Introduction to spin foams and background independent renormalization

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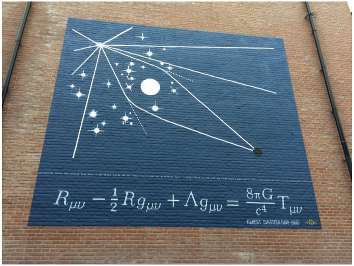
Gravity and quantum matter?

- General relativity $\Longrightarrow$ **dynamical space-time**
  - **Curvature** of space-time $\Longleftrightarrow$ gravitation
  - Matter curves space-time, curvature determines motion of matter
- **Quantum matter** (quantum field theory)
  - Visible matter content of universe
  - Space-time acts as a fixed **"stage"**



- Opposing assumptions, but theories relevant at different scales?

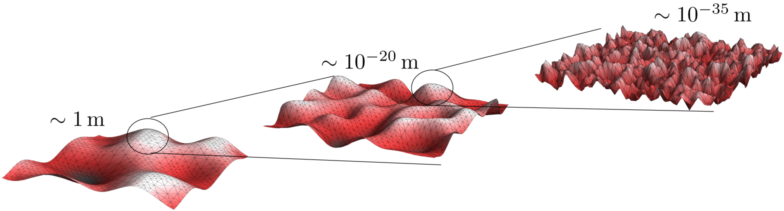
What happens at **smallest length scales?**

Quantum matter on classical space-time **inconsistent**.

Quantum nature of gravity cannot be ignored!

"Zooming in" to the Planck scale

- What is space-time at the **Planck scale** $l_P = \sqrt{\frac{\hbar G}{c^3}}$?

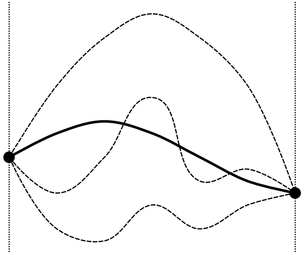


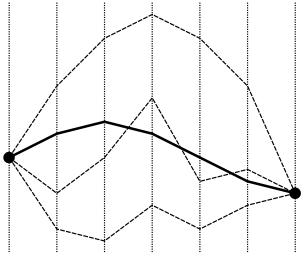
- **Bottom-up approach**: ansatz for UV physics
- Use **renormalization group** as "microscope"
  - "Zoom out" by relating theories at different scales

Replace space-time by a **"quantum space-time"**.

One promising approach: **spin foam quantum gravity**

Path integral for gravity





- Formal **path integral of geometries**

$$Z = \int \mathcal{D}[g] \; e^{\frac{i}{\hbar} S_{\mathrm{EH}}[g]}$$

  - S_{EH}: Einstein-Hilbert action
  - $\mathcal{D}[g]$: Measure on space of geometries

- Regulator: **Lattice**
  - **Diffeomorphism symmetry?**
  - **Continuum limit?**
  - Renormalizability?

Spin foam quantum gravity:

Background independent, **non-perturbative** quantization of gravity

Outline

Riemannian continuum gravity in 3D

- **Reviews** on spin foams and BF theory [Baez '99, Perez '12]
- Compact, orientable 3D **Riemannian** manifold $\mathcal{M}$
- 3D gravity described by **(topological)** SU(2) **BF theory**:

$$S[B,\omega] = \int_{\mathcal{M}} \mathrm{Tr}[B \wedge F(\omega)]$$

  - B $\mathfrak{su}(2)$ valued 1-form, F curvature 2-form of connection ω
  - ω connection on SU(2) principal bundle

- **Equations of motion**:

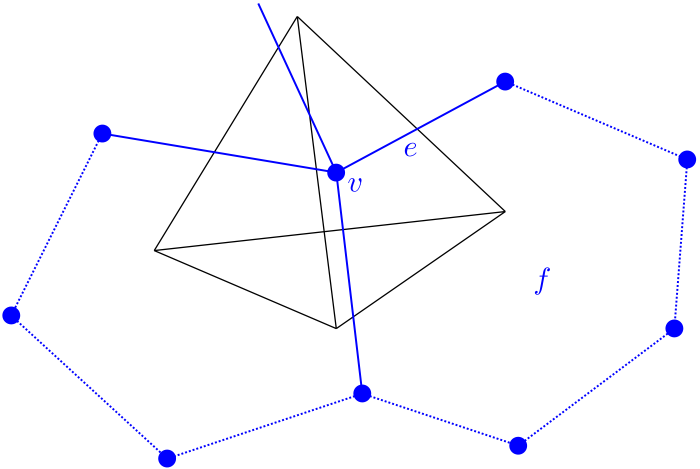
$$F(\omega) = 0 \quad \text{and} \quad d_\omega B = 0$$

- Path integral:

$$Z = \int \mathcal{D}[B]\mathcal{D}[\omega] \, e^{i \int_{\mathcal{M}} \mathrm{Tr}[B \wedge F(\omega)]} \text{ ``} = \int \mathcal{D}[\omega] \, \delta(F(\omega))\text{''}$$

Triangulation and its dual complex

- **Triangulation** Δ (of $\mathcal{M}$) and its **dual** Δ^*
  - $v \in \Delta^* \leftrightarrow \tau \in \Delta$
  - $e \in \Delta^* \leftrightarrow T \in \Delta$
  - $f \in \Delta^* \leftrightarrow E \in \Delta$



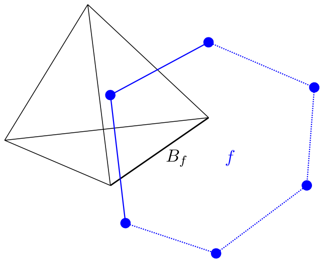
Discretizing BF theory

- **Discretize** B-field:
  - $(d-2)$-dim face $f \in \Delta^*$

$$B_f := \int_{\text{1-cell}} B$$

  - B-field smeared over an edge

Edge vector

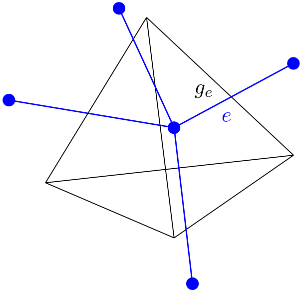


- **Discretize** connection ω:
  - 1-dim edge $e \in \Delta^*$

$$g_e := \mathcal{P}\exp\left(-\int_e \omega\right)$$

  - $g_e \in \mathrm{SU}(2)$: parallel transport along e

Parallel transport to another tetrahedron

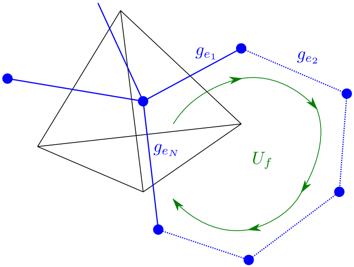


Discretized curvature and partition function

- **Discretize** curvature $F(\omega)$:
  - Holonomy around **closed loop** $f \in \Delta^*$:

$$U_f := \vec{\prod}_{e \subset f} g_e$$

  - Matching orienation assumend



- **Partition function**:

$$Z(\Delta) = \int \prod_{e \in \Delta^*} dg_e \prod_{f \in \Delta^*} dB_f \, e^{iB_f U_f} = \int \prod_{e \in \Delta^*} dg_e \prod_{f \in \Delta^*} \delta\left(\vec{\prod}_{e \subset f} g_e\right)$$

Curvature measured around edges $E \in \Delta$.

Flatness imposed: closed holonomies must be equal to identity.

Toward the spin foam representation

- **Peter-Weyl's theorem:** $\delta(g) = \sum_j d_j \,\mathrm{Tr}[\, D^j(g)]$
  - $j \in \frac{\mathbb{N}}{2}$: labels irrep of SU(2)
  - D^j_{mn}: Wigner-D matrix for irrep j
- **Expand** each δ-function in Z:
  - Irrep j_f per face $f \in \Delta^*$
  - Use $D^j(g_1\, g_2) = D^j(g_1) D^j(g_2)$

$$Z(\Delta) = \sum_{j_f} \int \prod_{e \in \Delta^*} dg_e \prod_{f \in \Delta^*} d_{j_f} \,\mathrm{Tr}[D^{j_f}(g_{e_1})\, D^{j_f}(g_{e_2}) \,\ldots\, D^{j_f}(g_{e_N})]$$

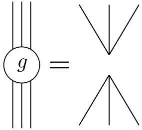
$$= \sum_{j_f} \prod_{f \in \Delta^*} d_{j_f} \mathrm{Tr} \underbrace{\left[\int \prod_{e \in \Delta^*} dg_e \bigotimes_{f \supset e} D^{j_f}(g_e) \right]}_{\mathbf{P}_{\text{Haar}}(\{j_f\})}$$

$\mathbf{P}_{\text{Haar}}$: **Haar-projector** onto **invariant subspace** $\mathrm{Inv}(\bigotimes_f j_f)$.

Towards spin networks

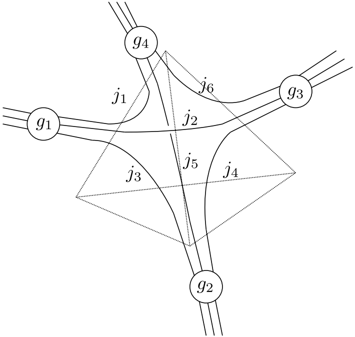
- 3D triangulation: $e \in \Delta^*$ is 3-valent
  - Triangle is bound by 3 edges
  - $e \in \Delta^*$ part of 3 faces $f \in \Delta^*$

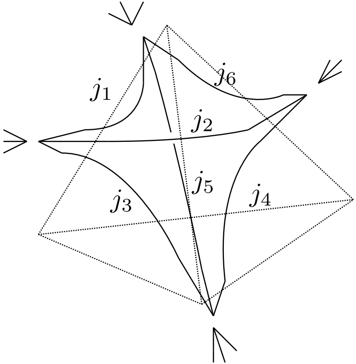
$$\mathbf{P}_{\mathrm{Haar}} = \int dg\, D^{j_{f_1}}(g) \otimes D^{j_{f_2}}(g) \otimes D^{j_{f_3}}(g) = $$



$$= |\iota\rangle\langle\iota|$$

  - Intertwiner ι: normalized unique vector in $\mathrm{Inv}(j_1 \otimes j_2 \otimes j_3)$

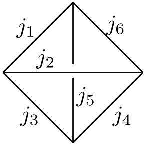




Ponzano-Regge model

- **Ingredients**:
  - Irrep j_f: length of edge E (dual to f)
  - Intertwiner ι_e: "shape" of triangle (dual to e)
- Final form of **partition function**:

$$Z(\Delta) = \sum_{j_f} \prod_{f\in\Delta^*} d_{j_f} \prod_{e\in\Delta^*} (-1)^{\cdots} \prod_{v\in\Delta^*}$$

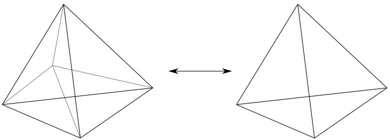


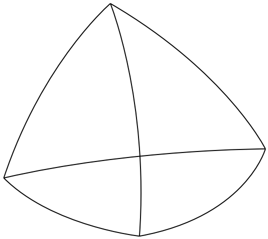
- Local assignment of amplitudes
  - **Vertex amplitude**: evaluation of spin network function
  - $\{6j\}$-symbol: "quantum tetrahedron"

Sum over **quantum geometric building blocks**

Derived from (constrained) **topological quantum field theory**

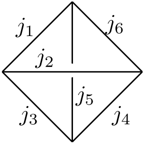
Properties of Ponzano-Regge model





- **Triangulation independent**
  - Invariant under Pachner moves
  - Discrete remnant of **diffeomorphism symmetry** [Freidel, Louapre '02]
- **Turaev-Viro** model: $\Lambda > 0$ [Turaev, Viro '92]
  - Quantum deformed $\mathrm{SU}(2)_\mathrm{k}$
  - $j_{\mathrm{max}} = \frac{\mathrm{k}}{2}$
  - $\Lambda = \frac{1}{G^2 \hbar^2 \mathrm{k}^2}$
- **Asymptotic expansion** of vertex amplitude

  [Ponzano, Regge '68]



$$\sim \frac{1}{\sqrt{12\pi V}} \cos\left(S_\mathrm{R} + \frac{\pi}{4}\right)$$

S_R: Regge / **discrete gravity action** for a single tetrahedron

Outline

Plebanski-Holst action

- 4D gravity as a **constrained BF theory**:

$$S[B,\omega,\lambda] = \frac{1}{\kappa}\int_{\mathcal{M}} \left[\left(*B^{IJ} + \frac{1}{\gamma}B^{IJ}\right)\wedge F_{IJ}(\omega) + \lambda_{IJKL}\, B^{IJ}\wedge B^{KL}\right]$$

  - $\lambda_{IJKL} = -\lambda_{JIKL} = -\lambda_{IJLK} = \lambda_{KLIJ}$ Lagrange multipliers
  - $\rightarrow B = \pm * (e\wedge e)$ or $B = \pm e\wedge e$
  - γ: Immirzi parameter
- Compare to **BF Yang-Mills**:

$$S_{\text{BFYM}}[B,A] = \int_{\mathcal{M}} \text{Tr}(B\wedge F(A)) + g^2\int_{\mathcal{M}} \text{Tr}(B\wedge \star B)$$

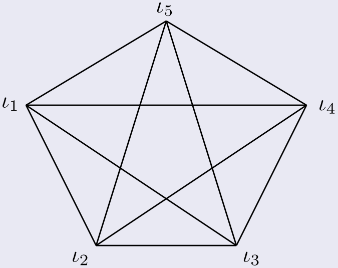
BF theory corresponds to Yang-Mills theory in **weak coupling limit**

Understand spin foam models as **generalized lattice gauge theories**

4D spin foam strategy

- As before, **discretize and quantise** $\mathrm{SU}(2) \times \mathrm{SU}(2)$ BF theory
  - B_f associated to faces $f \in \Delta^*$ (dual to triangles)
  - g_e associated to edges $e \in \Delta^*$ (dual to tetrahedra)

$$Z_{\mathrm{BF}}(\Delta) = \sum_{j_f, \iota_e} \prod_{f \in \Delta^*} d_{j_f} \prod_{e \in \Delta^*} ||\iota_e||^{-1} \prod_{v \in \Delta^*}$$



- Interpretation of variables:
  - j_f: **area of triangle** (dual to f)
  - ι_e: **shape of tetrahedron** (dual to e)

Different 4D spin foam models

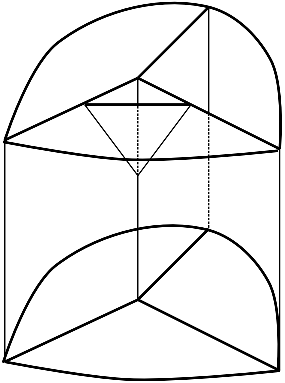
- Implement **simplicity constraints** to break **topological symmetry**
  - Restrictions on irreps (j^+, j^-) and intertwiners ι
- **Barrett-Crane model** [Barrett, Crane '98]
  - Simple bivector $\rightarrow j^+ = j^-$
- **EPRL-FK model** [Engle, Pereira, Rovelli, Livine '08, Freidel, Krasnov '08]
  - Weak imposition of constraints (in discrete):

$$j^{\pm} = \frac{1}{2} |1 \pm \gamma| \, j$$

  - Contact with **LQG Hilbert space** (spin networks)

Simplicity constraints: **restrictions on representations and intertwiners**

Spin foams: interpretation and results



- **Amplitude functional** $\mathcal{A} : \mathcal{H}_b \to \mathbb{C}$
  - $\mathcal{H}_b$: boundary Hilbert space on a discretisation b
- **Transition amplitude** for $\mathcal{H}_b = \mathcal{H}_i \otimes \mathcal{H}_f^*$
  - $\langle s' | s \rangle_{\mathcal{A}} = \mathcal{A}(s \otimes s')$
  - Evolution of 3D geometries
- **Asymptotic expansion** of vertex amplitude

  [Conrady, Freidel '08, Barrett, Dowdall, Fairbarn, Gomes, Hellmann '09]

$$\mathcal{A}_v \sim \cos(S_{\mathrm{R}}) + \dots$$

- Implementation of **Cosmological Constant**

  [Han '11, Fairbarn, Meusburger '11, Haggard, Han, Kaminski, Riello '15]

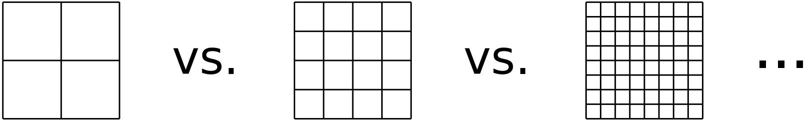
At discrete level: relation to **classical gravity**

Can we go **beyond** a **few simplices**?

Outline

Why we need renormalization

- Definition **not unique**
- **Lattice** is a **choice**.
  - Should the lattice be "fine" or "coarse"?
  - Can we **relate** the **physics** on different lattices?
  - **Diffeomorphism symmetry?** [Bahr, S.St. '15]



Physics must be **independent** of choice of regulator!

- How do we **bridge** the enormous **gap** to **observable physics**?
- Do we recover **general relativity** in a suitable limit?

Relating theories across discretisations

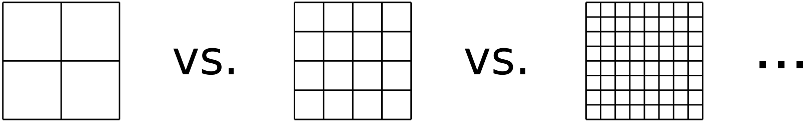
- **Wilsonian renormalization** in **absence of a scale**
  - No background geometry
- Use the **lattice** itself as a **relative scale**
  - “Fine” vs. “coarse” degrees of freedom
- **Coarse grain** by summing over “fine” degrees of freedom
  - E.g. decimation procedure
  - **Regge calculus** for $\Lambda \neq 0$ in 3D [Bahr, Dittrich ’09]
- **Interactiong** theories: **non-local** interactions arise
  - Example: Ising model in 2D
  - Difficult to iterate

Allow for **more general boundary data**

Complex building blocks that **interact locally**

Consistent boundary formulation I [Dittrich, S.St. '14]

- Recall: **amplitude functionals** $\mathcal{A}_b : \mathcal{H}_b \to \mathbb{C}$
  - Simplest example: single simplex
  - Different boundaries b, b' $\to$ different $\mathcal{H}_b$, $\mathcal{H}_{b'}$
- How can we **compare** $\mathcal{A}_b$ and $\mathcal{A}'_{b'}$?

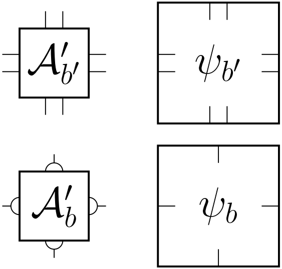


- Idea: **relate states** in $\mathcal{H}_b$ to states $\mathcal{H}_{b'}$
  - Embedding maps: $\iota_{b'b} : \mathcal{H}_b \to \mathcal{H}_{b'}$
  - Boundaries partially ordered set $b \prec b'$

Represent the **same state** on **different Hilbert spaces**

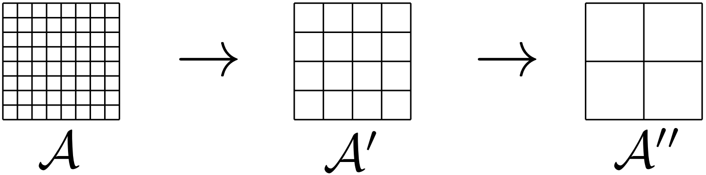
$$\mathcal{H}_b \ni \psi_b \to \iota_{b'b}(\psi_b) \in \mathcal{H}_{b'}$$

Consistent boundary formulation II [Dittrich, S.St. '14]



- Compare amplitude functionals
  - $\mathcal{A}_b : \mathcal{H}_b \to \mathbb{C}$
  - $\mathcal{A}'_{b'} : \mathcal{H}_{b'} \to \mathbb{C}$
- **Renormalization group equation**

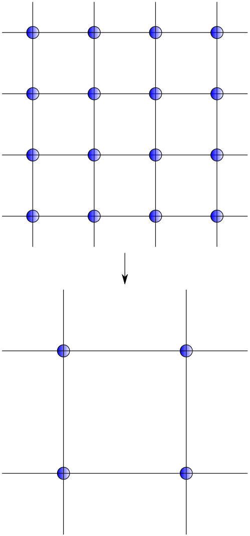
$$\mathcal{A}'_b(\psi_b) := \mathcal{A}'_{b'}(\iota_{b'b}(\psi_b))$$

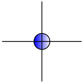


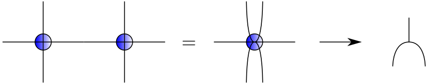
Assign **different amplitudes** to **different lattices**

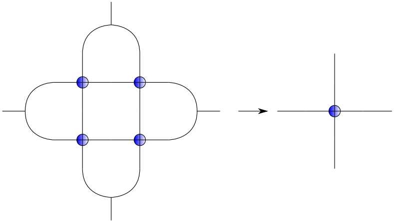
Renormalization group flow of spin foam amplitudes

Tensor network renormalization [Levin, Nave '07]



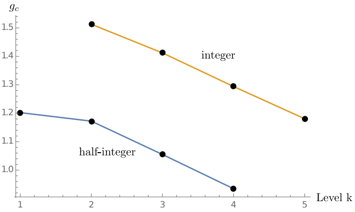
- Partition function as **tensor network**
  -  $\leftrightarrow$ local amplitude
- **Dynamical embedding map** from tensors
  - Singular value decomposition
  - **Local network**

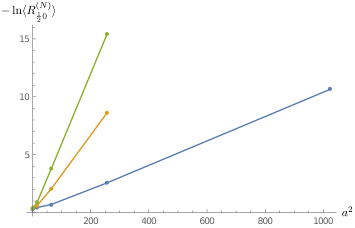




Study **flow of tensors** under coarse graining

Results from Tensor networks



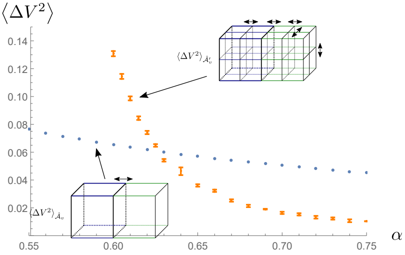


- **Numerical algorithm**
  - Works for **oscillating amplitudes**
  - Finite dimensional Hilbert spaces
- 2D analogue models for $\mathrm{SU}(2)_\mathrm{k}$ [Dittrich, Martin-Benito, S.St. '13, Cameron, Dittrich, Schnetter, S.St. '16]
- 3D lattice gauge theories for $\mathbb{Z}_2$ and S_3 [Dittrich, Mizera, S.St. '14, Delcamp, Dittrich '16]
- **Fusion basis** for 3D $\mathrm{SU}(2)_\mathrm{k}$ **lattice gauge theory** [Dittrich, Geiller '16, Cunningham, Dittrich, S.St. '20]
  - Phase transition: **confining** and **deconfing** phase
  - **Critical** g_c **drops** as we increase k.
  - Expectation values of Wilson loops

Hint towards **no deconfining phase** for **continuous group** SU(2).

Results from 4D restricted spin foam models

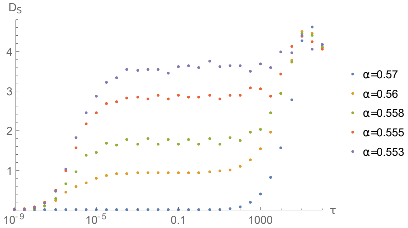
- Complementary route: Study **restricted path integral** [Bahr, S.St. '15, Bahr, Klöser, Rabuffo '17]



- **Proof of principle** [Bahr, S.St. PRL '16]
  - 4D restricted spin foam model
  - Several simplifications / assumptions

> Indication for a **UV-attractive fixed point** and restored **diffeomorphism symmetry**

- **Diffusion process** on space-time
  - **Spectral dimension** D_S of restricted spin foam [S.St., Thürigen '18]
  - $D_S \leq 4$ due to **superposition of geometries**



Outline

Summary

- **Spin foam quantisation strategy**
  - Discretisation and quantisation of **topological BF theory**
  - Derivation of spin foam representation
  - **4D** spin foam models: **simplicity constraint**
  - **Quantum geometric building blocks**
- **Background independent renormalization**
  - Tackle open questions (**diffeomorphism symmetry**, discretisation dependence,...)
  - Assign a **family of amplitudes** across lattices
- Example: **Tensor network renormalization**
  - **Fusion basis** for 3D lattice gauge theories

Promising results in this line of research

Intriguing open questions remains

Outlook

- **Renormalization**
  - Apply **tensor networks** (or simplified algorithms) to **4D spin foams**
  - Which other numerical methods useful (Monte Carlo)?
- **Numerical methods**
  - Develop algorithms to compute spin foam amplitudes
  - **Hybrid algorithm**, using semi-classical results
- **Matter and observables**
  - **Couple matter** to spin foams
  - **Renormalize both gravity and matter**
  - Geometric observables (spectral dimension, curvature)

Thank you for your attention!