# The Operator Product Expansion as a structural property of Quantum Field Theories 

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1 Composite operators in Quantum Field Theory (QFT)

2 The Operator Product Expansion

3 Euclidean field theories: the Renormalisation Group (RG) Flow Equation framework

4 Euclidean field theories: OPE in the RG Flow Equation framework

5 Application to Conformal Field Theories (CFTs)
6 Outlook: The OPE in curved spacetimes and application in the AdS/CFT correspondence

## Composite operators in Quantum Field Theory (QFT)

Composite operators in QFT 1/3

- Quantities of interest: time-ordered correlation functions in the interacting theory $\left\langle\mathcal{T}_{I} \phi\left(x_{1}\right) \cdots \phi\left(x_{n}\right)\right\rangle_{\Omega}=\frac{\left\langle\mathcal{T} \phi\left(x_{1}\right) \cdots \phi\left(x_{n}\right) \exp (i L / \hbar)\right\rangle_{0}}{\langle\mathcal{T} \exp (i L / \hbar)\rangle_{0}}$ (Gell-Mann-Low formula with
$\Omega$ : interacting vacuum, 0 : free vacuum, $L$ : interaction, $\mathcal{T}$ : time-ordering)
■ Finite results after renormalisation of parameters in the Lagrangian: masses $m$, couplings $\lambda$, wave-functions $Z$ : renormalised correlation functions are well-defined distributions
- However, coinciding point limit is singular:
$\left\langle\phi^{2}(x) \phi\left(y_{1}\right) \cdots \phi\left(y_{n}\right)\right\rangle_{0}=\lim _{x^{\prime} \rightarrow x}\left\langle\phi(x) \phi\left(x^{\prime}\right) \phi\left(y_{1}\right) \cdots \phi\left(y_{n}\right)\right\rangle_{0}=\infty!$
- In free theory, obtain finite composite operators by normal ordering:
$: \phi(x) \phi\left(x^{\prime}\right): 0 \equiv \phi(x) \phi\left(x^{\prime}\right)-\left\langle\phi(x) \phi\left(x^{\prime}\right)\right\rangle_{0} \mathbb{1}$
■ $\left\langle: \phi^{2}(x):_{0} \phi\left(y_{1}\right) \cdots \phi\left(y_{n}\right)\right\rangle_{0}=\lim _{x^{\prime} \rightarrow x}\left\langle: \phi(x) \phi\left(x^{\prime}\right): 0 \phi\left(y_{1}\right) \cdots \phi\left(y_{n}\right)\right\rangle_{0}<\infty$

Composite operators in QFT 2/3
■ In flat Minkowski spacetime, unique Poincaré-invariant vacuum state 0
■ In curved spacetime, only family of states $\omega$ with the same high-energy / short-distance behaviour as in flat space (Hadamard states)

- Characterised by properties of two-point function: $G_{\omega}^{+}\left(x, x^{\prime}\right) \equiv-\mathrm{i}\left\langle\phi(x) \phi\left(x^{\prime}\right)\right\rangle_{\omega}$
- For $x, x^{\prime}$ in a normal geodesic neighborhood: Hadamard expansion (in 4D) $G_{\omega}^{+}\left(x, x^{\prime}\right)=-\mathrm{i}\left[H^{+}\left(x, x^{\prime}\right)+W_{\omega}\left(x, x^{\prime}\right)\right]$ with $H^{+}\left(x, x^{\prime}\right)=\frac{U\left(x, x^{\prime}\right)}{\sigma^{+}\left(x, x^{\prime}\right)}+V\left(x, x^{\prime}\right) \ln \sigma^{+}\left(x, x^{\prime}\right)$ and $\sigma^{+}\left(x, x^{\prime}\right)=\sigma\left(x, x^{\prime}\right)+\mathrm{i} \epsilon t\left(x, x^{\prime}\right)$, where $\sigma\left(x, x^{\prime}\right)$ : half of square of geodesic distance between $x$ and $x^{\prime}, t$ : time function, $U / V / W_{\omega}$ : smooth/analytic biscalars
- $U$ and $V$ (singular part) are uniquely determined by the geometry of the spacetime, $W_{\omega}$ (regular part) depends on the state, in higher dimensions more singular terms
- Global alternative: microlocal spectrum/wave front set condition ( $G^{+}$is singular only if $x$ and $x^{\prime}$ are connected by lightlike geodesic and cotangent vectors are future-pointing)

Composite operators in QFT 3/3

■ Normal ordering can be done with respect to any Hadamard state:

$$
: \phi(x) \phi\left(x^{\prime}\right): \omega \equiv \phi(x) \phi\left(x^{\prime}\right)-\left\langle\phi(x) \phi\left(x^{\prime}\right)\right\rangle_{\omega} \mathbb{1}
$$

■ Since state is not unique, : $\bullet:_{\omega}$ does not transform covariantly under coordinate transformations $\Rightarrow$ normal order w.r.t. singular part only (Hadamard parametrix) $: \phi(x) \phi\left(x^{\prime}\right): H \equiv \phi(x) \phi\left(x^{\prime}\right)-H\left(x, x^{\prime}\right) \mathbb{1}$

- To define composite operators with derivatives, apply them before taking coincidence limit (point-splitting method)
■ Example: $T_{\mu \nu}(x)=\lim _{x^{\prime} \rightarrow x}\left[\left(\nabla_{\mu}^{x} \nabla_{\nu}^{x^{\prime}}-\frac{1}{2} g_{\mu \nu}(x) \nabla_{\alpha}^{x} \nabla_{x^{\prime}}^{\alpha}-\frac{1}{2} g_{\mu \nu}(x) m^{2}\right): \phi(x) \phi\left(x^{\prime}\right): H\right]$
- Renormalisation freedom is restricted by covariance requirement, example: $: \phi^{2}(x):_{H} \rightarrow: \phi^{2}(x):_{H}+\left(c_{1} m^{2}+c_{2} R\right) \mathbb{1}$ with $c_{1}, c_{2}$ constant
- Anomalies arise from terms which vanish classically by the EOM, but not after subtraction $\rightarrow$ talk by S . Theisen tomorrow morning


## The Operator Product Expansion

## The Operator Product Expansion 1/3

■ Rearrange terms to get $\phi(x) \phi\left(x^{\prime}\right)=H\left(x, x^{\prime}\right) \mathbb{1}+: \phi(x) \phi\left(x^{\prime}\right): H$, and instead of taking the limit $x^{\prime} \rightarrow x$, expand (formally) around $x^{\prime}=x$

$$
\begin{aligned}
\phi(x) \phi\left(x^{\prime}\right)= & H\left(x, x^{\prime}\right) \mathbb{1}+: \phi^{2}(x): H+\left(x^{\prime}-x\right)^{\mu}: \phi(x) \nabla_{\mu} \phi(x): H \\
& +\frac{1}{2}\left(x^{\prime}-x\right)^{\mu}\left(x^{\prime}-x\right)^{\nu}: \phi(x) \nabla_{\mu} \nabla_{\nu} \phi(x): H+\ldots
\end{aligned}
$$

■ Family of composite operators multiplied by (possibly singular) spacetime distributions
■ Free theory OPE: $\phi(x) \phi\left(x^{\prime}\right)=\sum_{B} \mathcal{C}_{\phi \phi}^{B}\left(x, x^{\prime}\right): \mathcal{O}_{B}(x): H$ with $\mathcal{C}_{\phi \phi}^{\mathbb{I}}\left(x, x^{\prime}\right)=H\left(x, x^{\prime}\right)$, $\mathcal{C}_{\phi \phi}^{\phi}\left(x, x^{\prime}\right)=0, \mathcal{C}_{\phi \phi}^{\nabla_{\mu} \phi}\left(x, x^{\prime}\right)=0, \mathcal{C}_{\phi \phi}^{\phi^{2}}\left(x, x^{\prime}\right)=1, \mathcal{C}_{\phi \phi}^{\phi \nabla_{\mu} \phi}\left(x, x^{\prime}\right)=\left(x^{\prime}-x\right)^{\mu}, \ldots$
■ Sum over all composite operators, ordered by dimension: $[\phi]=(D-2) / 2,[\nabla]=1$

- Scaling degree of $\mathcal{C}_{\phi \phi}^{B}$ is $D-2-\left[\mathcal{O}_{B}\right]$ (determines how $\operatorname{singular} \mathcal{C}_{\phi \phi}^{B}\left(x, x^{\prime}\right)$ is as $x^{\prime} \rightarrow x$ )

The Operator Product Expansion 2/3
■ In general, OPE for product of arbitrary composite operators

$$
: \mathcal{O}_{A_{1}}\left(x_{1}\right): H \cdots: \mathcal{O}_{A_{n}}\left(x_{n}\right): H=\sum_{B} \mathcal{C}_{A_{1} \cdots A_{n}}^{B}\left(x_{1}, \ldots, x_{n}\right): \mathcal{O}_{B}\left(x_{n}\right): H
$$

$■$ Each OPE coefficient $\mathcal{C}_{A_{1} \ldots A_{n}}^{B}$ has scaling degree $\left[\mathcal{O}_{A_{1}}\right]+\cdots+\left[\mathcal{O}_{A_{n}}\right]-\left[\mathcal{O}_{B}\right]$

- Conventional wisdom says that OPE is an asymptotic relation:

$$
\begin{aligned}
& \lim _{\tau \rightarrow 0} \tau^{\left[\mathcal{O}_{A_{1}}\right]+\cdots+\left[\mathcal{O}_{A_{n}}\right]-\Delta}\left[\left\langle: \mathcal{O}_{A_{1}}\left(\tau x_{1}\right): H \cdots: \mathcal{O}_{A_{n}}\left(\tau x_{n}\right): H\right\rangle_{\omega}\right. \\
&\left.-\sum_{B:\left[\mathcal{O}_{B}\right] \leq \Delta} \mathcal{C}_{A_{1} \cdots A_{n}}^{B}\left(\tau x_{1}, \ldots, \tau x_{n}\right)\left\langle: \mathcal{O}_{C}\left(\tau x_{n}\right): H\right\rangle_{\omega}\right]=0
\end{aligned}
$$

in any Hadamard state $\omega$, as we scale the points together

- Coefficients $\mathcal{C}_{A_{1} \ldots A_{n}}^{B}$ are independent of state, one-point functions determine state
- Analogy to Lie algebra relation: $\left[\mathfrak{t}_{A}, \mathfrak{t}_{B}\right]=f_{A B}{ }^{{ }^{t}}{ }_{C}$, where the explicit expressions for the generators $\mathfrak{t}_{A}$ depend on representation, but structure constants $f_{A B}{ }^{C}$ do not

The Operator Product Expansion 3/3

■ Factorisation condition from performing an OPE in two different ways:

$$
\begin{aligned}
& \left\langle: \mathcal{O}_{A}(x): H: \mathcal{O}_{B}(y): н: \mathcal{O}_{C}(z): H\right\rangle_{\omega} \\
& =\sum_{E} \mathcal{C}_{A B C}^{E}(x, y, z)\left\langle: \mathcal{O}_{E}(z): H\right\rangle_{\omega} \\
& =\sum_{E} \sum_{D} \mathcal{C}_{A B}^{D}(x, y) \mathcal{C}_{D C}^{E}(y, z)\left\langle: \mathcal{O}_{E}(z): H\right\rangle_{\omega} \\
& =\sum_{E} \sum_{D} \mathcal{C}_{B C}^{D}(y, z) \mathcal{C}_{A D}^{E}(x, z)\left\langle: \mathcal{O}_{E}(z): H\right\rangle_{\omega} \text { for suitable } x, y, z
\end{aligned}
$$

■ Commutation with derivatives: $\nabla_{\mu}^{x_{1}} \mathcal{C}_{A_{1} \cdots A_{n}}^{B}\left(x_{1}, \ldots, x_{n}\right)=\mathcal{C}_{\left(\nabla A_{1}\right) \ldots A_{n}}^{B}\left(x_{1}, \ldots, x_{n}\right)$
■ Further algebraic relations between coefficients stemming from the underlying theory/perturbation, example: interacting field equation $\left(\nabla_{x}^{2}-m^{2}\right) \mathcal{C}_{\phi A_{1} \cdots A_{n}}^{B}\left(x, x_{1}, \ldots, x_{n}\right)=\lambda \mathcal{C}_{\phi^{3} A_{1} \cdots A_{n}}^{B}\left(x, x_{1}, \ldots, x_{n}\right)+$ contact terms

- Up to now, all that was said was for free quantum field theory, and we need to generalise to (perturbatively) interacting theory


## Euclidean field theories: the Renormalisation Group (RG) Flow Equation framework

The RG flow equation framework $1 / 4$

■ Concrete calculation of $\left\langle\mathcal{T}_{1} \mathcal{O}_{A_{1}} \cdots \mathcal{O}_{A_{n}} \phi_{1} \cdots \phi_{k}\right\rangle_{\Omega}$ via regularised path integral

- Define $L^{\Lambda, \Lambda_{0}}\left(\left\{\mathcal{O}_{A_{i}}\right\} ; \Phi\right) \equiv-\hbar \ln \int \exp \left(-\frac{1}{\hbar} \phi G_{\Lambda, \Lambda_{0}}^{-1} \phi-\frac{1}{\hbar} L(\phi+\Phi)\right) \prod_{i=1}^{n} \mathcal{O}_{A_{i}}\left(\phi+\Phi, x_{i}\right) \mathcal{D} \phi$
- $L^{\wedge, \Lambda_{0}}$ is the generating functional of connected, amputated correlation functions with insertions of composite operators: $L^{\wedge, \Lambda_{0}}\left(\left\{\mathcal{O}_{A_{i}}\right\} ; \Phi\right)=\sum_{k} \Phi^{k}\left\langle\mathcal{O}_{A_{1}} \cdots \mathcal{O}_{A_{n}} \phi_{1} \cdots \phi_{k}\right\rangle_{c, a}$
- Formal perturbation parameter $\hbar$ (loop counting parameter)
- L is the perturbation (example: $\lambda \phi^{4}$ ) plus counterterms needed for renormalisation (example: $\hbar \lambda \Lambda_{0}^{2} \phi^{2}$ )
- $\mathcal{O}_{A_{i}}$ is the classical expression for to the operator $\mathcal{O}_{A_{i}}$ plus counterterms - $G^{\Lambda, \Lambda_{0}}(k)=\left(k^{2}+m^{2}\right)^{-1}\left[e^{-\frac{k^{2}+m^{2}}{\Lambda_{0}^{2}}}-e^{-\frac{k^{2}+m^{2}}{\Lambda^{2}}}\right]$ is a regularised propagator with infrared cutoff $\Lambda$ and ultraviolet cutoff $\Lambda_{0}$, physical limit is $\Lambda_{0} \rightarrow \infty, \Lambda \rightarrow 0$

The RG flow equation framework $2 / 4$

- $L^{\wedge, \Lambda_{0}}=0$ for free theories $\rightarrow$ measures non-triviality
- Take derivative w.r.t. $\Lambda$ and obtain flow equations: $\partial_{\Lambda} L^{\Lambda, \Lambda_{0}}=\frac{\hbar}{2} \frac{\delta}{\delta \Phi} \cdot\left(\partial_{\Lambda} C^{\Lambda, \Lambda_{0}}\right) \cdot \frac{\delta}{\delta \Phi} L^{\Lambda, \Lambda_{0}}-\frac{1}{2}\left(\frac{\delta}{\delta \Phi} L^{\Lambda, \Lambda_{0}}\right) \cdot\left(\partial_{\Lambda} C^{\Lambda, \Lambda_{0}}\right) \cdot\left(\frac{\delta}{\delta \Phi} L^{\Lambda, \Lambda_{0}}\right)$
- $\partial_{\Lambda} L^{\Lambda, \Lambda_{0}}\left(\mathcal{O}_{A}\right)=\frac{\hbar}{2} \frac{\delta}{\delta \Phi} \cdot\left(\partial_{\Lambda} C^{\Lambda, \Lambda_{0}}\right) \cdot \frac{\delta}{\delta \Phi} L^{\Lambda, \Lambda_{0}}\left(\mathcal{O}_{A}\right)$

$$
-\left(\frac{\delta}{\delta \Phi} L^{\Lambda, \Lambda_{0}}\right) \cdot\left(\partial_{\Lambda} C^{\Lambda, \Lambda_{0}}\right) \cdot\left(\frac{\delta}{\delta \Phi} L^{\Lambda, \Lambda_{0}}\left(\mathcal{O}_{A}\right)\right), \text { etc. }
$$

- Boundary conditions: for $\Lambda=\Lambda_{0}$ we recover "bare" theory: $L^{\Lambda_{0}, \Lambda_{0}}=L$
- Formal expansion in $\hbar$ (perturbation theory, loop order $\ell$ ) and number $n$ of $\phi$ fields gives $\mathcal{L}_{n}^{\wedge, \Lambda_{0}, \ell}(\mathbf{k})$ in momentum space
- No Feynman diagrams, no forest formula

The RG flow equation framework 3/4

- Flow equation:

$$
\begin{aligned}
& \partial_{\Lambda} \mathcal{L}_{n}^{\Lambda, \Lambda_{0}, \ell}(\mathbf{k})=\frac{1}{2} \int\left(\partial_{\Lambda} C^{\Lambda, \Lambda_{0}}(p)\right) \mathcal{L}_{n+2}^{\Lambda, \Lambda_{0}, \ell-1}(\mathbf{k}, p,-p) \frac{d^{4} p}{(2 \pi)^{4}} \\
& -\frac{1}{2} \sum_{n^{\prime}=0}^{n} \sum_{\ell^{\prime}=0}^{\ell} \mathcal{L}_{n^{\prime}+1}^{\Lambda, \Lambda_{0}, \ell^{\prime}}(\mathbf{k},-q)\left(\partial_{\Lambda} C^{\Lambda, \Lambda_{0}}(q)\right) \mathcal{L}_{n-n^{\prime}+1}^{\Lambda, \Lambda_{0}, \ell-\ell^{\prime}}(q, \mathbf{k})
\end{aligned}
$$

- Suitable for induction: right-hand side either lower $\ell$ or lower $n$ since $\mathcal{L}_{1}^{\Lambda, \Lambda_{0}, 0}=\mathcal{L}_{2}^{\Lambda, \Lambda_{0}, 0}=0$ (reason for connected \& amputated)
- Can also give boundary conditions at $\Lambda=0$ instead of $\Lambda=\Lambda_{0}$ :

$$
\mathcal{L}_{n}^{0, \Lambda_{0}, \ell}(\mathbf{k})=\mathcal{L}_{n}^{\Lambda_{0}, \Lambda_{0}, \ell}(\mathbf{k})-\int_{0}^{\Lambda_{0}} \partial_{\lambda} \mathcal{L}_{n}^{\lambda, \Lambda_{0}, \ell}(\mathbf{k}) \mathrm{d} \lambda
$$

■ For $n+|w| \leq 4$ arbitrary conditions at $\Lambda=0$ and $\mathbf{k}=0$, for $n+|w|>4$ we have $\partial^{w} \mathcal{L}_{n}^{\Lambda_{0}, \Lambda_{0}, \ell}(\mathbf{k})=0$

- Additional momentum derivatives $\partial^{W}$ are needed to close induction $\left(\mathcal{L}_{n}^{\Lambda, \Lambda_{0}, \ell}\right.$ smooth in $\mathbf{k}$ for all $\Lambda<\Lambda_{0}$ )

The RG flow equation framework 4/4

- Bounds: $\left\|\partial^{w} \mathcal{L}_{n}^{\Lambda, \Lambda_{0}, \ell}(\mathbf{k})\right\| \leq \sup (\Lambda, m)^{4-n-|w|} \mathcal{P}\left(\ln _{+} \frac{\Lambda}{m}, \ln +\frac{|\mathbf{k}|}{m}\right)$ $\left\|\partial^{w} \partial_{\Lambda_{0}} \mathcal{L}_{n}^{\Lambda, \Lambda_{0}, \ell}(\mathbf{k})\right\| \leq \Lambda_{0}^{-2} \sup (\Lambda, m)^{5-n-|w|} \mathcal{P}\left(\ln _{+} \frac{\Lambda_{0}}{m}, \ln +\frac{|\mathbf{k}|}{m}\right)$
■ Uniformly bounded and convergent as $\Lambda_{0} \rightarrow \infty$, finite as $\Lambda \rightarrow 0$
- Functionals with operator insertions have similar flow equation, induction scheme works if one ascends in the number of insertions
- Boundary conditions for one insertion at $\Lambda=0$ and $\mathbf{k}=0$ for $n+|w| \leq\left[\mathcal{O}_{A}\right]$, at $\Lambda=\Lambda_{0}$ for the rest
- Bounds: $\left\|\partial^{w} \mathcal{L}_{n}^{\Lambda, \Lambda_{0}, \ell}\left(\mathcal{O}_{A}(0) ; \mathbf{k}\right)\right\| \leq \sup (\Lambda, m)^{\left[\mathcal{O}_{A}\right]-n-|w|} \mathcal{P}\left(\ln _{+} \frac{\Lambda}{m}, \ln +\frac{|\mathbf{k}|}{m}\right)$ $\left\|\partial^{w} \partial_{\Lambda_{0}} \mathcal{L}_{n}^{\Lambda, \Lambda_{0}, \ell}\left(\mathcal{O}_{A}(0) ; \mathbf{k}\right)\right\| \leq \Lambda_{0}^{-2} \sup (\Lambda, m)^{\left[\mathcal{O}_{A}\right]+1-n-|w|} \mathcal{P}\left(\ln +\frac{\Lambda_{0}}{m}, \ln +\frac{|\mathbf{k}|}{m}\right)$
- $\mathcal{L}_{n}^{\Lambda, \Lambda_{0}, \ell}\left(\mathcal{O}_{A}(x) ; \mathbf{k}\right)=\mathrm{e}^{-\mathrm{i} x \sum_{j} k_{j}} \mathcal{L}_{n}^{\Lambda, \Lambda_{0}, \ell}\left(\mathcal{O}_{A}(0) ; \mathbf{k}\right)$ is smooth in $x$


## Euclidean field theories: OPE in the RG Flow Equation framework

OPE in the RG flow equation framework 1/2

- Functionals with multiple insertions are distributions in the insertion points $x_{i}$, scaling degree controlled by additional counterterms of dimension $\Delta$
- Bounds for functionals with two insertions: $\left\|\partial^{w} \mathcal{L}_{n, \Delta}^{\Lambda, \Lambda_{0}, \ell}\left(\mathcal{O}_{A}(x) \mathcal{O}_{B}(y) ; \mathbf{k}\right)\right\| \leq$ $|x-y|^{\Delta-\left[\mathcal{O}_{A}\right]-\left[\mathcal{O}_{B}\right]} \sup (\Lambda, m)^{\Delta-n-|w|} \mathcal{P}\left(\ln _{+} \frac{\Lambda}{m}, \ln +\frac{|\mathbf{k}|}{m}\right)$
- Define OPE coefficients $\mathcal{C}_{A B}^{C}(x) \propto \partial_{\mathbf{k}}^{w} \mathcal{L}_{n,\left[\mathcal{O}_{C}\right]-1}^{\wedge, \Lambda_{0} \ell}\left(\mathcal{O}_{A}(x) \mathcal{O}_{B}(0) ; \mathbf{0}\right)$ for $\left[\mathcal{O}_{C}\right]<\left[\mathcal{O}_{A}\right]+\left[\mathcal{O}_{B}\right]$, and by additional derivatives w.r.t. $x$ otherwise
- Prove bounds for the OPE remainder

$$
\left\|\mathcal{L}^{\wedge, \Lambda_{0}, \ell}\left(\mathcal{O}_{A}(x) \mathcal{O}_{B}(0) ; \Phi\right)-\sum_{C:[C] \leq N} \mathcal{C}_{A B}^{C}(x) \mathcal{L}^{\wedge, \Lambda_{0}, \ell}\left(\mathcal{O}_{C}(0) ; \Phi\right)\right\|
$$

$$
\leq(N!)^{-\frac{1}{2}} \sup (m, \Lambda)^{n+N} K^{N}|x|^{N-\left[\mathcal{O}_{A}\right]-\left[\mathcal{O}_{B}\right]}\|\Phi\|_{(N+2)(\ell+5) / 2}
$$

- OPE is actually convergent for arbitrary (spacelike) separations!

OPE in the RG flow equation framework 2/2

- Prove bounds on appropriate differences of finite sums, and show that they converge as $N \rightarrow \infty$ at each fixed perturbation order $\ell$
■ Factorisation: $\mathcal{C}_{A_{1} \cdots A_{n}}^{C}\left(x_{1}, \ldots, x_{n}\right)=\sum_{B} \mathcal{C}_{A_{1} \cdots A_{k}}^{B}\left(x_{1}, \ldots, x_{k}\right) \mathcal{C}_{B A_{k+1} \cdots A_{n}}^{C}\left(x_{k}, \ldots, x_{n}\right)$ holds for all $\frac{\max _{1 \leq i \leq k}\left|x_{i}-x_{k}\right|}{\min _{k<j \leq n}\left|x_{i}-x_{k}\right|}<1$
- Coupling constant derivative (for $\lambda \phi^{4}$ interaction):

$$
\begin{aligned}
& \partial_{\lambda} \mathcal{C}_{A B}^{C}(x, y)=-\int\left[\mathcal{C}_{\phi^{4} A B}^{C}(z, x, y)-\sum_{D:\left[\mathcal{O}_{D}\right] \leq\left[\mathcal{O}_{A}\right]} \mathcal{C}_{\phi^{4} A}^{D}(z, x) \mathcal{C}_{D B}^{C}(x, y)\right. \\
& \left.\quad-\sum_{D:\left[\mathcal{O}_{D}\right] \leq\left[\mathcal{O}_{B}\right]} \mathcal{C}_{\phi^{4} B}^{D}(z, y) \mathcal{C}_{A D}^{C}(x, y)-\sum_{D:\left[\mathcal{O}_{D}\right] \leq\left[\mathcal{O}_{C}\right]} \mathcal{C}_{A B}^{D}(x, y) \mathcal{C}_{\phi^{4} D}^{C}(z, y)\right] \mathrm{d}^{4} z
\end{aligned}
$$

■ Explicit formula in perturbation theory in $\lambda$ (OPE coefficients of free theory known from previous section)

- Finite formula in fixed (BPHZ-like) renormalisation scheme with explicit counterterms that ensure convergence


## Application to Conformal Field Theories (CFTs)

Conformal Field Theories $1 / 3$

- In addition to Poincaré covariance, the correlation functions of CFTs are covariant under scale transformations $x^{\mu} \rightarrow x^{\prime \mu}=\alpha x^{\mu}$ and special conformal transformations $x^{\mu} / x^{2} \rightarrow x^{\prime \mu} / x^{\prime 2}=x^{\mu} / x^{2}-b^{\mu}$
■ Classically scale-invariant theories: massless $\lambda \phi^{4}$ or Yang-Mills theory in 4D
■ In the quantum theory scale invariance is mostly anomalous, but examples of CFTs known (or at least conjectured): critical Ising model or Liouville theory in 2D, $\mathrm{O}(N)$ model or QED with $N_{\mathrm{f}}>2$ fermions in 3D, $\mathcal{N}=4$ super-Yang-Mills in 4D
- In CFT, all composite operators classified into primaries and descendants (derivatives of primaries)
- Two- and three-point functions of primaries completely fixed by conformal symmetry: $\left\langle\mathcal{O}_{A}(x) \mathcal{O}_{B}(y)\right\rangle_{0}=\frac{\delta_{A B} t_{A B}(x, y)}{|x-y|^{2\left[\mathcal{O}_{A}\right]}}$, where $t_{A B}$ : fixed tensor structure

Conformal Field Theories 2/3

$$
\left\langle\mathcal{O}_{A}(x) \mathcal{O}_{B}(y) \mathcal{O}_{C}(z)\right\rangle_{0}=\frac{\sum_{\alpha} \lambda_{A B C}^{\alpha} t_{A B C}^{\alpha}(x, y, z)}{|x-y|^{\left[\mathcal{O}_{A}\right]+\left[\mathcal{O}_{B}\right]-\left[\mathcal{O}_{C}\right]}|y-z|^{\left[\mathcal{O}_{B}\right]+\left[\mathcal{O}_{C}\right]-\left[\mathcal{O}_{A}\right]}|x-z|^{\left[\mathcal{O}_{A}\right]+\left[\mathcal{O}_{C}\right]-\left[\mathcal{O}_{B}\right]}},
$$

where $t_{A B C}^{\alpha}$ : fixed tensor structure, $\lambda_{A B C}^{\alpha}$ : constant determining the theory (one constant for each possible tensor structure $\alpha$ )

- OPE is very powerful for CFTs, since is is convergent even non-perturbatively (physical argument: scale invariance implies that asymptotic equality is equal to convergence)
- Higher n-point functions are thus in principle known, since they are determined by OPE with factorisation condition, tensor structures (conformal blocks) which are fixed by representation theory of conformal group, and three-point OPE coefficients $\lambda_{A B C}^{\alpha}$
- In practice, already determination of conformal blocks in closed form is difficult for higher $D$ (in particular $D=4$ ) and operators with spin

Conformal Field Theories 3/3

- Assume CFT with marginal interaction $g \mathcal{V}$ (that is, conformal for all $g$ )

■ Formula for coupling-constant dependence of OPE coefficients translates into a formula for the coupling-constant dependence of the dimensions of primary operators $\left[\mathcal{O}_{A}\right]$ and three-point OPE coefficients $\lambda_{A B C}^{\alpha}$ :

- $\lambda_{A A \mathbb{1}} \partial_{g}\left[\mathcal{O}_{A}\right]=\sum_{\alpha} \mathcal{D}_{A}^{\alpha} \lambda_{A A \mathcal{V}}$ and
$\partial_{g} \lambda_{A B C}^{\alpha}=\sum_{D} \sum_{\beta, \gamma}\left[{ }^{a} \mathcal{T}_{A B C D}^{\beta \gamma \alpha} \lambda_{\mathcal{V A D}}^{\beta} \lambda_{B C D}^{\gamma}+{ }^{b} \mathcal{T}_{A B C D}^{\beta \gamma \alpha} \lambda_{A B D}^{\beta} \lambda_{\mathcal{V} C D}^{\gamma}+{ }^{c} \mathcal{T}_{A B C D}^{\beta \gamma \alpha} \lambda_{\mathcal{V} B D}^{\beta} \lambda_{A C D}^{\gamma}\right]$, where constants $\mathcal{D}$ and ${ }^{\circ} \mathcal{T}$ are fixed by conformal symmetry
■ Infinite-dimensional coupled system of ordinary differential equations for the conformal data, consistent under non-degeneracy condition on the $\left[\mathcal{O}_{A}\right]$
■ In principle, system is (approximately) solvable once initial conditions are given, but the obvious condition (free theory for $g=0$ ) is highly degenerate


## Outlook: The OPE in curved spacetimes and application in the AdS/CFT correspondence

## Outlook (work in progress)

- Generalisation of results for the (perturbatively interacting) OPE to curved spacetime
- Proof of convergence and factorisation free theory in analytic spacetimes such as (Anti-)de Sitter (AdS) almost complete
- Proof of gauge invariance of the OPE for gauge theories more complicated

■ Formulas for coupling-constant dependence of OPE coefficients can be obtained after a redefinition of composite operators (use renormalisation freedom to pass to specific scheme)
■ Idea of my DFG project: show that the OPE and the formula for the coupling-constant dependence of the coefficients are valid in AdS, can be restricted to a time-like hypersurface and moved to the conformal boundary of AdS

- Use the AdS/CFT correspondence to map this restriction from the boundary of AdS to the dual CFT, obtaining a dual system of ODEs for the conformal data

Thank you for your attention

## Questions?

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Literature: arXiv:1511.09425, arXiv:1603.08012, arXiv:1710.05601 and references therein

