Theory of quantum entanglement

Mark M. Wilde

Hearne Institute for Theoretical Physics,
Department of Physics and Astronomy,
Center for Computation and Technology,
Louisiana State University,
Baton Rouge, Louisiana, USA

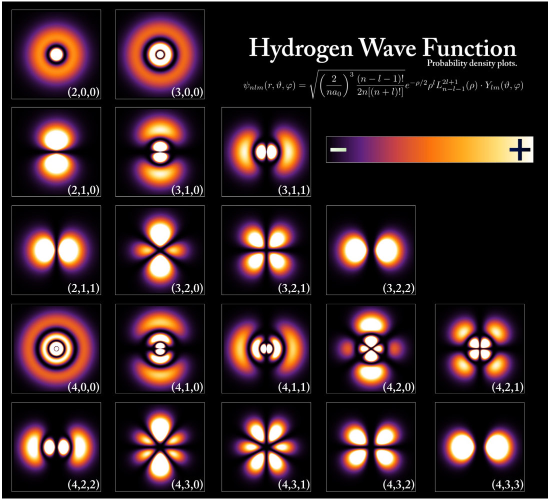
mwilde@lsu.edu

On Sabbatical with
Stanford Institute for Theoretical Physics,
Stanford University,
Stanford, California 94305

Mitteldeutsche Physik Combo, University of Leipzig, Leipzig, Germany, September 17, 2020

Motivation

- **Quantum entanglement** is a resource for quantum information processing (basic protocols like teleportation and super-dense coding)
- Has applications in quantum computation, quantum communication, quantum networks, quantum sensing, quantum key distribution, etc.
- Important to understand entanglement in a **quantitative** sense
- Adopt **axiomatic** and **operational** approaches
- This tutorial focuses on entanglement in non-relativistic quantum mechanics and finite-dimensional quantum systems



(Image courtesy of `https://en.wikipedia.org/wiki/Quantum_state`)

Quantum states

- The state of a quantum system is given by a square matrix called the **density matrix**, usually denoted by ρ, σ, τ, ω, etc. (also called *density operator*)

- It should be **positive semi-definite** and have **trace equal to one**. That is, all of its eigenvalues should be non-negative and sum up to one. We write these conditions symbolically as $\rho \geq 0$ and $\text{Tr}\{\rho\} = 1$. Can abbreviate more simply as $\rho \in \mathcal{D}(\mathcal{H})$, to be read as "$\rho$ is in the set of density matrices."

- The dimension of the matrix indicates the number of distinguishable states of the quantum system.

- For example, a physical *qubit* is a quantum system with dimension two. A classical bit, which has two distinguishable states, can be embedded into a qubit.

Interpretation of density operator

- The density operator, in addition to a description of an experimental procedure, is all that one requires to predict the (probabilistic) outcomes of a given experiment performed on a quantum system.

- It is a generalization of (and subsumes) a probability distribution, which describes the state of a classical system.

- All probability distributions can be embedded into a quantum state by placing the entries along the diagonal of the density operator.

Mixtures of quantum states

- A **probabilistic mixture** of two quantum states is also a quantum state. That is, for $\sigma_0, \sigma_1 \in \mathcal{D}(\mathcal{H})$ and $p \in [0,1]$, we have

$$p\sigma_0 + (1-p)\sigma_1 \in \mathcal{D}(\mathcal{H}).$$

- The set of density operators is thus **convex**.

Mixed states and pure states

- A density operator can have dimension ≥ 2 and can be written as

$$\rho = \sum_{i,j} \rho^{i,j} |i\rangle\langle j|,$$

where $\{|i\rangle \equiv e_i\}$ is the standard basis and $\rho^{i,j}$ are the matrix elements.

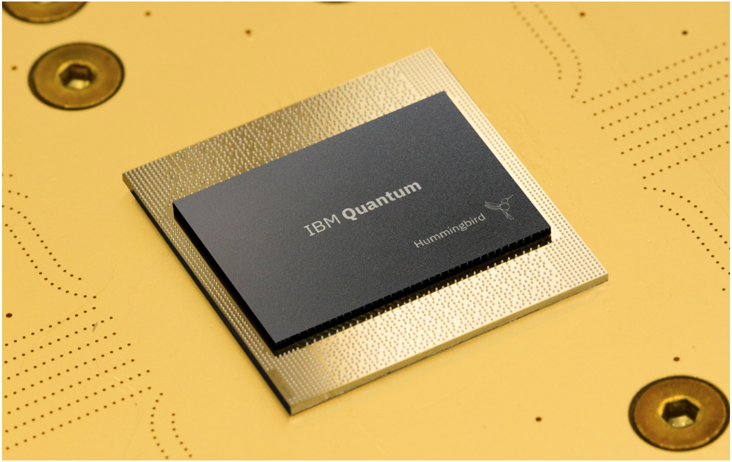
- Since every density operator is positive semi-definite and has trace equal to one, it has a **spectral decomposition** as

$$\rho = \sum_{x} p_X(x) |\phi_x\rangle\langle \phi_x|,$$

where $\{p_X(x)\}$ are the non-negative eigenvalues, summing to one, and $\{|\phi_x\rangle\}$ is a set of orthonormal eigenvectors.

- A density operator ρ is **pure** if there exists a unit vector $|\psi\rangle$ such that $\rho = |\psi\rangle\langle\psi|$ (rank $= 1$) and otherwise it is **mixed** (rank > 1).

Multiple quantum systems...



IBM 65-qubit universal quantum computer (released September 2020)

Composite quantum systems

- If the state of Alice's system is ρ and the state of Bob's system is σ and they have never interacted in the past, then the state of the joint Alice-Bob system is

$$\rho_A \otimes \sigma_B.$$

- We use the system labels to say who has what.

Composite quantum systems (ctd.)

- More generally, a generic state ρ_{AB} of a bipartite system AB acts on a tensor-product Hilbert space $\mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$.

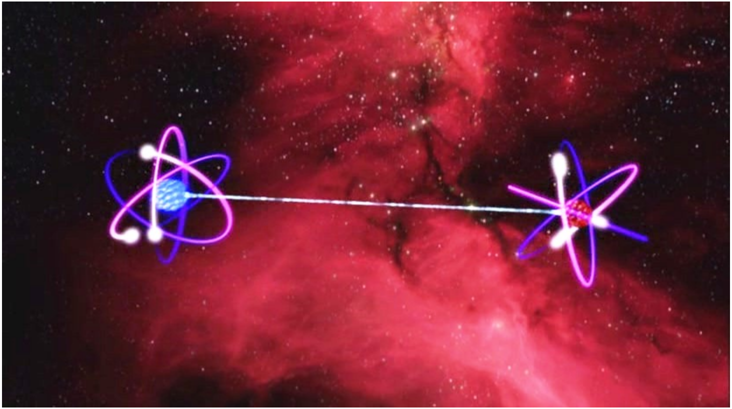
- If $\{|i\rangle_A\}_i$ is an orthonormal basis for $\mathcal{H}_A$ and $\{|j\rangle_B\}_j$ is an orthonormal basis for $\mathcal{H}_B$, then $\{|i\rangle_A \otimes |j\rangle_B\}_{i,j}$ is an orthonormal basis for $\mathcal{H}_{AB}$.

- **Generic state** ρ_{AB} can be written as

$$\rho_{AB} = \sum_{i,j,k,l} \rho^{i,k,j,l} |i\rangle\langle k|_A \otimes |j\rangle\langle l|_B$$

where $\rho^{i,k,j,l}$ are matrix elements

Quantum entanglement...



Depiction of quantum entanglement taken from
http://thelifeofpsi.com/2013/10/28/bertlmanns-socks/

Separable states and entangled states

- If Alice and Bob prepare states ρ_A^x and σ_B^x based on a random variable X with distribution p_X, then the state of their systems is

$$\sum_x p_X(x) \rho_A^x \otimes \sigma_B^x.$$

- Such states are called **separable states** [Wer89] and can be prepared using local operations and classical communication (**LOCC**). No need for a quantum interaction between A and B to prepare these states.
- By spectral decomposition, every separable state can be written as

$$\sum_z p_Z(z) |\psi^z\rangle\langle\psi^z|_A \otimes |\phi^z\rangle\langle\phi^z|_B,$$

where, for each z, $|\psi^z\rangle_A$ and $|\phi^z\rangle_B$ are unit vectors.
- **Entangled states** are states that cannot be written in the above form.

Example of entangled state

- A prominent example of an entangled state is the *ebit* (eee · bit):

$$|\Phi\rangle\langle\Phi|_{AB},$$

where $|\Phi\rangle_{AB} \equiv \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$.

- In matrix form, this is

$$|\Phi\rangle\langle\Phi|_{AB} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

- To see that this is entangled, consider that for every $|\psi\rangle_A$ and $|\phi\rangle_B$

$$\left|\langle\Phi|_{AB}|\psi\rangle_A \otimes |\phi\rangle_B\right|^2 \leq \frac{1}{2}$$

- $\Rightarrow$ impossible to write $|\Phi\rangle\langle\Phi|_{AB}$ as a separable state.

Tool: Schmidt decomposition

Schmidt decomposition theorem

Given a two-party unit vector $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, we can express it as

$$|\psi\rangle_{AB} \equiv \sum_{i=0}^{d-1} \sqrt{p_i}\, |i\rangle_A\, |i\rangle_B, \text{ where}$$

- probabilities p_i are real, strictly positive, and normalized $\sum_i p_i = 1$.
- $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ are orthonormal bases for systems A and B.
- $\left[\sqrt{p_i}\right]_{i\in\{0,\ldots,d-1\}}$ is the vector of Schmidt coefficients.
- Schmidt rank d of $|\psi\rangle_{AB}$ is equal to the number of Schmidt coefficients p_i in its Schmidt decomposition and satisfies

$$d \leq \min\left\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\right\}.$$

- Pure state $|\psi\rangle\langle\psi|_{AB}$ is entangled iff $d \geq 2$.

Tool: Partial trace

- The **trace** of a matrix X can be realized as

$$\mathrm{Tr}[X] = \sum_i \langle i|X|i\rangle,$$

  where $\{|i\rangle\}_i$ is an orthonormal basis.

- **Partial trace** of a matrix Y_{AB} acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ can be realized as

$$\mathrm{Tr}_A[Y_{AB}] = \sum_i (\langle i|_A \otimes I_B) Y_{AB} (|i\rangle_A \otimes I_B),$$

  where $\{|i\rangle_A\}_i$ is an orthonormal basis for $\mathcal{H}_A$ and I_B is the identity matrix acting on $\mathcal{H}_B$.

- Both trace and partial trace are **linear operations**.

Interpretation of partial trace

- Suppose Alice and Bob possess quantum systems in the state ρ_{AB}. We calculate the density matrix for Alice's system using partial trace:

$$\rho_A \equiv \mathrm{Tr}_B[\rho_{AB}].$$

- We can then use ρ_A to predict the outcome of any experiment performed on Alice's system alone.
- Partial trace generalizes marginalizing a probability distribution:

$$\begin{aligned}
&\mathrm{Tr}_Y\left[\sum_{x,y} p_{X,Y}(x,y)|x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y\right] \\
&= \sum_{x,y} p_{X,Y}(x,y)|x\rangle\langle x|_X \,\mathrm{Tr}[|y\rangle\langle y|_Y] \\
&= \sum_x \left[\sum_y p_{X,Y}(x,y)\right] |x\rangle\langle x|_X = \sum_x p_X(x)|x\rangle\langle x|_X,
\end{aligned}$$

where $p_X(x) \equiv \sum_y p_{X,Y}(x,y)$.

Purification of quantum noise...

Artistic rendering of the notion of purification
(Image courtesy of seaskylab at FreeDigitalPhotos.net)

Tool: Purification of quantum states

- A **purification** of a state ρ_S on system S is a pure quantum state $|\psi\rangle\langle\psi|_{RS}$ on systems R and S, such that

$$\rho_S = \mathrm{Tr}_R[|\psi\rangle\langle\psi|_{RS}].$$

- **Simple construction**: take $|\psi\rangle_{RS} = \sum_x \sqrt{p(x)}|x\rangle_R \otimes |x\rangle_S$ if ρ_S has spectral decomposition $\sum_x p(x)|x\rangle\langle x|_S$.

- Two different states $|\psi\rangle\langle\psi|_{RS}$ and $|\phi\rangle\langle\phi|_{RS}$ purify ρ_S iff they are related by a unitary U_R acting on the reference system. Necessity:

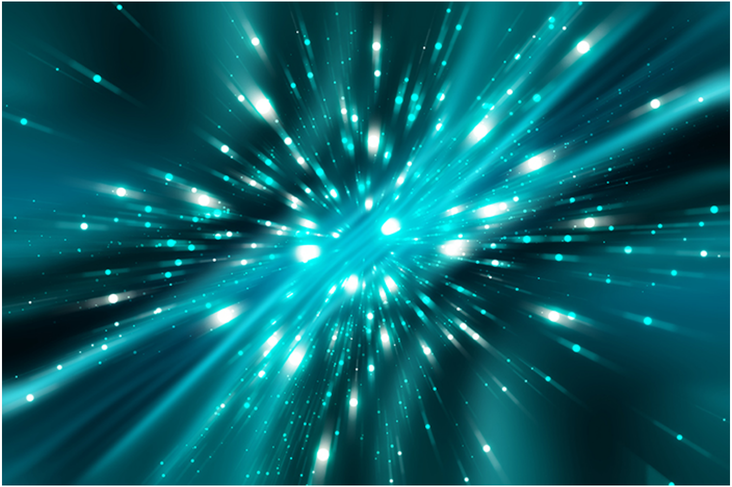
$$\begin{aligned}\mathrm{Tr}_R[(U_R \otimes I_S)|\psi\rangle\langle\psi|_{RS}(U_R^\dagger \otimes I_S)] &= \mathrm{Tr}_R[(U_R^\dagger U_R \otimes I_S)|\psi\rangle\langle\psi|_{RS}] \\ &= \mathrm{Tr}_R[|\psi\rangle\langle\psi|_{RS}] \\ &= \rho_S.\end{aligned}$$

To prove sufficiency, use Schmidt decomposition.

Uses and interpretations of purification

- The concept of purification is one of the most often used tools in quantum information theory.
- This concept does not exist in classical information theory and represents a **radical departure** (i.e., in classical information theory it is not possible to have a definite state of two systems such that the reduced systems are individually indefinite).
- Physical interpretation: Noise or mixedness in a quantum state is due to entanglement with an inaccessible reference / environment system.
- Cryptographic interpretation: In the setting of quantum cryptography, we assume that an eavesdropper Eve has access to the full purification of a state ρ_{AB} that Alice and Bob share. This means physically that Eve has access to every other system in the universe that Alice and Bob do not have access to!
- Advantage: only need to characterize Alice and Bob's state in order to understand what Eve has.

Quantum channels...



Artistic rendering of a quantum channel
(Image courtesy of Shutterstock / Serg-DAV)

Classical channels

- Classical channels model evolutions of classical systems.
- What are the requirements that we make for classical channels?
- 1) They should be **linear maps**, which means they respect convexity.
- 2) They should take probability distributions to probability distributions (i.e., they should output a legitimate state of a classical system when a classical state is input).
- These requirements imply evolution of a classical system is specified by a **conditional probability matrix** N with entries $p_{Y|X}(y|x)$, so that the input-output relationship of a classical channel is given by

$$p_Y = N\, p_X \qquad \Longleftrightarrow \qquad p_Y(y) = \sum_x p_{Y|X}(y|x) p_X(x).$$

Quantum channels

- Quantum channels model evolutions of quantum systems.
- We make similar requirements:
- A quantum channel $\mathcal{N}$ is a **linear map** acting on the space of (density) matrices:

$$\mathcal{N}(p\rho + (1-p)\sigma) = p\mathcal{N}(\rho) + (1-p)\mathcal{N}(\sigma),$$

  where $p \in [0,1]$ and $\rho, \sigma \in \mathcal{D}(\mathcal{H})$.
- We demand that a quantum channel should take quantum states to quantum states.
- This means that it should be **trace (probability) preserving**:

$$\text{Tr}[\mathcal{N}(X)] = \text{Tr}[X]$$

  for all $X \in \mathcal{L}(\mathcal{H})$ (linear operators, i.e., matrices).

Complete positivity

- Other requirement is **complete positivity**.
- We can always expand $X_{RS} \in \mathcal{L}(\mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{S})$ as

$$X_{RS} = \sum_{i,j} |i\rangle\langle j|_R \otimes X_S^{i,j},$$

and then define

$$(\mathrm{id}_R \otimes \mathcal{N}_S)(X_{RS}) = \sum_{i,j} |i\rangle\langle j|_R \otimes \mathcal{N}_S\left(X_S^{i,j}\right),$$

with the interpretation being that "nothing (identity channel) happens on system R while the channel $\mathcal{N}$ acts on system S."
- A quantum channel should also be **completely positive**:

$$(\mathrm{id}_R \otimes \mathcal{N}_S)(X_{RS}) \geq 0,$$

where id_R denotes the identity channel acting on system R of arbitrary size and $X_{RS} \in \mathcal{L}(\mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{S})$ is such that $X_{RS} \geq 0$.

Quantum channels: completely positive, trace-preserving

- A map $\mathcal{N}$ satisfying the requirements of linearity, trace preservation, and complete positivity takes all density matrices to density matrices and is called a **quantum channel**.

- To check whether a given map is completely positive, it suffices to check whether

$$(\mathrm{id}_R \otimes \mathcal{N}_S)(|\Phi\rangle\langle\Phi|_{RS}) \geq 0,$$

where

$$|\Phi\rangle_{RS} = \frac{1}{\sqrt{d}} \sum_i |i\rangle_R \otimes |i\rangle_S$$

and $d = \dim(\mathcal{H}_\mathcal{R}) = \dim(\mathcal{H}_\mathcal{S})$.

- Interpretation: the state resulting from a channel acting on one share of a maximally entangled state completely characterizes the channel.

Choi-Kraus representation theorem

Structure theorem for quantum channels

Every quantum channel $\mathcal{N}$ can be written in the following form:

$$\mathcal{N}(X) = \sum_i K_i X K_i^\dagger, \tag{1}$$

where $\{K_i\}_i$ is a set of Kraus operators, with the property that

$$\sum_i K_i^\dagger K_i = I. \tag{2}$$

The form given in (1) corresponds to complete positivity and the condition in (2) to trace (probability) preservation. This decomposition is not unique, but one can find a minimal decomposition by taking a spectral decomposition of $(\mathrm{id}_R \otimes \mathcal{N}_S)(|\Phi\rangle\langle\Phi|_{RS})$.

Unitary channels

- If a channel has one Kraus operator (call it U), then it satisfies $U^\dagger U = I$ and is thus a unitary matrix.[1]
- Unitary channels are **ideal, reversible channels**.
- Instruction sequences for quantum algorithms (to be run on quantum computers) are composed of ideal, unitary channels.
- So if a quantum channel has more than one Kraus operator (in a minimal decomposition), then it is non-unitary.

[1] It could also be part of a unitary matrix, in which case it is called an "isometry."

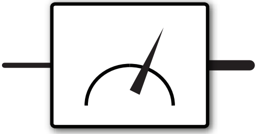
Measurement channels

- **Measurement channels** take quantum systems as input and produce classical systems as output.
- A measurement channel $\mathcal{M}$ has the following form:

$$\mathcal{M}(\rho) = \sum_x \operatorname{Tr}[M^x \rho] |x\rangle\langle x|,$$

where $M_x \geq 0$ for all x and $\sum_x M^x = I$.

- Can also interpret a measurement channel as returning the classical value x with probability $\operatorname{Tr}[M^x \rho]$.
- We depict them as



Quantum instrument

- A **quantum instrument** is a quantum channel with a quantum input and two outputs: one classical and one quantum [DL70, Oza84].
- It is a measurement in which we record not only the classical data, but also keep the post-measurement state in a quantum system
- Evolves an input state ρ as follows:

$$\rho \to \sum_x \mathcal{M}^x(\rho) \otimes |x\rangle\langle x|$$

  where $\{\mathcal{M}^x\}_x$ is a set of completely positive maps such that the sum map $\sum_x \mathcal{M}^x$ is trace preserving.
- Probability of obtaining outcome x:

$$\text{Tr}[\mathcal{M}^x(\rho)]$$

  and post-measurement state in this case is

$$\frac{\mathcal{M}^x(\rho)}{\text{Tr}[\mathcal{M}^x(\rho)]}$$

Purifications of quantum channels

- Recall that we can purify quantum states and understand noise as arising due to entanglement with an inaccessible reference system.
- We can also purify quantum channels and understand a noisy process as arising from a unitary interaction with an inaccessible environment.

Stinespring's theorem [Sti55]

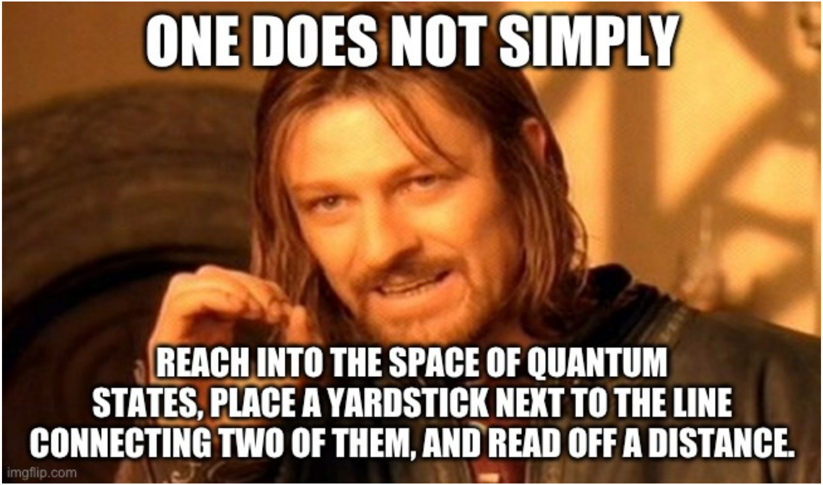
For every quantum channel $\mathcal{N}_{A\to B}$, there exists a pure state $|0\rangle\langle 0|_E$ and a unitary matrix $U_{AE\to BE'}$, acting on input systems A and E and producing output systems B and E', such that

$$\mathcal{N}_{A\to B}(\rho_A) = \mathrm{Tr}_{E'}[U_{AE\to BE'}(\rho_A \otimes |0\rangle\langle 0|_E)(U_{AE\to BE'})^\dagger].$$

Summary of quantum states and channels

- Every quantum state is a positive, semi-definite matrix with trace equal to one.
- Quantum states of multiple systems can be separable or entangled.
- Quantum states can be purified (this notion does not exist in classical information theory).
- Quantum channels are completely positive, trace-preserving maps.
- Preparation channels take classical systems to quantum systems, and measurement channels take quantum systems to classical systems.
- Quantum channels can also be purified (i.e., every quantum channel can be realized by a unitary interaction with an environment, followed by partial trace). This notion also does not exist in classical information theory.

Distance measures...



(Image courtesy of https://imgflip.com/memegenerator/One-Does-Not-Simply and quote from page 1 of Fuchs' thesis: https://arxiv.org/pdf/quant-ph/9601020)

Function of a diagonalizable matrix

- If an $n \times n$ matrix D is diagonal with entries $d_1, \ldots, d_n$, then for a function f, we define

$$f(D) = \begin{bmatrix} g(d_1) & 0 & \cdots & 0 \\ 0 & g(d_2) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & g(d_n) \end{bmatrix}$$

  where $g(x) = f(x)$ if $x \in \operatorname{dom}(f)$ and $g(x) = 0$ otherwise.
- If a matrix A is diagonalizable as $A = KDK^{-1}$, then for a function f, we define

$$f(A) = Kf(D)K^{-1}.$$

- Evaluating the function only on the support of the matrix allows for functions such as $f(x) = x^{-1}$ and $f(x) = \log x$.

Trace distance

- Define the **trace norm** of a matrix X by $\|X\|_1 := \text{Tr}[\sqrt{X^\dagger X}]$.
- Trace norm induces **trace distance** between two matrices X and Y:

$$\|X - Y\|_1 .$$

- For two density matrices ρ and σ, the following bounds hold

$$0 \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq 1.$$

  LHS saturated iff $\rho = \sigma$ and RHS iff ρ is orthogonal to σ.
- For commuting ρ and σ, normalized trace distance reduces to variational distance between probability distributions along diagonals.
- Has an operational meaning as the bias of the optimal success probability in a hypothesis test to distinguish ρ from σ [Hel67, Hol72].
- Does not increase under the action of a quantum channel:

$$\|\rho - \sigma\|_1 \geq \|\mathcal{N}(\rho) - \mathcal{N}(\sigma)\|_1 .$$

Fidelity

- **Fidelity** $F(\rho,\sigma)$ between density matrices ρ and σ is [Uhl76]

$$F(\rho,\sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2.$$

- For pure states $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$, reduces to squared overlap:

$$F(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = |\langle\psi|\phi\rangle|^2.$$

- For density matrices ρ and σ, the following bounds hold:

$$0 \leq F(\rho,\sigma) \leq 1.$$

  LHS saturated iff ρ and σ are orthogonal and RHS iff $\rho = \sigma$.
- Fidelity does not decrease under the action of a quantum channel $\mathcal{N}$:

$$F(\rho,\sigma) \leq F(\mathcal{N}(\rho), \mathcal{N}(\sigma)).$$

Uhlmann's theorem

- **Uhlmann's theorem** [Uhl76] states that

$$F(\rho_S, \sigma_S) = \max_{U_R} \left| \langle \psi |_{RS} U_R \otimes I_S | \phi \rangle_{RS} \right|^2,$$

  where $|\psi\rangle_{RS}$ and $|\phi\rangle_{RS}$ purify ρ_S and σ_S, respectively.

- A core theorem used in quantum Shannon theory, and in other areas such as quantum complexity theory and quantum error correction.

- Since it involves purifications, this theorem has no analog in classical information theory.

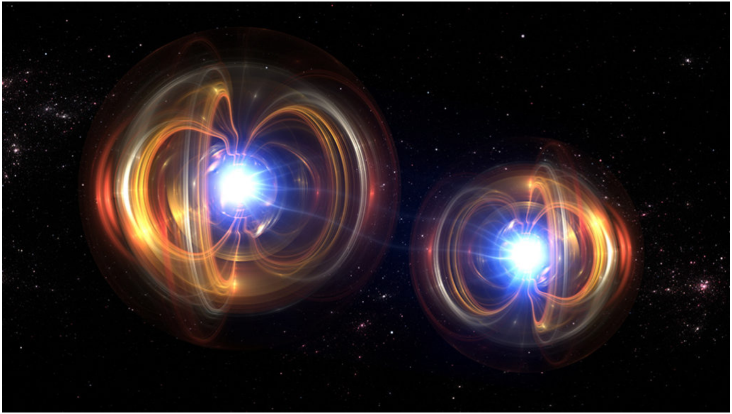
Relations between fidelity and trace distance

- Trace distance is useful because it obeys the triangle inequality, and fidelity is useful because we have Uhlmann's theorem.
- The following inequalities relate the two measures [FvdG98], which allow for going back and forth between them:

$$1 - \sqrt{F(\rho,\sigma)} \leq \frac{1}{2}\left\|\rho - \sigma\right\|_1 \leq \sqrt{1 - F(\rho,\sigma)}.$$

- **Sine distance** $\sqrt{1 - F(\rho,\sigma)}$ [Ras06] has both properties (triangle inequality and Uhlmann's theorem).

Entanglement theory...



(Image courtesy of Jurik Peter / Shutterstock)

Reminder of separable and entangled states

- If Alice and Bob prepare states ρ_A^x and σ_B^x based on a random variable X with distribution p_X, then the state of their systems is

$$\sum_x p_X(x) \rho_A^x \otimes \sigma_B^x.$$

- Such states are called **separable states** [Wer89] and can be prepared using local operations and classical communication (no need for a quantum interaction between A and B to prepare these states).

- Pure state entangled iff Schmidt rank ≥ 2 (thus, easy to decide if a pure state is entangled)

Motivation from Bell experiment

- **Bell experiment** [Bel64] consists of spatially separated parties A and B performing local measurements on a state ρ_{AB}.
- A flips a coin and gets outcome x.
- Then performs a measurement $\{\Gamma_a^{(x)}\}_a$ with outcome a.
- B flips a coin and gets outcome y.
- Then performs a measurement $\{\Omega_b^{(y)}\}_b$ with outcome b.
- Conditional probability of observing a and b given x and y:

$$p(a, b|x, y) = \mathrm{Tr}[(\Gamma_a^{(x)} \otimes \Omega_b^{(y)})\rho_{AB}]$$

Motivation from Bell experiment (ctd.)

- Separable states have **local hidden variable theory** in a Bell experiment (AKA shared randomness strategy)
- Suppose that ρ_{AB} is separable, so that

$$\rho_{AB} = \sum_\lambda p(\lambda)\sigma_A^\lambda \otimes \omega_B^\lambda$$

- Then conditional probability given by

$$p(a, b|x, y) = \sum_\lambda p(\lambda)p(a|x, \lambda)p(b|y, \lambda)$$

$$\text{where } p(a|x, \lambda) = \mathrm{Tr}[\Gamma_a^{(x)}\sigma_A^\lambda], \qquad p(b|y, \lambda) = \mathrm{Tr}[\Omega_b^{(y)}\omega_A^\lambda]$$

- Using a separable state, correlations achievable are simulable by a classical strategy

Loophole-free Bell test...

Picture of loophole-free Bell test at TU Delft
(Image taken from http://hansonlab.tudelft.nl/loophole-free-bell-test/)

NP-hardness of deciding entanglement

- **Computationally hard** to decide if a state is separable or entangled.
- More precisely, consider the following computational decision problem:
- Given a mathematical description of the density operator ρ_{AB} as a matrix of rational entries and $\varepsilon > 0$. Decide whether

$$\rho_{AB} \in \text{SEP}(A\!:\!B)$$
$$\text{or} \quad \inf_{\sigma_{AB} \in \text{SEP}(A:B)} \frac{1}{2} \left\| \rho_{AB} - \sigma_{AB} \right\|_1 \geq \varepsilon.$$

- Decision problem NP-hard to solve for $\varepsilon \leq \frac{1}{\text{poly}(d_A, d_B)}$ [Gur04, Gha10].
- This means that if widely believed conjectures in theoretical computer science are true, the best classical or quantum algorithms will have running time exponential in $d_A \times d_B$.

Positive partial transpose criterion

- Even if it is NP-hard to decide whether a state is separable or entangled, we can look for one-way criteria.
- **Positive partial transpose** criterion [Per96, HHH96]
- The following transpose map is a positive map:

$$T(X) := \sum_{i,j} |i\rangle\langle j| X |i\rangle\langle j|$$

- That is, $T(X) \geq 0$ if $X \geq 0$
- The transpose map is called "partial transpose" if it acts on one share of a bipartite operator X_{AB}:

$$T_B(X_{AB}) := (\mathrm{id}_A \otimes T_B)(X_{AB}) = \sum_{i,j} (I_A \otimes |i\rangle\langle j|_B) X_{AB} (I_A \otimes |i\rangle\langle j|_B)$$

- A state that has a positive partial transpose is said to be a PPT state

Positive partial transpose criterion (ctd.)

- A separable state σ_{AB} has a positive partial transpose because

$$T_B(\sigma_{AB}) = T_B\left(\sum_\lambda p(\lambda)\rho_A^\lambda \otimes \omega_B^\lambda\right)$$
$$= \sum_\lambda p(\lambda)\rho_A^\lambda \otimes T_B(\omega_B^\lambda) \geq 0$$

- Thus, SEP $\subset$ PPT
- Containment is **strict** because $\exists$ PPT entangled states [Hor97]
- Contrapositive: if ρ_{AB} has a **negative partial transpose**, then it is entangled
- Example: Applying T_B to maximally entangled state gives an operator proportional to unitary SWAP operator, which has negative eigenvalues

Computational complexity and PPT states

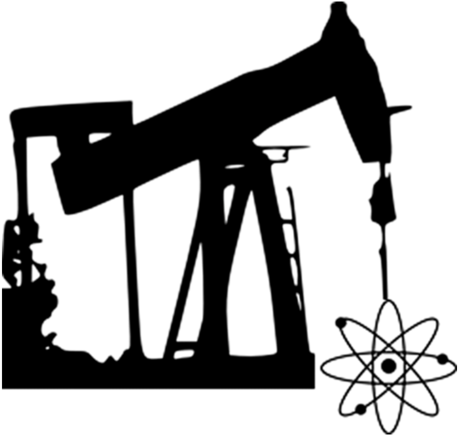
- Much easier computationally to work with PPT than with SEP
- For a Hermitian operator M_{AB}, consider the optimizations

$$\max_{\sigma_{AB} \in \text{SEP}} \text{Tr}[M_{AB}\sigma_{AB}] \qquad \text{vs.} \qquad \max_{\sigma_{AB} \in \text{PPT}} \text{Tr}[M_{AB}\sigma_{AB}]$$

- The first is NP-hard, while the second is efficiently computable as a **semi-definite program**:

$$\max_{\sigma_{AB} \in \text{PPT}} \text{Tr}[M_{AB}\sigma_{AB}] = \max_{\sigma_{AB}} \{ \text{Tr}[M_{AB}\sigma_{AB}] : \sigma_{AB} \geq 0, T_B(\sigma_{AB}) \geq 0, \text{Tr}[\sigma_{AB}] = 1 \}$$

Resource theory of entanglement...



(Image courtesy of https://journals.aps.org/rmp/abstract/10.1103/RevModPhys.91.025001)

Resource theory of entanglement

- Idea of general **resource theory** is that some states are free and others are resourceful [CG19].
- **Resource theory of entanglement** was the first resource theory considered in QIT [BDSW96]
- Entanglement is useful for tasks like teleportation [BBC$^+$93], super-dense coding [BW92], and quantum key distribution [Eke91], so these are the resourceful states
- Separable states are the free states
- What are the free operations?

LOCC channel

- In the theory of entanglement and quantum communication, one often assumes that Alice and Bob can communicate classical data for free.
- Paradigm is local op.'s and classical comm. (LOCC) [BDSW96].
- A **one-way LOCC channel** from Alice to Bob consists of Alice performing a quantum instrument, sending classical outcome to Bob, who performs a quantum channel conditioned on the classical data.
- An **LOCC channel** consists of finite, but arbitrarily large number of 1-way LOCC channels from Alice to Bob and then from Bob to Alice.

LOCC channel (ctd.)

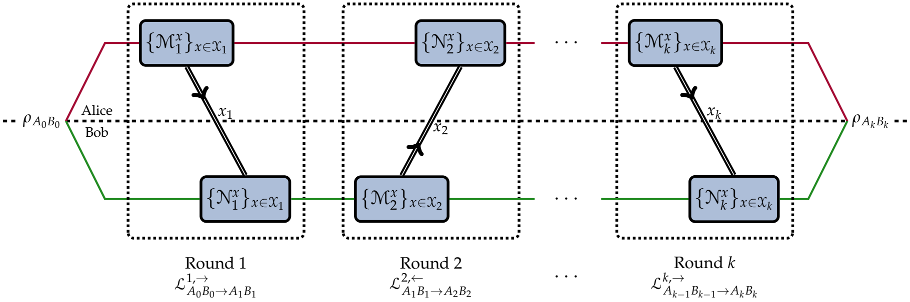
- An **LOCC channel** can be written as a separable channel $\mathcal{L}_{AB\to A'B'}$:

$$\mathcal{L}_{AB\to A'B'}(\rho_{AB}) = \sum_z (\mathcal{E}^z_{A\to A'} \otimes \mathcal{F}^z_{B\to B'})(\rho_{AB}),$$

  where $\{\mathcal{E}^z_{A\to A'}\}_z$ and $\{\mathcal{F}^z_{B\to B'}\}_z$ are sets of completely positive, trace non-increasing maps, such that $\mathcal{L}_{AB\to A'B'}$ is a completely positive, trace-preserving map (quantum channel).

- However, the converse is not true. There exist separable channels that are not LOCC channels [BDF$^+$99]

Depiction of LOCC



(Figure designed by Sumeet Khatri)

LOCC channels preserve the set of separable states

- If σ_{AB} is separable and an LOCC channel $\mathcal{L}_{AB\to A'B'}$ acts on it, the resulting state is separable because

$$\mathcal{L}_{AB\to A'B'}(\sigma_{AB}) = \sum_{z}(\mathcal{E}^{z}_{A\to A'} \otimes \mathcal{F}^{z}_{B\to B'}) \left(\sum_{\lambda} p(\lambda)\rho^{\lambda}_{A} \otimes \omega^{\lambda}_{B}\right)$$
$$= \sum_{z,\lambda} p(\lambda)\, \mathcal{E}^{z}_{A\to A'}(\rho^{\lambda}_{A}) \otimes \mathcal{F}^{z}_{B\to B'}(\omega^{\lambda}_{B})$$

- Thus, one cannot create entanglement by the action of LOCC on separable states.

- So it is reasonable for LOCC to be the set of free operations in the resource theory of entanglement, due to this property in addition to the physical motivation

Axiomatic approach to quantifying entanglement

Basic axiom for entanglement measure [HHHH09]

An **entanglement measure** E is a function that does not increase under the action of LOCC. That is, E is an entanglement measure if the following inequality holds for every state ρ_{AB} and LOCC channel $\mathcal{L}_{AB\to A'B'}$:

$$E(A;B)_\rho \geq E(A';B')_\omega$$

where $\omega_{A'B'} := \mathcal{L}_{AB\to A'B'}(\rho_{AB})$.

Consequence of basic axiom [HHHH09]

- Implies that E is minimal and constant on separable states because one can get from one separable state to another by LOCC:

$$E(A;B)_\sigma = c \qquad \forall \sigma_{AB} \in \mathrm{SEP}(A\!:\!B)$$

- Conventional to set $c = 0$
- Thus, $E(A;B)_\rho \geq 0$ for every state ρ_{AB}
- and $E(A;B)_\sigma = 0$ if $\sigma_{AB} \in \mathrm{SEP}(A\!:\!B)$

Other desirable properties for an entanglement measure

Faithfulness

$E(A;B)_\sigma = 0$ if and only if $\sigma_{AB} \in \mathrm{SEP}(A\!:\!B)$

Invariance under classical communication

$$E(AX;B)_\rho = E(A;BX)_\rho = \sum_x p(x) E(A;B)_{\rho^x}$$

for a classical–quantum state:

$$\rho_{XAB} := \sum_x p(x) |x\rangle\langle x|_X \otimes \rho^x_{AB}$$

Other desirable properties (ctd.)

Convexity

For a convex combination $\bar{\rho} := \sum_x p(x)\rho^x_{AB}$ of states,

$$\sum_x p(x) E(A;B)_{\rho^x} \geq E(A;B)_{\bar{\rho}}$$

Additivity

For a tensor-product state $\rho_{A_1A_2B_1B_2} = \tau_{A_1B_1} \otimes \omega_{A_2B_2}$,

$$E(A_1A_2;B_1B_2)_\rho = E(A_1;B_1)_\tau + E(A_2;B_2)_\omega$$

Selective LOCC monotonicity

- Let $\{\mathcal{L}^{x}_{AB\to A'B'}\}_x$ be a collection of maps, such that $\mathcal{L}^{\leftrightarrow}_{AB\to A'B'}$ is an LOCC channel of the form:

$$\mathcal{L}^{\leftrightarrow}_{AB\to A'B'} = \sum_x \mathcal{L}^{x}_{AB\to A'B'},$$

  where each map $\mathcal{L}^{x}_{AB\to A'B'}$ is completely positive such that the sum map $\mathcal{L}^{\leftrightarrow}_{AB\to A'B'}$ is trace preserving.

- Furthermore, each map $\mathcal{L}^{x}_{AB\to A'B'}$ can be written as follows:

$$\mathcal{L}^{x}_{AB\to A'B'} = \sum_y \mathcal{E}^{x,y}_{A\to A'} \otimes \mathcal{F}^{x,y}_{B\to B'},$$

  where $\{\mathcal{E}^{x,y}_{A\to A'}\}_x$ and $\{\mathcal{F}^{x,y}_{B\to B'}\}_x$ are sets of completely positive maps.

Selective LOCC monotonicity (ctd.)

- Set $p(x) := \text{Tr}[\mathcal{L}^x_{AB\to A'B'}(\rho_{AB})]$, and for x such that $p(x) \neq 0$, set

$$\omega^x_{AB} := \frac{1}{p(x)}\mathcal{L}^x_{AB\to A'B'}(\rho_{AB}).$$

- If the classical value of x is not discarded, then the given state ρ_{AB} is transformed to the ensemble $\{(p(x),\omega^x_{AB})\}_x$ via LOCC.

- E satisfies **selective LOCC monotonicity** if

$$E(\rho_{AB}) \geq \sum_{x\in\mathcal{X}:p(x)\neq 0} p(x)E(\omega^x_{AB}),$$

  for every ensemble $\{(p(x),\omega^x_{AB})\}_{x\in\mathcal{X}}$ that arises from ρ_{AB} via LOCC as specified above.

- Interpretation: *Entanglement does not increase on average under LOCC*

Proving convexity and selective LOCC monotonicity

- Let E be a function that, for every bipartite state ρ_{AB}, is
  1. invariant under classical communication and
  2. obeys data processing under local channels, in the sense that

$$E(A;B)_\rho \geq E(A';B')_\omega,$$

  for all channels $\mathcal{N}_{A\to A'}$ and $\mathcal{M}_{B\to B'}$, where

$$\omega_{A'B'} := (\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(\rho_{AB}).$$

- Then E is **convex** and is a **selective LOCC monotone**.

Operational approach to quantifying entanglement

- **Axiomatic approach** to quantifying entanglement starts with the basic axiom and lists out properties that are desirable for an entanglement measure to obey

- A conceptually different approach is the **operational approach**: Certain information-theoretic tasks quantify the amount of entanglement present in a quantum state

- These approaches intersect when trying to establish bounds on the optimal rates of the operational tasks

- Most prominent tasks are **entanglement distillation** and **entanglement dilution**

Entanglement distillation

- **One-shot distillable entanglement** of a bipartite state ρ_{AB}:

$$E_D^{\varepsilon}(A;B)_\rho := \sup_{d\in\mathbb{N},\, \mathcal{L}\in\text{LOCC}} \left\{ \log_2 d : \frac{1}{2}\left\| \mathcal{L}_{AB\to\hat{A}\hat{B}}(\rho_{AB}) - \Phi^d_{\hat{A}\hat{B}} \right\|_1 \leq \varepsilon \right\},$$

where $\Phi^d_{\hat{A}\hat{B}} := \frac{1}{d}\sum_{i,j} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B$

- **Distillable entanglement** of ρ_{AB}:

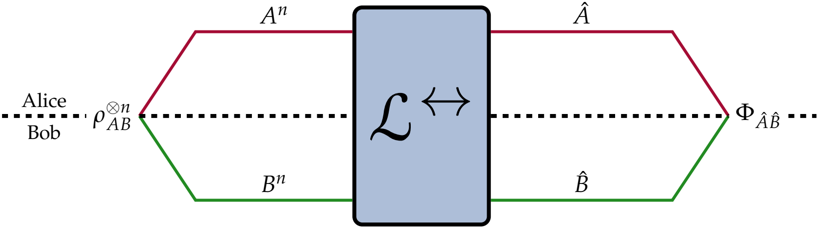
$$E_D(A;B)_\rho := \inf_{\varepsilon\in(0,1)} \liminf_{n\to\infty} \frac{1}{n} E_D^{\varepsilon}(A^n;B^n)_{\rho^{\otimes n}}$$

where $E_D^{\varepsilon}(A^n;B^n)_{\rho^{\otimes n}}$ evaluated on $\rho_{AB}^{\otimes n}$

- **Strong converse distillable entanglement** of ρ_{AB}:

$$\widetilde{E}_D(A;B)_\rho := \sup_{\varepsilon\in(0,1)} \limsup_{n\to\infty} \frac{1}{n} E_D^{\varepsilon}(A^n;B^n)_{\rho^{\otimes n}}$$

Depiction of entanglement distillation



(Figure designed by Sumeet Khatri)

Entanglement dilution

- **One-shot entanglement cost** of a bipartite state ρ_{AB}:

$$E_C^{\varepsilon}(A;B)_\rho := \inf_{d\in\mathbb{N},\,\mathcal{L}\in\text{LOCC}} \left\{ \log_2 d : \frac{1}{2}\left\|\mathcal{L}_{\hat{A}\hat{B}\to AB}(\Phi^d_{\hat{A}\hat{B}}) - \rho_{AB}\right\|_1 \leq \varepsilon \right\},$$

- **Entanglement cost** of ρ_{AB}:

$$E_C(A;B)_\rho := \sup_{\varepsilon\in(0,1)} \limsup_{n\to\infty} \frac{1}{n} E_C^{\varepsilon}(A^n;B^n)_{\rho^{\otimes n}}$$

- **Strong converse entanglement cost** of ρ_{AB}:

$$\widetilde{E}_C(A;B)_\rho := \inf_{\varepsilon\in(0,1)} \liminf_{n\to\infty} \frac{1}{n} E_C^{\varepsilon}(A^n;B^n)_{\rho^{\otimes n}}$$

Relating distillable entanglement and entanglement cost

- For all $\varepsilon_1, \varepsilon_2 \geq 0$ such that $\varepsilon_1, \varepsilon_2 \leq 1$, [Wil20]

$$E_D^{\varepsilon_1}(A; B)_\rho \leq E_C^{\varepsilon_2}(A; B)_\rho + \log_2\left(\frac{1}{1-\varepsilon'}\right),$$

  where $\varepsilon' := (\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2})^2$

- **Second law like statement**: *cannot get out much more entanglement than we invest*

- Applying definitions, conclude that

$$E_D(A; B)_\rho \leq E_C(A; B)_\rho.$$

- Asymptotically, cannot get out more than we invest

Bounding distillable entanglement and entanglement cost

- **Difficult to compute** distillable entanglement and entanglement cost
- Next best approach: Establish lower and upper bounds
- Upper bound on E_C: **Entanglement of formation** [BDSW96]
- Upper bound on E_D: **Rains relative entropy** [Rai01, ADMVW02]
- Upper bound on E_D & lower bound on E_C: **Squashed entanglement**
- Each of these is an entanglement measure

Entanglement entropy

- Recall **Schmidt-rank entanglement criterion** for pure bipartite states ψ_{AB}: ψ_{AB} is entangled if and only its Schmidt rank ≥ 2

- Use entropy of reduced state to decide whether ψ_{AB} is entangled:

$$H(A)_\psi := -\operatorname{Tr}[\psi_A \log_2 \psi_A]$$

  where $\psi_A = \operatorname{Tr}_B[\psi_{AB}]$. Called **entropy of entanglement**

- $H(A)_\psi = 0$ if ψ_{AB} is separable and $H(A)_\psi > 0$ if ψ_{AB} is entangled

- For pure bipartite states, entanglement theory simplifies immensely:

$$E_D(A;B)_\psi = E_C(A;B)_\psi = H(A)_\psi$$

  for every pure bipartite state ψ_{AB} [BBPS96].

Entanglement of formation

- To get an entanglement measure for a mixed state ρ_{AB}, take so-called convex roof of entanglement entropy [BDSW96]:

$$E_F(A;B)_\rho := \inf_{\{(p(x),\psi^x_{AB})\}_x} \left\{ \sum_x p(x) H(A)_{\psi^x} : \rho_{AB} = \sum_x p(x) \psi^x_{AB} \right\}$$

  Decompose ρ_{AB} into a convex combination of pure states and evaluate expected entanglement entropy.

- Called **entanglement of formation**

Properties of entanglement of formation

- E_F monotone under selective LOCC, convex, subadditive, and faithful.
- Reduces to entropy of entanglement for pure bipartite states
- E_F is **non-additive** [Sho04, Has09].
- Also NP-hard to compute in general [Hua14], but can calculate it for certain special classes of states.

Entanglement of formation and entanglement cost

- Entanglement of formation is an upper bound on entanglement cost:

$$E_F(A; B)_\rho \geq E_C(A; B)_\rho,$$

for every bipartite state ρ_{AB} [BDSW96].

- Regularized entanglement of formation is equal to entanglement cost [HHT01]:

$$E_C(A; B)_\rho := E_F^{\text{reg}}(A; B)_\rho,$$

where

$$E_F^{\text{reg}}(A; B)_\rho := \lim_{n\to\infty} \frac{1}{n} E_F(A^n; B^n)_{\rho^{\otimes n}}$$

Logarithmic negativity

Another simple entanglement measure for a bipartite state ρ_{AB} is **logarithmic negativity** [VW02, Ple05]:

$$E_N(A;B)_\rho := \log_2 \| T_B(\rho_{AB}) \|_1$$

Properties of logarithmic negativity

- Non-negative: $E_N(A;B)_\rho \geq 0$ for every state ρ_{AB}
- Faithful on PPT states: $E_N(A;B)_\rho = 0$ if and only if ρ_{AB} is PPT
- Selective LOCC monotone [Ple05]
- Upper bound on distillable entanglement [VW02]:

$$E_D(A;B)_\rho \leq E_N(A;B)_\rho$$

- This implies that PPT states have no distillable entanglement!
- However, there are better entanglement-measure upper bounds for the distillable entanglement

Quantum relative entropy

- **Quantum relative entropy** of a state ρ and a positive semi-definite operator σ is defined as [Ume62]

$$D(\rho\|\sigma) := \mathrm{Tr}[\rho(\log_2 \rho - \log_2 \sigma)]$$

- Standard definition with operational meaning [HP91, ON00]

Properties of quantum relative entropy

- **Data-processing inequality** for quantum relative entropy: Let ρ be a state, σ a positive semi-definite operator, and $\mathcal{N}$ a quantum channel. Then [Lin75]

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

- For every state ρ and positive semi-definite operator σ satisfying $\text{Tr}[\sigma] \leq 1$,

$$D(\rho\|\sigma) \geq 0$$

  and

$$D(\rho\|\sigma) = 0 \text{ if and only if } \rho = \sigma$$

Relative entropy of entanglement

Relative entropy of entanglement of a bipartite state ρ_{AB} [VPRK97, VP98]:

$$E_R(A;B)_\rho := \inf_{\sigma_{AB} \in \text{SEP}(A:B)} D(\rho_{AB} \| \sigma_{AB})$$

Properties of relative entropy of entanglement

- LOCC monotone: consequence of data-processing of relative entropy and set of separable states preserved by LOCC
- Selective LOCC monotone: from data-processing inequality, joint convexity of relative entropy
- Convex
- Faithful on separable states
- Reduces to entropy of entanglement for pure bipartite states
- Upper bound on distillable entanglement

Entang. of formation and relative entropy of entang.

- For every bipartite state ρ_{AB},

$$E_F(A;B)_\rho \geq E_R(A;B)_\rho$$

- Simple proof: Let $\{(p(x), \psi^x_{AB})\}_x$ be an arbitrary pure-state decomposition of ρ_{AB}. Then

$$\sum_x p(x) H(A)_{\psi^x} = \sum_x p(x) E_R(A;B)_{\psi^x} \geq E_R(A;B)_\rho$$

  We used the fact that REE equals entropy of entanglement for pure states and REE is convex

Rains relative entropy

Rains relative entropy [Rai01, ADMVW02]:

$$R(A;B)_\rho := \inf_{\sigma_{AB} \in \mathrm{PPT}'(A:B)} D(\rho_{AB} \| \sigma_{AB})$$

where

$$\mathrm{PPT}'(A:B) := \{\sigma_{AB} : \sigma_{AB} \geq 0,\ E_N(\sigma_{AB}) \leq 0\}$$

Properties of Rains relative entropy

- LOCC monotone: follows from data processing for relative entropy and the fact that $\mathrm{PPT}'(A:B)$ is preserved under LOCC

- Reduces to entropy of entanglement for pure bipartite states

- Rains relative entropy is a tighter upper bound on distillable entanglement than other entanglement measures [Rai01]:

$$E_D(A;B)_\rho \leq R(A;B)_\rho \leq \min\{E_R(A;B)_\rho, E_N(A;B)_\rho\}$$

  with $R(A;B)_\rho \leq E_R(A;B)_\rho$ following because

$$\mathrm{SEP}(A:B) \subset \mathrm{PPT}(A:B) \subset \mathrm{PPT}'(A:B),$$

Calculating Rains relative entropy

Can be calculated efficiently using Matlab, with CVX, CVXQuad, and QuantInf packages [Wil18]:

```
na = 2; nb = 2;
rho = randRho(na*nb);  % Generate random state

cvx_begin sdp
    variable tau(na*nb,na*nb) hermitian ;
    minimize ( quantum_rel_entr(rho, tau)/ log(2) );
    tau >= 0;
    norm_nuc(Tx(tau, 2, [ na nb ])) <= 1;
cvx_end

rains_rel_ent = cvx_optval;
```

Squashed entanglement...

(Image courtesy of https://levelup.gitconnected.com/quantum-key-distribution-for-everyone-f08dd5646f33)

Mutual information

- **Mutual information** of a bipartite state ρ_{AB} defined as [Str65]

$$I(A;B)_\rho := D(\rho_{AB} \| \rho_A \otimes \rho_B)$$

- As it turns out, this has an equivalent expression

$$I(A;B)_\rho = \inf_{\sigma_A, \tau_B} D(\rho_{AB} \| \sigma_A \otimes \tau_B)$$

- Measures how distinguishable the state ρ_{AB} is from a product state

- This is not useful as a measure of entanglement because it measures all correlations, including classical correlations

Mutual information and separable states

- Suppose that ρ_{AB} is a separable state, so that we can write it as

$$\rho_{AB} = \sum_x p(x) \sigma_A^x \otimes \tau_B^x .$$

- There exists an extension of this state to a classical system X:

$$\omega_{ABX} := \sum_x p(x) \sigma_A^x \otimes \tau_B^x \otimes |x\rangle\langle x|_X$$

- Conditioned on value in classical system, state is product and thus

$$\sum_x p(x) I(A;B)_{\sigma_A^x \otimes \tau_B^x} = 0$$

Conditional mutual information

- **Conditional mutual information** of a tripartite state κ_{ABE} is defined as

$$I(A;B|E)_\kappa := H(AE)_\kappa + H(BE)_\kappa - H(ABE)_\kappa - H(E)_\kappa$$

- **Strong subadditivity** entropy inequality [LR73a, LR73b]:

$$I(A;B|E)_\kappa \geq 0$$

for every tripartite state κ_{ABE}.

Conditional mutual information with classical conditioning

For a classical–quantum state of the form κ_{ABX}, where

$$\kappa_{ABX} := \sum_x p(x) \kappa_{AB}^x \otimes |x\rangle\!\langle x|_X$$

the conditional mutual information evaluates to

$$I(A;B|X)_\kappa = \sum_x p(x) I(A;B)_{\kappa^x}$$

Squashed entanglement with classical extension

- Motivated by this, we could define an entanglement measure for a bipartite state ρ_{AB} as

$$E_{\text{sq},c}(A;B)_\rho := \frac{1}{2} \inf_{\omega_{ABX}} \{ I(A;B|X)_\omega : \text{Tr}_X[\omega_{ABX}] = \rho_{AB} \}$$

  where the extension system X is classical

- It is equal to zero for every separable state, by picking the extension system as we did previously

- It is also a selective LOCC monotone

Squashed entanglement

- **Squashed entanglement** measure for a bipartite state ρ_{AB} defined as

$$E_{sq}(A;B)_\rho := \frac{1}{2}\inf_{\omega_{ABE}} \{I(A;B|E)_\omega : \mathrm{Tr}_E[\omega_{ABE}] = \rho_{AB}\}$$

  where the extension system E is quantum [CW04] (see also [Tuc99, Tuc02])

- Infimum seems to be necessary because no known upper bound on size of quantum system E in the optimization

Properties of squashed entanglement

- Reduces to entanglement entropy for pure states [CW04]
- Selective LOCC monotone [CW04]
- Convex [CW04]
- Additive [CW04]
- Faithful [BCY11, LW18]
- Upper bound on distillable entanglement and lower bound on entanglement cost [CW04]:

$$E_D(A;B)_\rho \leq E_{\mathrm{sq}}(A;B)_\rho \leq E_C(A;B)_\rho$$

Example: Generalized amplitude damping channel (GADC)

Generalized amplitude damping channel (GADC) is a two-parameter family of channels described as follows:

$$\mathcal{A}_{\gamma,N}(\rho) = A_1 \rho A_1^\dagger + A_2 \rho A_2^\dagger + A_3 \rho A_3^\dagger + A_4 \rho A_4^\dagger,$$

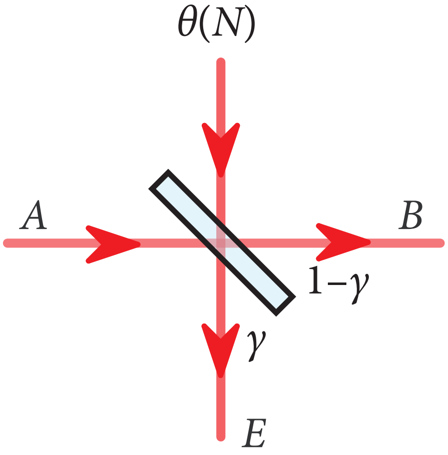
where $\gamma, N \in [0,1]$ and

$$A_1 = \sqrt{1-N}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \qquad A_2 = \sqrt{\gamma(1-N)}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$A_3 = \sqrt{N}\begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_4 = \sqrt{\gamma N}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The parameter γ characterizes loss, and N describes environmental noise.

Physical realization of GADC by beamsplitter interaction



for qubit thermal state $\theta(N) = (1 - N)|0\rangle\langle 0| + N|1\rangle\langle 1|$

Bounds on distillable entanglement of GADC [KSW20]

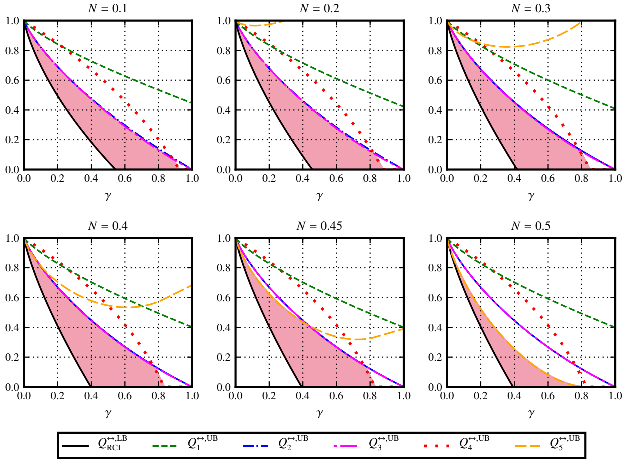


Figure: Distillable entanglement lies in the shaded region of each plot. Squashed entanglement bounds are in blue and magenta. Rains-like bound in gold (from approximately teleportation-simulable channel argument).

Example: Dephased Bell state

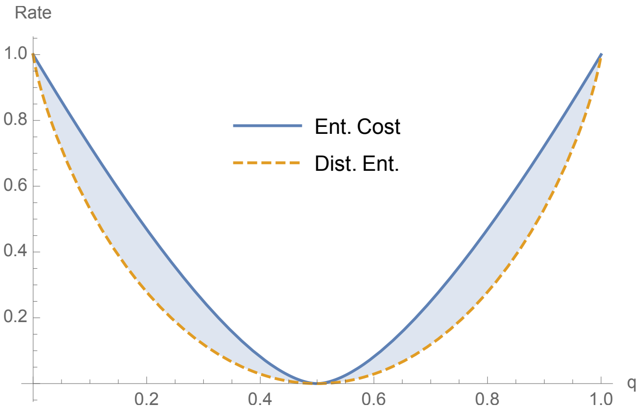


Figure: Plot of distillable entanglement and entanglement cost of the dephased Bell state $(1-q)\Phi_{AB} + qZ_B\Phi_{AB}Z_B$, where $\Phi_{AB} := \frac{1}{2}\sum_{i,j\in\{0,1\}} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B$.

Summary of entanglement measures

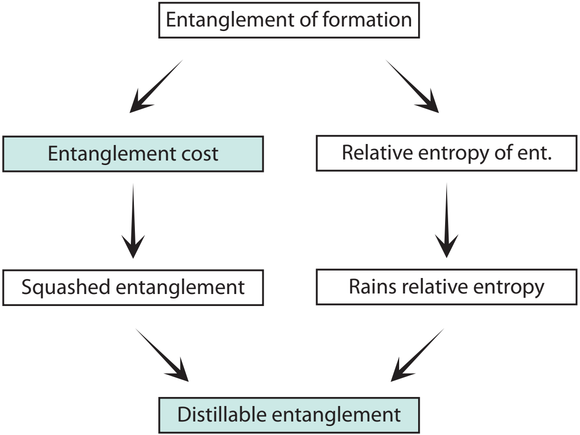


Figure: Arrow $\rightarrow$ indicates $\geq$ and light blue ones are operational measures

Conclusion

- Entanglement is a resource for operational tasks like teleportation, super-dense coding, and quantum key distribution
- Goal of entanglement theory is to quantify entanglement
- Two approaches: axiomatic and operational approaches
- Entanglement measures like entanglement of formation, squashed entanglement, and Rains relative entropy are useful and serve as bounds on operational entanglement measures like distillable entanglement and entanglement cost

Open questions

- Are there other interesting and useful entanglement measures? Efficiently computable and operationally meaningful (See [WW20] for recent progress)
- Find examples of states for which we can calculate squashed entanglement
- Squashed entanglement not even known to be computable. Is it uncomputable?
- What is the relevance of these entanglement measures in other areas of physics?

[ADMVW02] Koenraad Audenaert, Bart De Moor, Karl Gerd H. Vollbrecht, and Reinhard F. Werner. Asymptotic relative entropy of entanglement for orthogonally invariant states. *Physical Review A*, 66(3):032310, September 2002. arXiv:quant-ph/0204143.

[BBC+93] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Physical Review Letters*, 70(13):1895–1899, March 1993.

[BBPS96] Charles H. Bennett, Herbert J. Bernstein, Sandu Popescu, and Benjamin Schumacher. Concentrating partial entanglement by local operations. *Physical Review A*, 53(4):2046–2052, April 1996. arXiv:quant-ph/9511030.

[BCY11] Fernando G.S.L. Brandao, Matthias Christandl, and Jon Yard. Faithful squashed entanglement. *Communications in Mathematical Physics*, 306(3):805–830, September 2011. arXiv:1010.1750.

References II

[BDF+99] Charles H. Bennett, David P. DiVincenzo, Christopher A. Fuchs, Tal Mor, Eric Rains, Peter W. Shor, John A. Smolin, and William K. Wootters. Quantum nonlocality without entanglement. *Physical Review A*, 59(2):1070–1091, February 1999. arXiv:quant-ph/9804053.

[BDSW96] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. *Physical Review A*, 54(5):3824–3851, November 1996. arXiv:quant-ph/9604024.

[Bel64] John Stewart Bell. On the Einstein-Podolsky-Rosen paradox. *Physics*, 1:195–200, 1964.

[BW92] Charles H. Bennett and Stephen J. Wiesner. Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. *Physical Review Letters*, 69(20):2881–2884, November 1992.

[CG19] Eric Chitambar and Gilad Gour. Quantum resource theories. *Reviews of Modern Physics*, 91(2):025001, April 2019. arXiv:1806.06107.

References III

[CW04] Matthias Christandl and Andreas Winter. "Squashed entanglement": An additive entanglement measure. *Journal of Mathematical Physics*, 45(3):829–840, March 2004. arXiv:quant-ph/0308088.

[DL70] Edward B. Davies and J. T. Lewis. An operational approach to quantum probability. *Communications in Mathematical Physics*, 17:239–260, 1970.

[Eke91] Artur K. Ekert. Quantum cryptography based on Bell's theorem. *Physical Review Letters*, 67(6):661–663, August 1991.

[FvdG98] Christopher A. Fuchs and Jeroen van de Graaf. Cryptographic distinguishability measures for quantum mechanical states. *IEEE Transactions on Information Theory*, 45(4):1216–1227, May 1998. arXiv:quant-ph/9712042.

[Gha10] Sevag Gharibian. Strong NP-hardness of the quantum separability problem. *Quantum Information and Computation*, 10(3):343–360, March 2010. arXiv:0810.4507.

References IV

[Gur04] Leonid Gurvits. Classical complexity and quantum entanglement. *Journal of Computer and System Sciences*, 69(3):448–484, 2004. arXiv:quant-ph/0303055.

[Has09] Matthew B. Hastings. Superadditivity of communication capacity using entangled inputs. *Nature Physics*, 5:255–257, April 2009. arXiv:0809.3972.

[Hel67] Carl W. Helstrom. Detection theory and quantum mechanics. *Information and Control*, 10(3):254–291, 1967.

[HHH96] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. Separability of mixed states: necessary and sufficient conditions. *Physics Letters A*, 223(1-2):1–8, November 1996. arXiv:quant-ph/9605038.

[HHHH09] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. Quantum entanglement. *Reviews of Modern Physics*, 81(2):865–942, June 2009. arXiv:quant-ph/0702225.

References V

[HHT01] Patrick M. Hayden, Michal Horodecki, and Barbara M. Terhal. The asymptotic entanglement cost of preparing a quantum state. *Journal of Physics A: Mathematical and General*, 34(35):6891, September 2001. arXiv:quant-ph/0008134.

[Hol72] Alexander S. Holevo. An analogue of statistical decision theory and noncommutative probability theory. *Trudy Moskovskogo Matematicheskogo Obshchestva*, 26:133–149, 1972.

[Hor97] Pawel Horodecki. Separability criterion and inseparable mixed states with positive partial transposition. *Physics Letters A*, 232(5):333–339, August 1997. arXiv:quant-ph/9703004.

[HP91] Fumio Hiai and Dénes Petz. The proper formula for relative entropy and its asymptotics in quantum probability. *Communications in Mathematical Physics*, 143(1):99–114, December 1991.

[Hua14] Yichen Huang. Computing quantum discord is NP-complete. *New Journal of Physics*, 16(3):033027, March 2014. arXiv:1305.5941.

References VI

[KSW20] Sumeet Khatri, Kunal Sharma, and Mark M. Wilde. Information-theoretic aspects of the generalized amplitude damping channel. *Physical Review A*, 102(1):012401, July 2020. arXiv:1903.07747.

[Lin75] Göran Lindblad. Completely positive maps and entropy inequalities. *Communications in Mathematical Physics*, 40(2):147–151, June 1975.

[LR73a] Elliott H. Lieb and Mary Beth Ruskai. A fundamental property of quantum-mechanical entropy. *Physical Review Letters*, 30(10):434–436, March 1973.

[LR73b] Elliott H. Lieb and Mary Beth Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *Journal of Mathematical Physics*, 14:1938–1941, 1973.

[LW18] Ke Li and Andreas Winter. Squashed entanglement, k-extendibility, quantum Markov chains, and recovery maps. *Foundations of Physics*, 48:910–924, February 2018. arXiv:1410.4184.

References VII

[ON00] Tomohiro Ogawa and Hiroshi Nagaoka. Strong converse and Stein's lemma in quantum hypothesis testing. *IEEE Transactions on Information Theory*, 46(7):2428–2433, November 2000. arXiv:quant-ph/9906090.

[Oza84] Masanao Ozawa. Quantum measuring processes of continuous observables. *Journal of Mathematical Physics*, 25(1):79–87, 1984.

[Per96] Asher Peres. Separability criterion for density matrices. *Physical Review Letters*, 77(8):1413–1415, August 1996. arXiv:quant-ph/9604005.

[Ple05] Martin B. Plenio. Logarithmic negativity: A full entanglement monotone that is not convex. *Physical Review Letters*, 95(9):090503, August 2005. arXiv:quant-ph/0505071.

[Rai01] Eric M. Rains. A semidefinite program for distillable entanglement. *IEEE Transactions on Information Theory*, 47(7):2921–2933, November 2001. arXiv:quant-ph/0008047.

[Ras06] Alexey E. Rastegin. Sine distance for quantum states. February 2006. arXiv:quant-ph/0602112.

References VIII

[Sho04] Peter W. Shor. Equivalence of additivity questions in quantum information theory. *Communications in Mathematical Physics*, 246(3):453–472, 2004. arXiv:quant-ph/0305035.

[Sti55] William F. Stinespring. Positive functions on C*-algebras. *Proceedings of the American Mathematical Society*, 6:211–216, 1955.

[Str65] Ruslan L. Stratonovich. Information capacity of a quantum communications channel. i. *Soviet Radiophysics*, 8(1):82–91, January 1965.

[Tuc99] Robert R. Tucci. Quantum entanglement and conditional information transmission. September 1999. arXiv: quant-ph/9909041.

[Tuc02] Robert R. Tucci. Entanglement of distillation and conditional mutual information. Februrary 2002. arXiv:quant-ph/0202144.

[Uhl76] Armin Uhlmann. The “transition probability” in the state space of a *-algebra. *Reports on Mathematical Physics*, 9(2):273–279, April 1976.

References IX

[Ume62] Hisaharu Umegaki. Conditional expectations in an operator algebra IV (entropy and information). *Kodai Mathematical Seminar Reports*, 14(2):59–85, 1962.

[VP98] Vlatko Vedral and Martin B. Plenio. Entanglement measures and purification procedures. *Physical Review A*, 57(3):1619–1633, March 1998. arXiv:quant-ph/9707035.

[VPRK97] Vlatko Vedral, Martin B. Plenio, M. A. Rippin, and Peter L. Knight. Quantifying entanglement. *Physical Review Letters*, 78(12):2275–2279, March 1997. arXiv:quant-ph/9702027.

[VW02] Guifré Vidal and Reinhard F. Werner. Computable measure of entanglement. *Physical Review A*, 65:032314, February 2002. arXiv:quant-ph/0102117.

[Wer89] Reinhard F. Werner. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Physical Review A*, 40(8):4277–4281, October 1989.

References X

[Wil18] Mark M. Wilde. Entanglement cost and quantum channel simulation. *Physical Review A*, 98(4):042338, October 2018. arXiv:1807.11939.

[Wil20] Mark M. Wilde. Relating one-shot distillable entanglement to one-shot entanglement cost. Unpublished note, September 2020.

[WW20] Xin Wang and Mark M. Wilde. Cost of quantum entanglement simplified. *Physical Review Letters*, 125(4):040502, July 2020. arXiv:2007.14270.