOCC channel** consists of finite, but arbitrarily large number of 1-way LOCC channels from Alice to Bob and then from Bob to Alice.

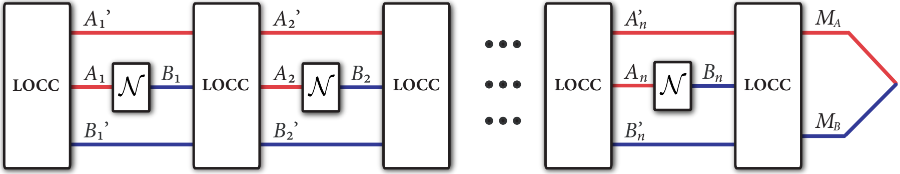
LOCC channel (ctd.)

An **LOCC channel** can be written as a separable channel $\mathcal{L}_{AB\to A'B'}$:

$$\mathcal{L}_{AB\to A'B'}(\rho_{AB}) = \sum_z (\mathcal{E}^z_{A\to A'} \otimes \mathcal{F}^z_{B\to B'})(\rho_{AB}),$$

where $\{\mathcal{E}^z_{A\to A'}\}_z$ and $\{\mathcal{F}^z_{B\to B'}\}_z$ are sets of completely positive, trace non-increasing maps, such that $\mathcal{L}_{AB\to A'B'}$ is a completely positive, trace-preserving map (quantum channel).

LOCC-assisted quantum communication



- An (n, M, ε) protocol for LOCC-assisted quantum communication over the quantum channel $\mathcal{N}$ calls the channel n times.
- In between every channel use, Alice and Bob are allowed to perform an LOCC channel for free.
- The final state $\omega_{M_A M_B}$ should have fidelity larger than $1 - \varepsilon$ with a maximally entangled state $\Phi_{M_A M_B}$ of Schmidt rank M:

$$\langle \Phi |_{M_A M_B} \omega_{M_A M_B} | \Phi \rangle_{M_A M_B} \geq 1 - \varepsilon.$$

LOCC assisted quantum capacity [BDSW96, TGW14]

- n-shot LOCC assisted quantum capacity:

$$Q^{n,\varepsilon}_{\leftrightarrow}(\mathcal{N}) := \sup_{\text{LOCC protocols}} \{\log_2 M : \exists\, (n, M, \varepsilon) \text{ protocol for } \mathcal{N}\}$$

- LOCC-assisted quantum capacity of $\mathcal{N}$:

$$Q_{\leftrightarrow}(\mathcal{N}) := \inf_{\varepsilon \in (0,1)} \liminf_{n \to \infty} \frac{1}{n} Q^{n,\varepsilon}_{\leftrightarrow}(\mathcal{N})$$

- Strong converse LOCC-assisted quantum capacity of $\mathcal{N}$ defined as

$$\widetilde{Q}_{\leftrightarrow}(\mathcal{N}) := \sup_{\varepsilon \in (0,1)} \limsup_{n \to \infty} \frac{1}{n} Q^{n,\varepsilon}_{\leftrightarrow}(\mathcal{N})$$

- Always have $Q_{\leftrightarrow}(\mathcal{N}) \leq \widetilde{Q}_{\leftrightarrow}(\mathcal{N})$

Upper bound on n-shot LOCC-a. quantum capacity [FF19]

- For final state $\omega_{M_A M_B}$, can show for all $\varepsilon \in [0,1)$ and $\alpha \in (1,\infty)$ that

$$\begin{aligned}\log_2 M &\leq R_H^{\varepsilon}(M_A; M_B)_\omega \\ &\leq \widehat{R}_\alpha(M_A; M_B)_\omega + \frac{\alpha}{\alpha - 1} \log_2\left(\frac{1}{1-\varepsilon}\right) \\ &\leq n\widehat{R}_\alpha(\mathcal{N}) + \frac{\alpha}{\alpha - 1} \log_2\left(\frac{1}{1-\varepsilon}\right)\end{aligned}$$

- It then follows for all $\alpha \in (1,\infty)$ that

$$\frac{1}{n} Q_{\leftrightarrow}^{n,\varepsilon}(\mathcal{N}) \leq \widehat{R}_\alpha(\mathcal{N}) + \frac{\alpha}{n(\alpha - 1)} \log_2\left(\frac{1}{1-\varepsilon}\right)$$

Upper bound on LOCC-assisted quantum capacity

- Now take the limit $n \to \infty$:

$$\limsup_{n\to\infty} \frac{1}{n} Q_{\leftrightarrow}^{n,\varepsilon}(\mathcal{N}) \leq \widehat{R}_{\alpha}(\mathcal{N})$$

- Since the bound holds for all $\alpha \in (1, \infty)$, conclude that

$$\limsup_{n\to\infty} \frac{1}{n} Q_{\leftrightarrow}^{n,\varepsilon}(\mathcal{N}) \leq \inf_{\alpha\in(1,\infty)} \widehat{R}_{\alpha}(\mathcal{N}) = \widehat{R}(\mathcal{N})$$

- Conclude bound on strong converse LOCC-assisted quantum capacity

$$\widetilde{Q}_{\leftrightarrow}(\mathcal{N}) \leq \widehat{R}(\mathcal{N})$$

Conclusion

Three quantum generalizations of Rényi relative entropy and their use:

- **Petz–Rényi relative entropy**:

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log_2 \operatorname{Tr}[\rho^\alpha \sigma^{1-\alpha}]$$

  Finds use as lower bound for distinguishability and comm. tasks

- **Sandwiched Rényi relative entropy**:

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log_2 \operatorname{Tr}[(\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha})^\alpha]$$

  Finds use as upper bound for distinguishability and comm. tasks

Conclusion (ctd.)

- **Geometric Rényi relative entropy**:

$$\widehat{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log_2 \operatorname{Tr}[\sigma(\sigma^{-1/2}\rho\sigma^{-1/2})^\alpha]$$

Finds use as upper bound for feedback-assisted distinguishability and communication tasks

Future directions

- Are there other interesting quantum generalizations of Rényi relative entropy?
- Do they have applications in quantum information theory?
- See α-z Rényi relative entropies [AD15] and their data-processing inequality [Zha20]

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